Finite Element Analysis

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Lecture No. # 38

We are looking at axisymmetric elasticity problems, in the last class we looked at axisymmetric linear triangular element. Basically, three-dimensional elasticity problems involving axisymmetric geometry, and loading can be treated as two-dimensional problems. And in the last class we looked at the governing equations, differential equations, and also we looked at how to derive finite element equations for a linear triangular element. We will continue with that before we do that, let us look back what we have done in the last class.

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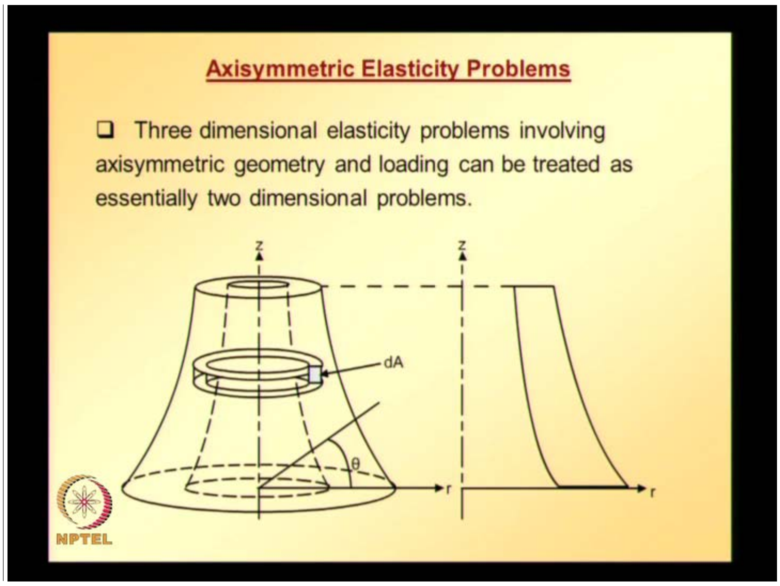

So, basically three-dimensional elasticity problems involving axisymmetric geometry, and loading as shown in the figure, here here a typical situation with z axis as axis of revolution is illustrated in the figure, for this kind of typical situation, three-dimensional elasticity problems can be modeled as two-dimensional problems.

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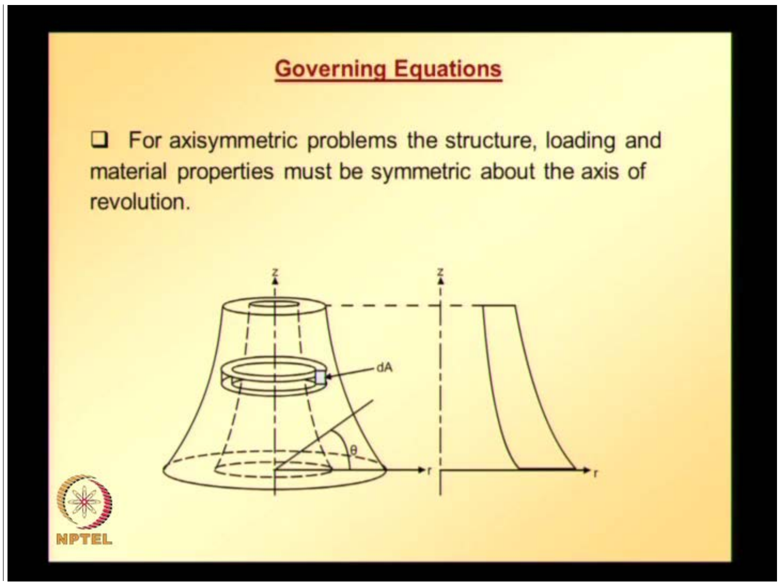

And one more important thing is for axisymmetric problems, the structure loading and material properties must be symmetric about axis of revolution. Here, in this particular situation that we are looking at z axis is axis of revolution. So, we can model this three dimensional problem as Axisymmetric problem, only if structure loading and material properties are all symmetric with respect to the axis of revolution. Even if geometry is symmetric about the axis of revolution or the structure is symmetric about the axis of revolution, if loading and material properties are not symmetric, then we cannot use this axisymmetric model.

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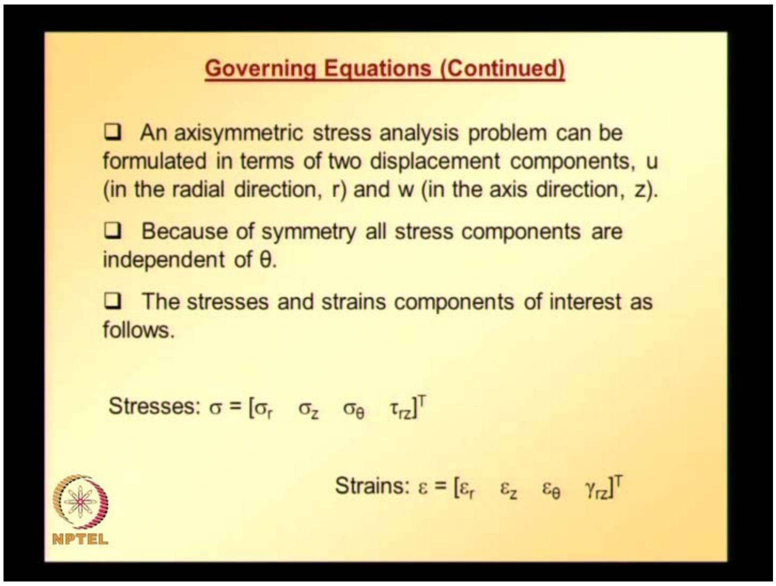

An Axisymmetric stress analysis problem can be formulated in terms of two displacement components u and w. u is displacement component in the radial direction or w is the displacement component in axial direction is z. These two are similar to u and v, u for displacement component in the x direction; v for displacement component in the y direction, that we used for plane stress plane strain problems. And here for axisymmetric problems because of symmetry all these stress components are independent of theta. Stresses and strain components of interest are as follows.

Four components of stresses sigma r, sigma z, sigma theta, tau r z all these components are put together in a vector denoted with letter sigma. Similarly, strains epsilon r, epsilon z, epsilon theta, gamma r z put together in a vector denoted with epsilon. So, four components of stresses, four components of strains under small displacements and strain assumptions, strain displacement relation can be written as follows.

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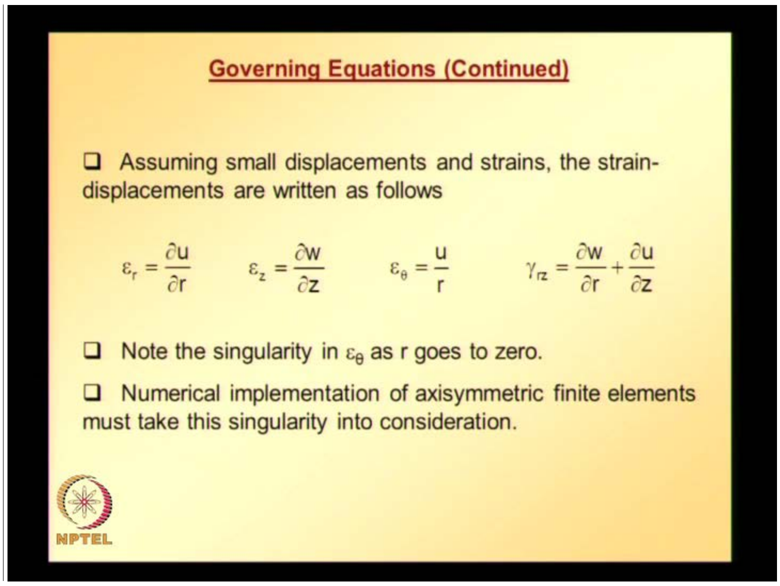

And here please note that, epsilon theta is having r in the denominator that is epsilon theta is going to go to infinity as r tends to zero. So, note the singularity in epsilon theta as r goes to zero and we need to take care of this during numerical implementation of axisymmetric finite elements. Numerical implementation of axisymmetric finite elements must take this singularity into consideration.

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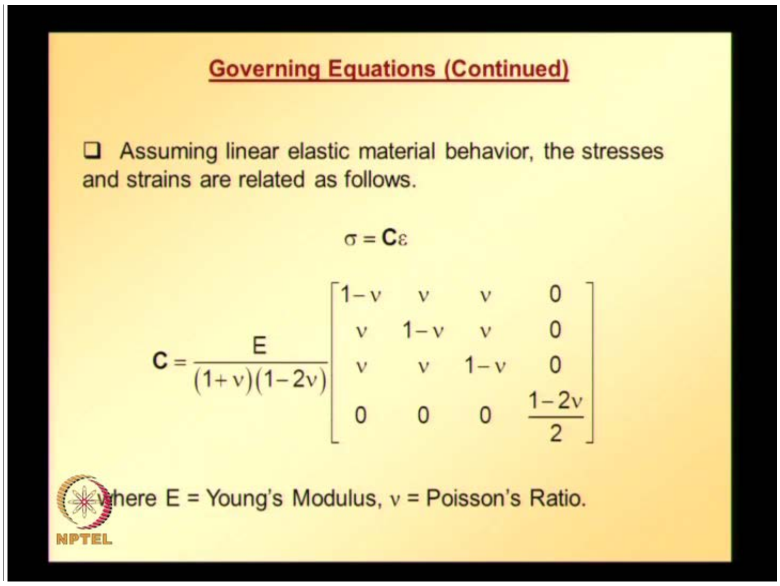

Next, we need to know the constitutive matrix, assuming linear elastic material behavior, stresses and strains are related via this equation sigma is equal to c times epsilon, where c is the constitute matrix which is going to be function of two material parameters. Youngs modules and Poissons ratio for axisymmetric problem constitutive the matrix is going to be four by four in dimension. Because, there are four stress components and four strain components.

And comparing these equations that we just looked at for axisymmetric problem with plane stress plane strain with the equations corresponding to plane stress plane strain conditions, it can be seen that problems are very similar. And only thing is primary unknowns of Axisymmetric problem are u and w whereas, for plane stress plane strain problems the primary unknowns are displacements in the x direction and displacements in the y directions u and v.

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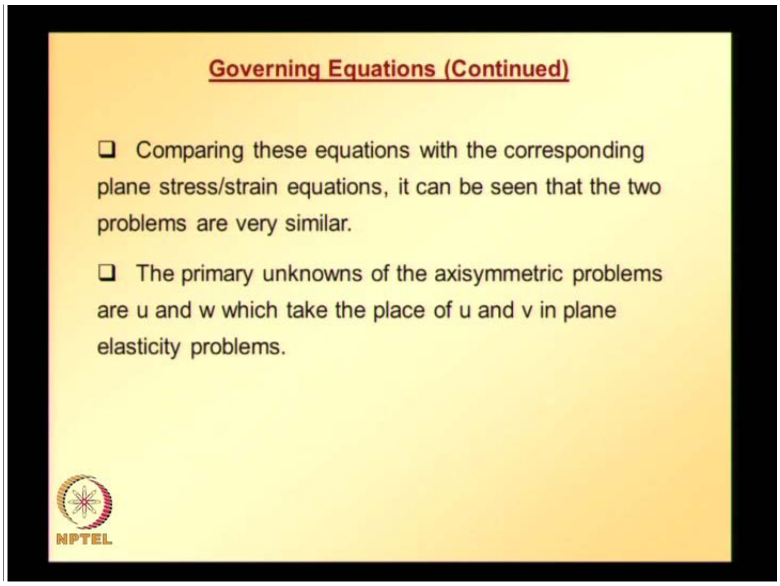

So, before we derived axisymmetric derived finite element equations for axisymmetric linear triangular element in the last class, we also looked at potential energy functional.

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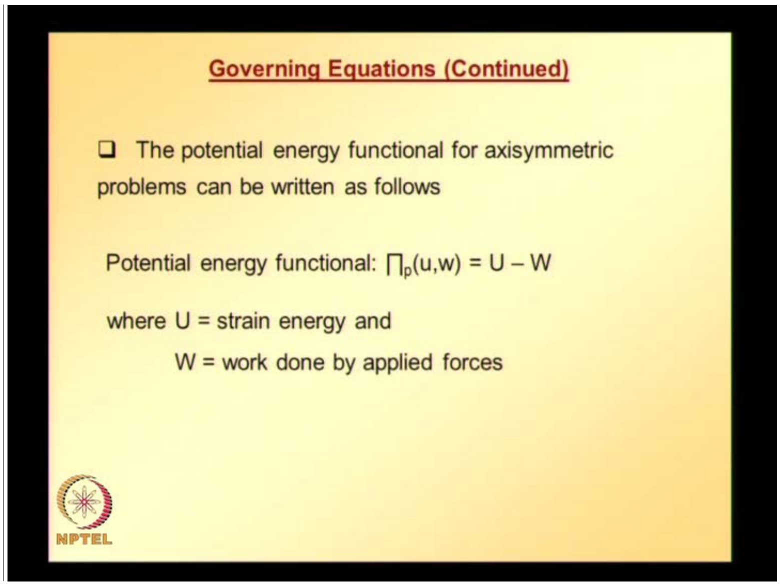

So, potential energy functional can be written similar to that, what we have written for plane stress plane strain problems. Because, both problems are looking similar except the displacement components are different. So, potential energy functional for axisymmetric problems can be written as follows. Potential energy functional pi is function of u and w, where u is displacement component in r direction, w is displacement component in z direction. So, potential energy functional pi is u minus w, the definition of u and w are similar to that for plane stress plane strain problems. u is strain energy; w is work done by the applied forces. So, now we need to look in detail, how to calculate strain energy for axisymmetric problems in work done, by the applied forces are axisymmetric problems.

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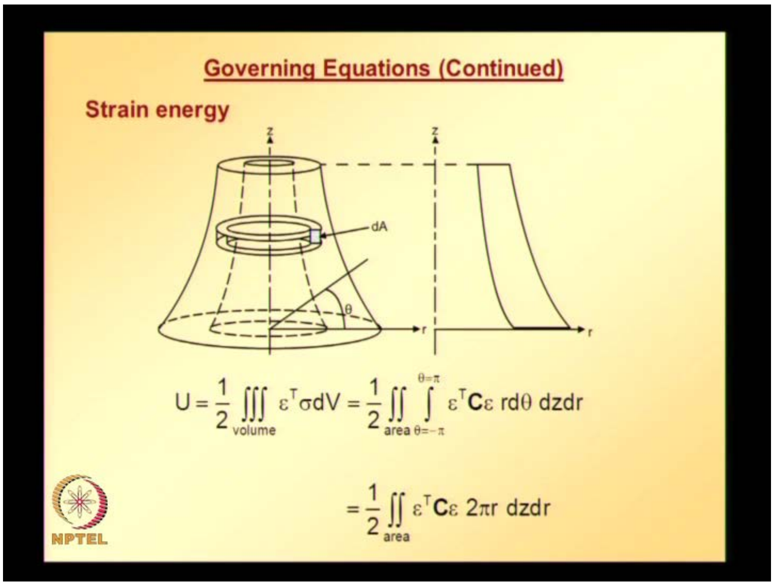

So, strain energy is given by this half volume integral epsilon transpose sigma and this is since, we are dealing with axisymmetric problems. We can integrate this with respect to theta going from minus pi to pi and when we do that this strain energy becomes an area integral or surface integral. So, work done by the next quantity that we require is work done by the applied surface forces.

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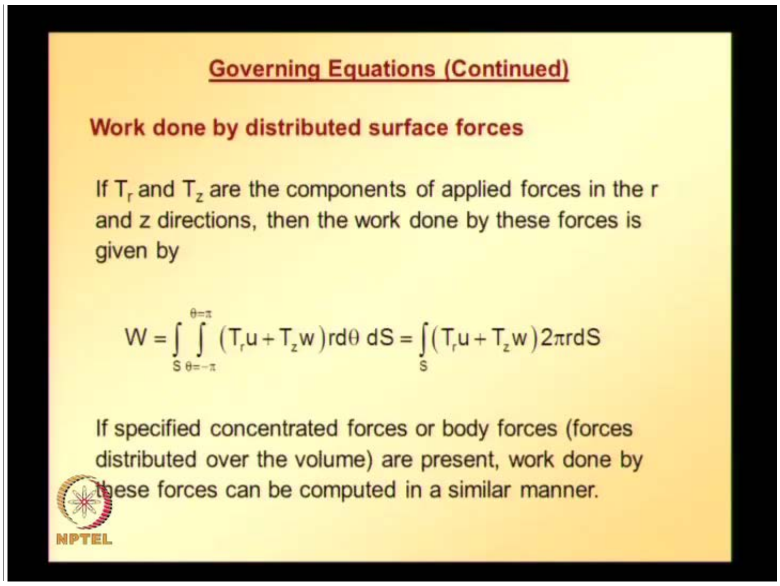

Work done is given by displacement component in r direction times, traction and r direction displacement component in z direction times, traction component in z directions. So, if t r t z is the components of applied forces in r and z directions work done by the forces is given by this. Again, since this is an Axisymmetric problem we can integrate with respect to theta going from minus pi to pi and then this integral of work done by the distributed surface forces reduces to a line integral. So, potential energy functional is given by strain energy minus work done by the applied forces strain energy is basically area integral.

We need to integrate over the surface area of axisymmetric model and work done by the distributed forces, we need to evaluate at line integral along the side or edge, along which surface forces are prescribed. If specified concentrated forces or body forces are present work done by this force can be computed in a similar manner. So, basically this entire portion we covered in the last class just for brief review, I went through this once again. So, now let us look at derivation of finite element equations for a quadrilateral element.

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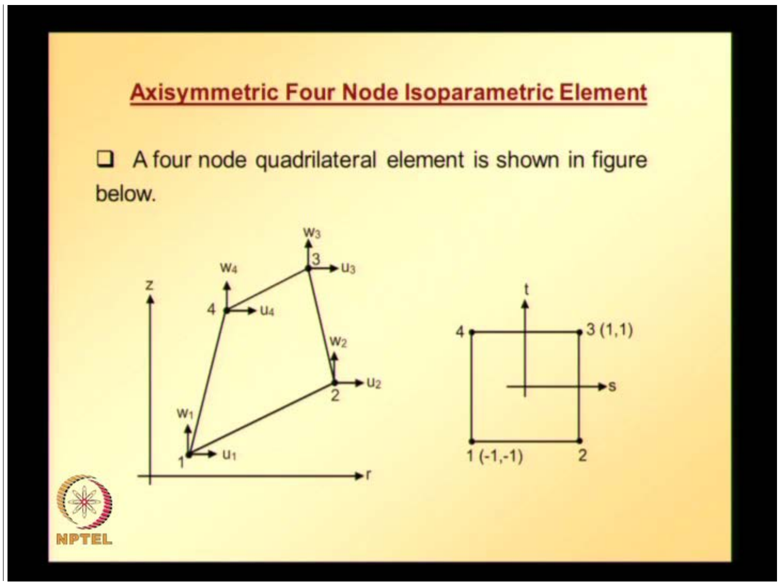

Axisymmetric four node Isoparametric element: A four node quadrilateral element is shown in the figure. All the displacement components, at all nodes are shown and each element each of this quadrilateral element is actually a ring with quadrilateral representing its cross section. Because we are dealing with problem, which is obtained by revolution about z axis; so, the element equations can be derived using Isoparametric mapping concepts, that we looked at every year and for that we require a parent element. So, on the right hand side a parent element is shown in the figure. That parent element is looking similar to what we used for plane stress plane strain problems. Four node parent element with nodes with all the four nodes at four vertices of square the coordinates of each of the nodes are also indicated in the figure.

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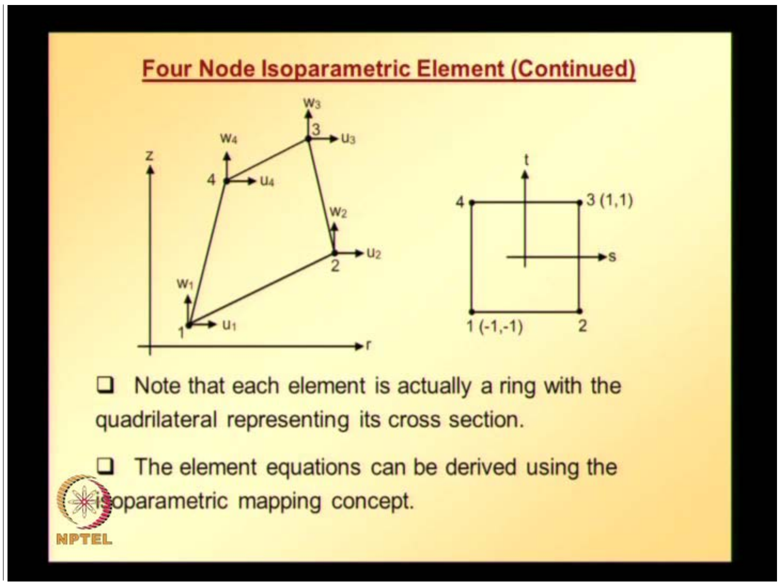

Note that each element is actually ring with quadrilateral representing its cross section element equations can be derived using Isoparametric mapping concept. So, the trial solutions can be expressed in terms of shape functions of the parent element and displacement components at each of the four nodes in r and z directions.

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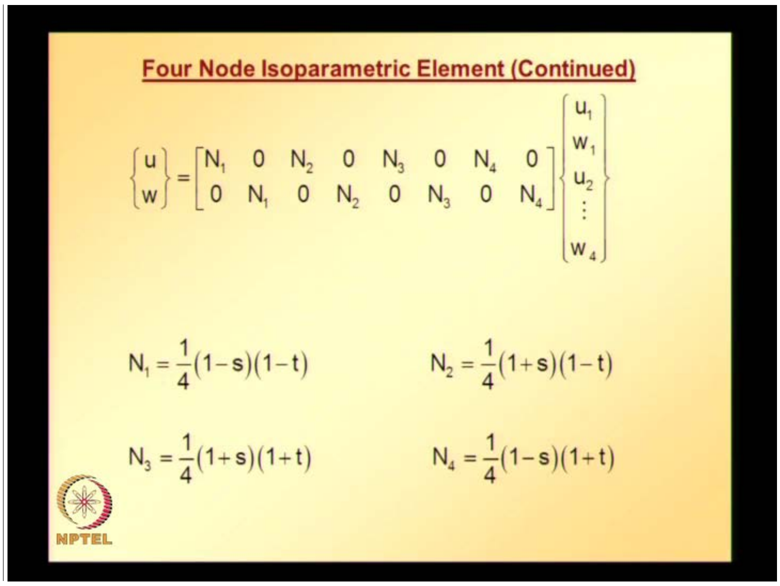

So, this is how we can calculate displacement component in r and z directions at any point. Once we know the displacement components at all the four nodes using parent element shape functions, where parent element shape functions are similar to that. We already looked at for plane stress plane strain problems. Basically these shape functions can be obtained using Lagrange interpolation formula in s and t directions.

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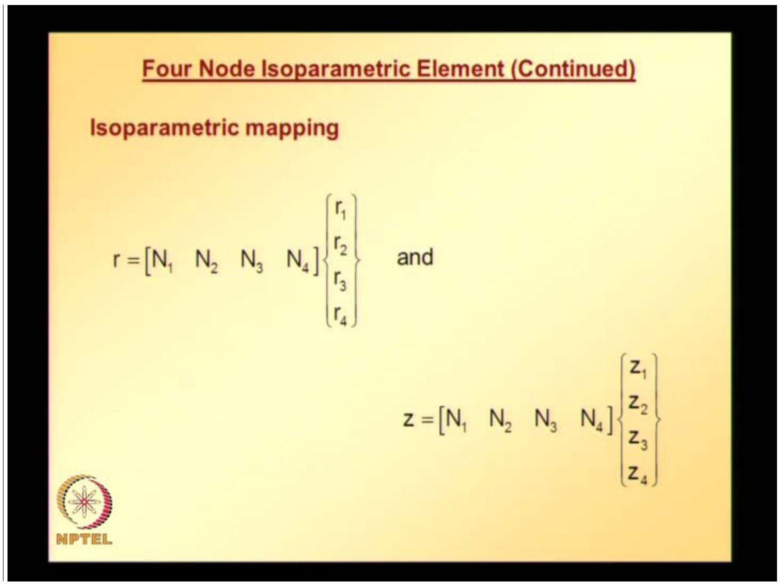

Now, Isoparametric mapping using Isoparametric mapping concept, we can write the coordinate r at any point, in terms of coordinates of all the four nodes radial coordinates of all the four nodes and also if you know the parent element shape functions. Similarly, z coordinate can be written in this manner and these two equations, we require for calculating Jacobian and determinant of Jacobian. Basically, Jacobian consists of partial derivative of r with respect to s and t partial derivative of z with respect to s and t. So, once we have these two equations, we need to take partial derivatives of these two equations with respect to s and t and then we can calculate what is Jacobian matrix j and also determinant of J.

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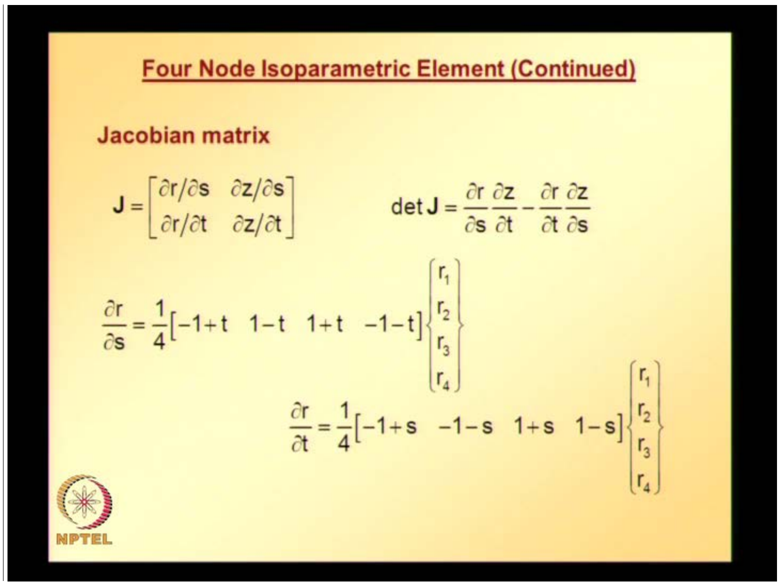

So, Jacobian matrix consists of components, partial derivative of r with respect s, partial derivative of z with respect s, partial derivative of r with respect to t, partial derivative of z with respect to t and determinant of J is given by this and for this we require partial derivative of r with respect to s, which can easily be obtained from the previous equations by taking partial derivative of shape functions N 1 to N 4 with respect to s.

Similarly, by taking derivatives of shape functions n 1 to n 4 with respect to t, we can easily write partial derivative of r with respect to t. So, similar expressions that are partial derivative of z with respect to s partial derivative of z with respect to t can also be easily written. Finally, we can calculate once we have these two equations or two expressions, we can easily calculate what is Jacobian matrix and determinant of Jacobian.

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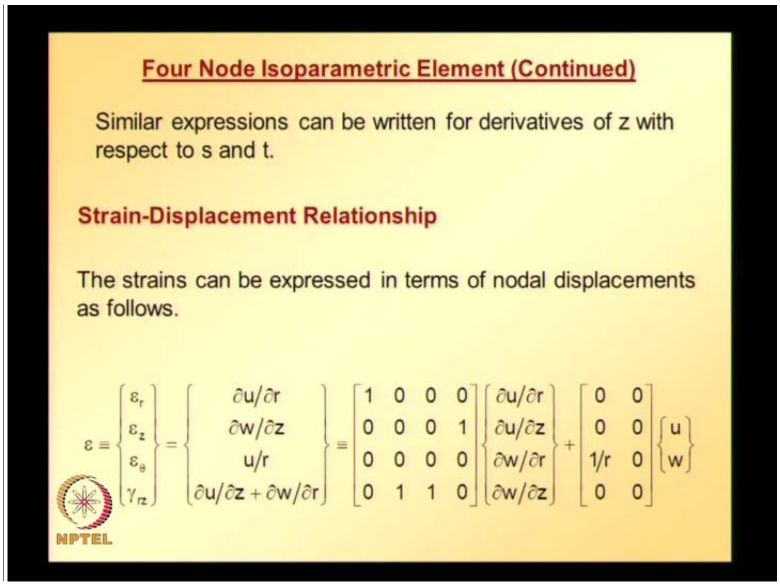

So, now we need to look at strain displacement relation, how to express strain displacement relation in terms of finite element shape functions? Strains can be expressed in terms of nodal displacements for axisymmetric problem the four components of strains are defined like this. We have seen this relation earlier; only thing is now we are putting them in a vector form all the components of strain and the strain vector, in terms of displacements can be rearranged and can be written as can be expressed in terms of matrix and vector form which is shown on the right hand side of the question.

So, to calculate strain vector epsilon, we need to know, what is partial derivative of u with respect r, partial derivative of u with respect to z, partial derivative of w with respect to r, partial derivative of w with respect to z. We know that since, we already have the trial solution in terms of nodal displacements and in terms of finite element shape functions. We know it is easy to calculate, partial derivative of u with respect to s and t, w with respect to s and t. So, we need to find a relation how to calculate partial derivative of u with respect to r and z in terms of partial in terms of partial derivative of u with respect to s and t. Similarly, we need to find the relation between partial derivative of w with respect to r and z and partial derivative of w with respect to s and t.

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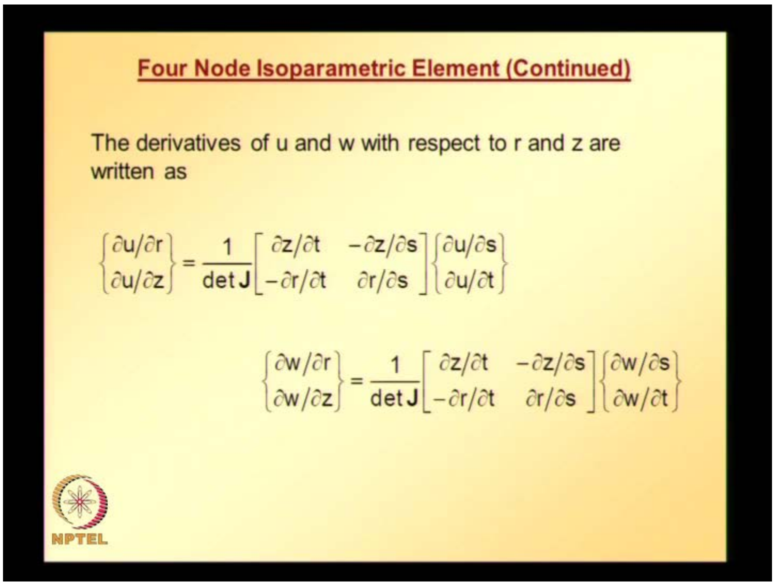

This can be obtained using these relations, which for we require to know what is determinant of J. So, using these two relations the partial displacement derivatives displacement component derivatives with respect to r and z can be replaced with respect to s and t. So, writing these two equations together, we get this equation we can substitute back into the equations that, we have seen for strains where we can replace partial derivatives of displacements with respect to r and z. In terms of partial derivative of displacement with respect to s and t and rearrange it.

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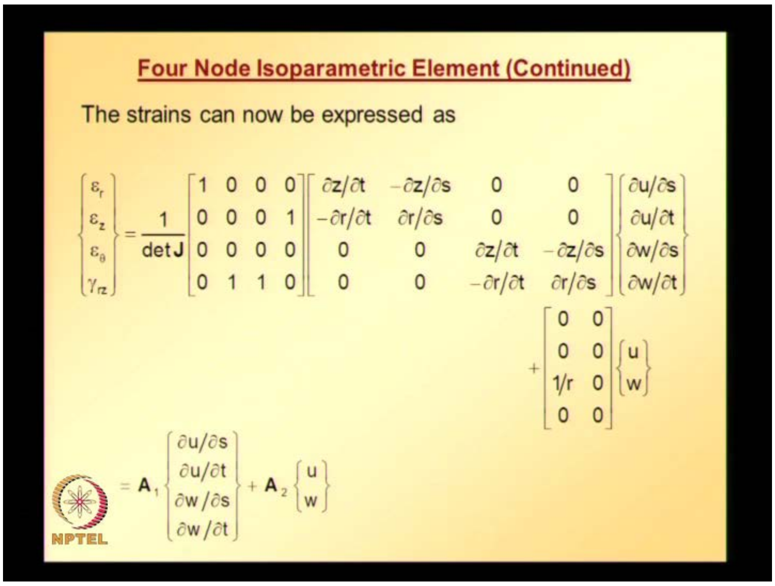

In this manner and we can define some intermediate quantities A 1 and A 2, where A 1 A 2 can easily be figured out by comparing the two equations.

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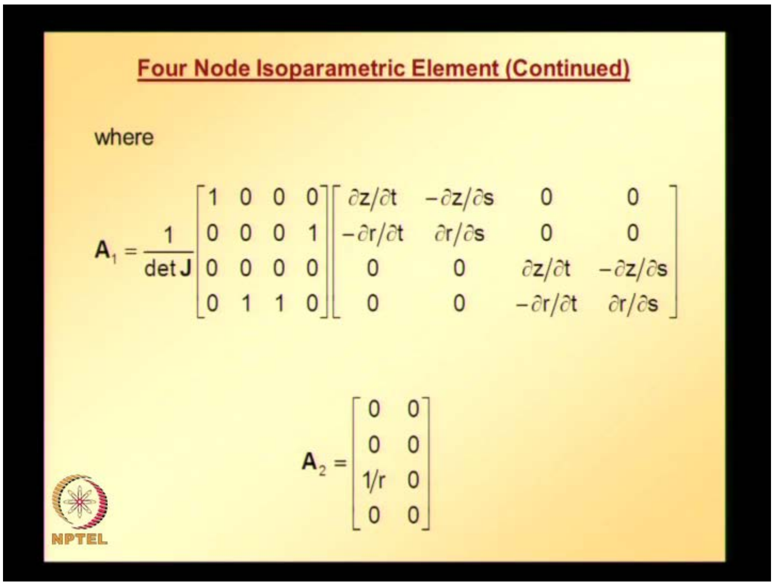

A 1 is defined like this; A 2 is defined in this manner. If you see the previous equation, it involves partial derivative of you with respect to s and t, partial derivative of w with respect to s and t.

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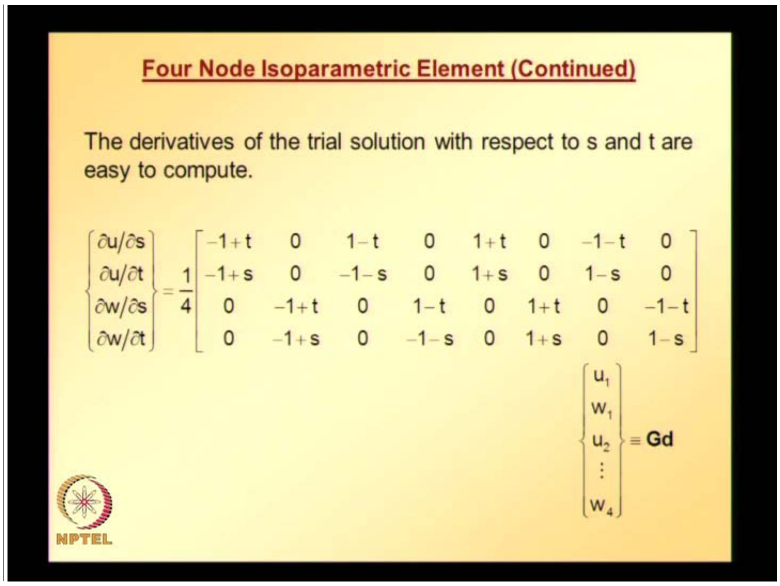

So, we need to make a note of how to calculate that, since we already know parent element shape functions, we can easily find how this equation look like and defining intermediate quantity g, which consists of partial derivative of shape functions with respect to s and t and d is nothing but, displacement component the vector of displacements consisting of all components of displacements at all nodes.

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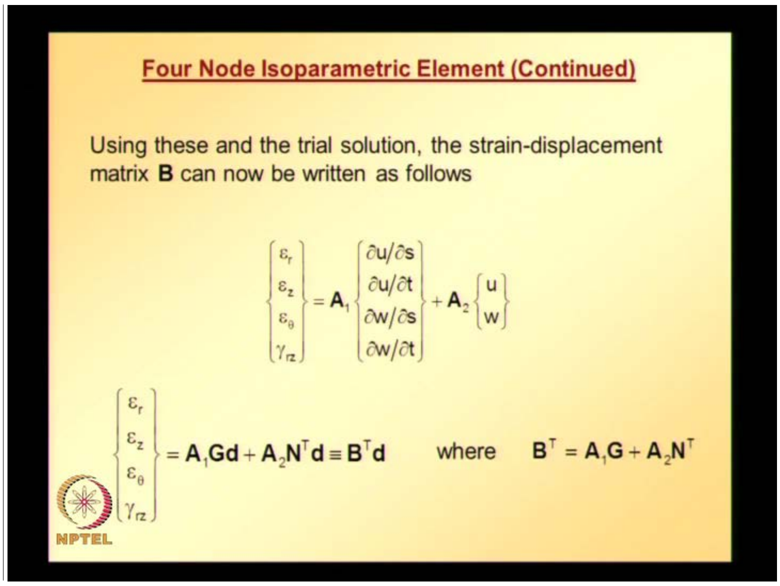

So, with this definitions strain displacement matrix can now be written like this. So, b transpose is equal to A 1 times G plus A 2 times N transpose. So, we already defined what is A 1 A 2 G and N transpose is nothing but, a vector consisting of finite element shape functions or matrix consisting of finite element shape functions. So, with this definition of strain, we can easily assemble what is element stiffness matrix.

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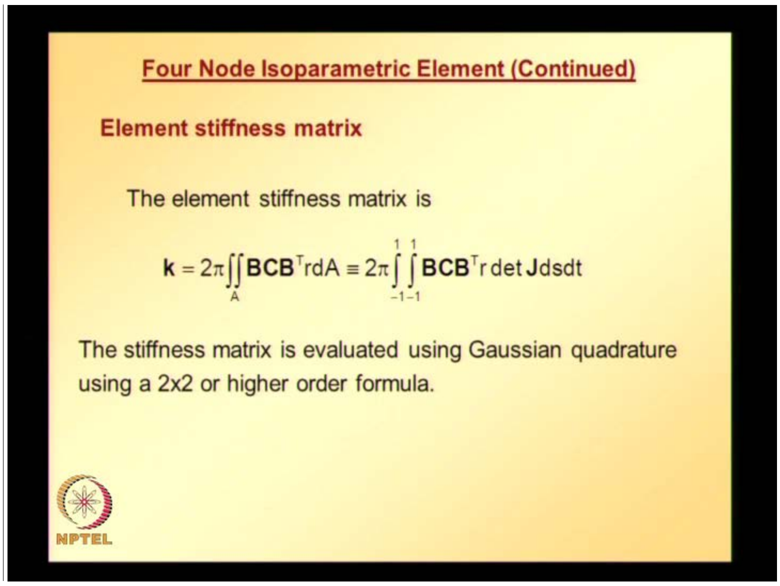

Element stiffness matrix is defined like this and since, b the strain displacement plane displacement matrix for the four node Isoparametric element. Four node Axisymmetric Isoparametric element that, we are looking at the strain displacement matrix p is not a constant. So, we need to adopt some kind of numerical integration scheme to evaluate this or to approximate this integral to get element stiffness matrix. So, we can use gauss Quadrature element stiffness matrix is evaluated using gauss Quadrature and using two by two or higher order formula if require.

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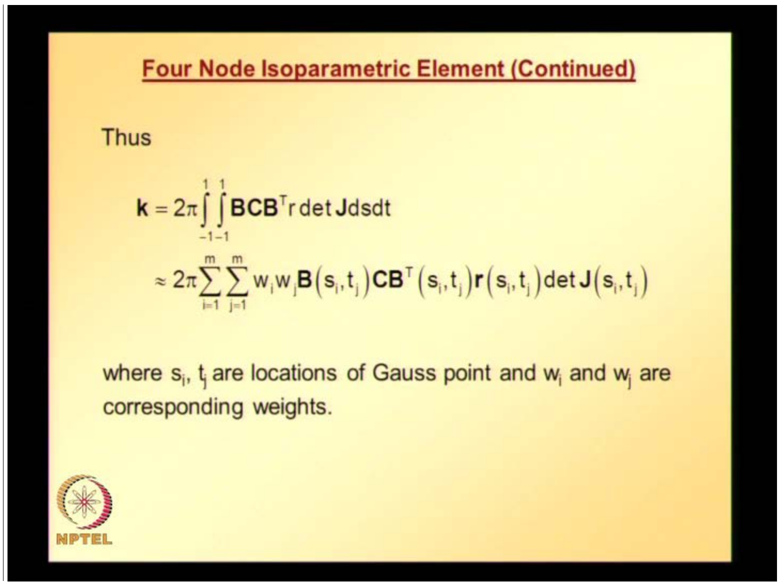

So, this integral for element stiffness matrix, once we decide how many integration points, we choose along s direction and along t direction we can approximate element stiffness matrix like this, which involves contribution from different integration points to the element stiffness matrix, where s i t j are locations of gauss points w i w j are corresponding weights. So, this is how we can assemble element stiffness matrix for four node Axisymmetric Isoparametric element. So, now we need to look at how to get equivalent nodal value vector, because finally if you want to solve the problem. We need to know, what is stiffness matrix? What is force vector equivalent nodal force? How to evaluate equivalent nodal forces?

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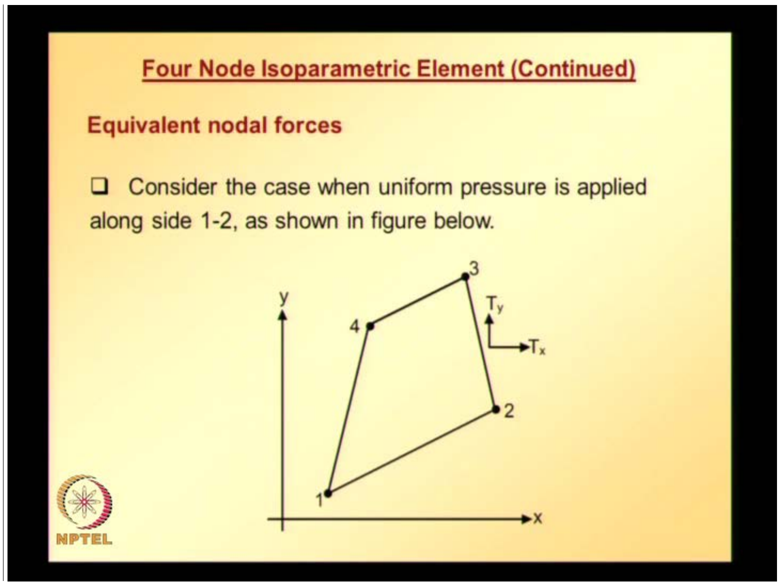

So, for that considers the case, when uniform pressure is applied along one of the sides. Let us say side 1 2 that is side connecting nodes 1 2 as shown in figure below. Here, there is a small typing mistake in the figure, x instead of r it is typed as x instead of z, it is typed as y. So read x and y as r and z. Similarly, T x is nothing but, T r T y is T z. So, now assuming uniform pressure is applied alongside 1 2 and also we should note that along side 1 2 shape functions of nodes 3 and 4 are 0 alongside 1 2. So, along this side 1 2 shape functions N 1 N 2 are only the shape functions, which are non zero and N 3 N 4 are 0 and also along side 1 2 since, there are only two nodes along side 1 2. The shape functions of nodes N 1 shape functions of nodes one and two that are N 1 and 2 are going to be linear functions.

So, this information is required because, we need to know what is the shape function matrix of all the four nodes along side 1 2 to assemble equivalent nodal force vector.

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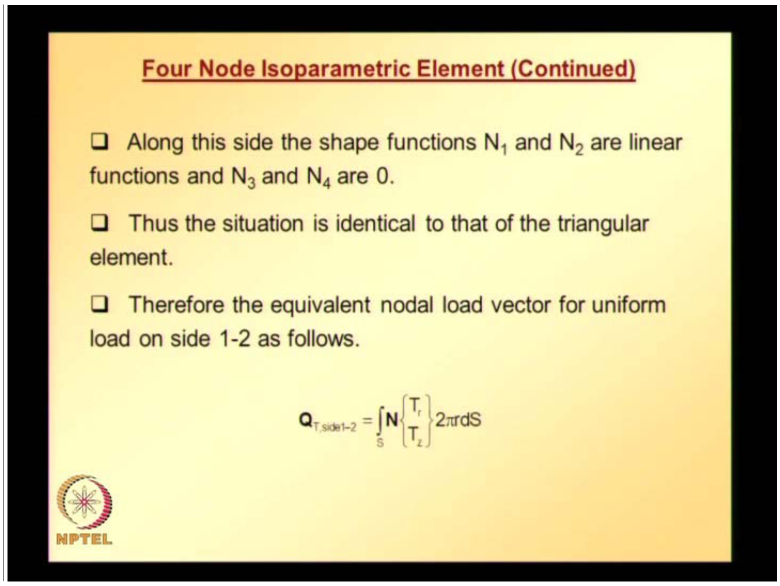

Whatever, we discussed now about the shape function matrix that is, not new that is similar to or that is identical to that, we already looked at when we are dealing with triangular elements. So, equivalent nodal force vector for uniform load along side 1 2 is given by this Q is the equivalent nodal vector load vector, Q T is for traction along side 1 2. So, it is written as Q T Q subscript t side 1 2, it is given by line integral shape function matrix or matrix consisting of shape functions along side 1 2. All the four shape functions alongside 1 2 times traction components along radial direction and axial direction.

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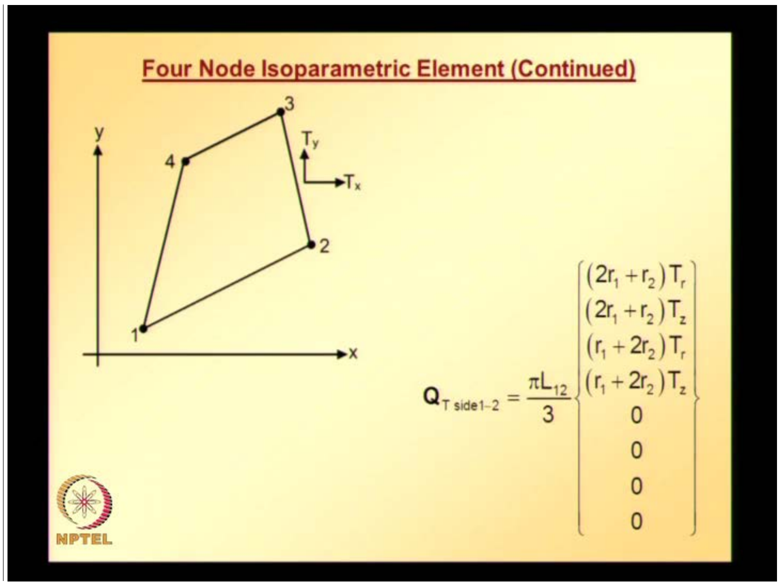

And by substituting all the quantities, please note that in the previous equation, we have 2 pi r can also the expressed in terms of finite element shape functions and the radial coordinates of all the four nodes. So, by doing that substitution and simplifying the previous integral, we get this equivalent nodal vector load vector. You can easily see that equivalent nodal vector load vector the components corresponding to nodes three and four are zero, because uniform pressure load is applied along side 1 2.

So, similar load vectors equivalent load a vectors can be assembled even if load is specified on the other sides 2 3 or 3 4 or 4 1. So, far we discussed how to assemble or how to get the elements stiffness matrix and also equivalent nodal force vector for four node Axisymmetric Isoparametric element. So, now we are ready to solve a problem so, what we will do is we will take the problem that we already looked at in the last class, where we used two axisymmetric linear triangular elements to solve the problem. So, here we are going to solve the same problem using only one four node Axisymmetric Isoparametric element using the equations that we just derived.

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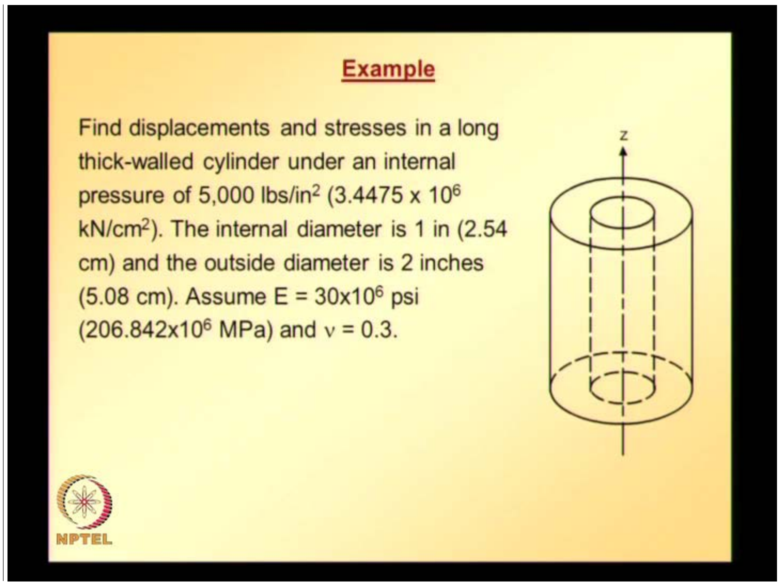

So, problems statement is as follows, find displacements and stresses in a long thick cylinder and under an internal pressure, internal diameter value is given and also outside diameter value is given material properties are specified. All the quantities are given both in SI units and FPS units. Since the figures corresponding to FPS units are round numbers, we will proceed and solve this problem with respect to the figures for FPS units. Since, this cylinder is long cylinder we can neglect the end effects and also since this problem, the structure is or geometry is Axisymmetric. It is satisfying Axisymmetric conditions and also loading is satisfying Axisymmetric conditions.

So, we can easily use axisymmetric model to solve this problem and also material properties are constant. So, they also satisfy axisymmetric conditions or symmetric all this that is, structure, geometry, material properties and also loading are symmetric with respect to axis of revolution. So, we can use axisymmetric model and we will be using one four node Axisymmetric Isoparametric elements element to solve this problem.

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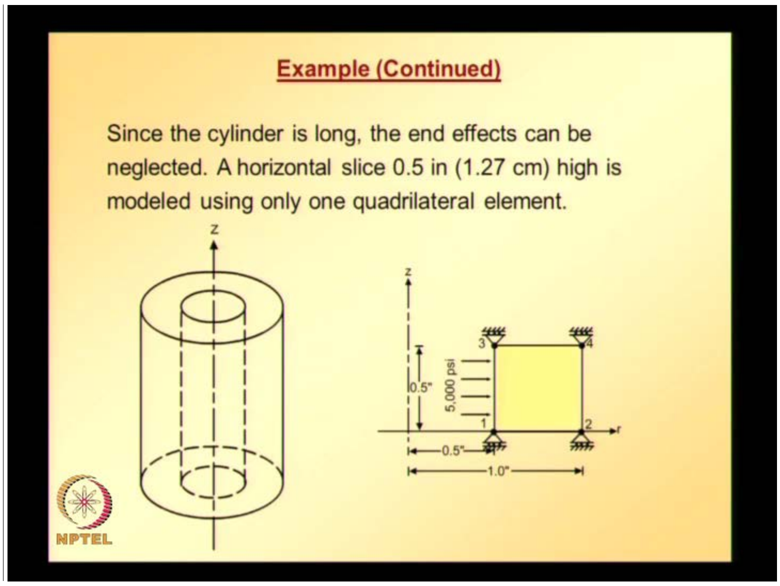

Since, this cylinder is long and f x can be neglected a horizontal slice of point one inch that is, 1.27 centimeters height is modeled using one quadrilateral element, that we discussed and this element the nodal numbers is also indicated and also since this problem is Axisymmetric problems and the nodes are constrained to have displacement only in radial direction. So, displacement component at all the four nodes in the axial direction is zero, also the traction is applied along side joining nodes three and one. Please note, that for the node numbering that is shown in the figure on the right hand side, local node one is same as global node one, local node two is same as global node two and local node three is global node four and local node four is global node three. So, we need to keep that in mind, when we are assembling the equations. So, moving in the counter clockwise directions element nodes are 1 2 4 3.

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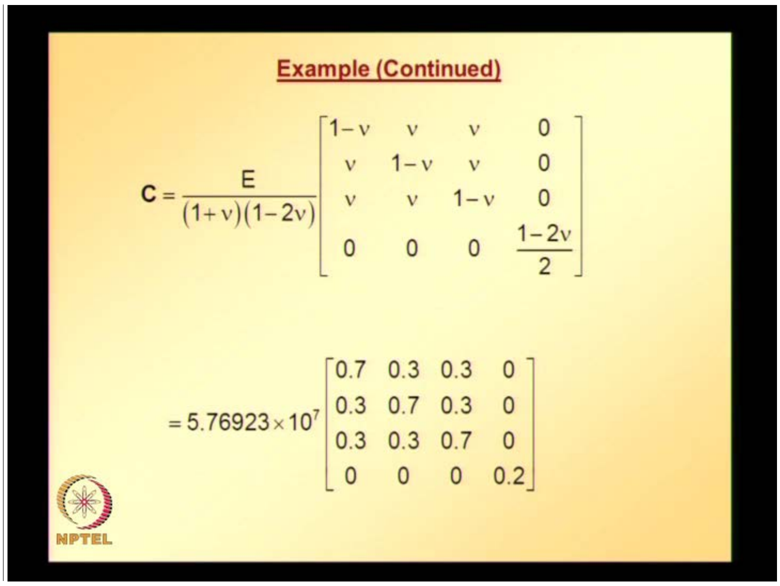

Accordingly, we can we need to note what are what are the a radial coordinates and also what are the axial coordinates of all the four nodes, before that constitutive matrix which is required for solving this problem is given by this e and Poissons ratio values are given Youngs modulus Poissons ratio values are given in the problem statements. So, using those values we can simplify and we get this constitutive matrix, the element node numbers are 1 2 4 3.

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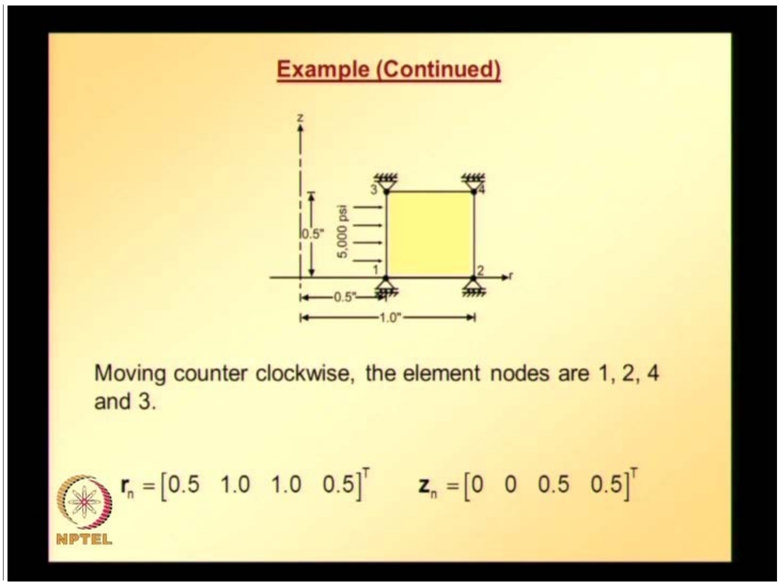

So, corresponding vector consisting of all radial coordinates in the radial direction and coordinates in the axial directions are given by these two vectors. Since, we are using only one element and we adopt two by two, if you adopt two by two Gaussian Quadrature. We know how to get the coordinates and weights of each of the integration points. So, here the calculation details corresponding to one integration points, we look at and similar operation can be repeated at other integration points and contribution from all integration points can be taken, to get the final element equation, final element stiffness.

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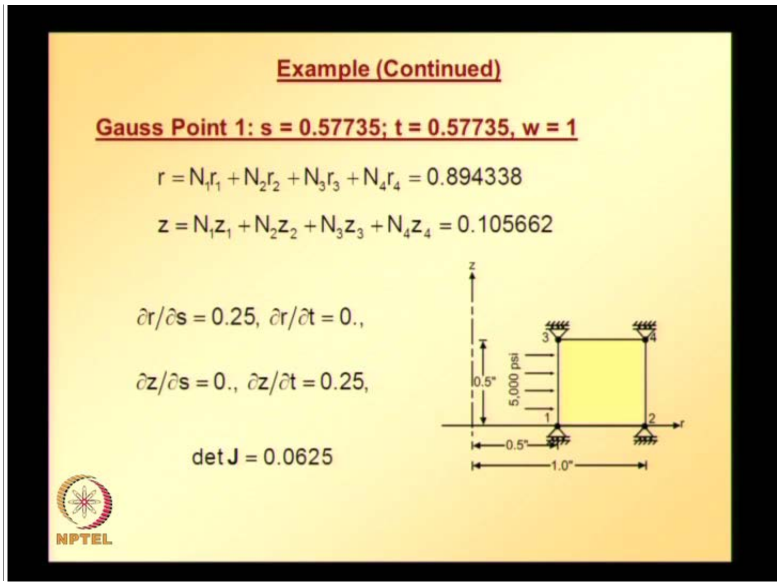

So, at integration point one s and t coordinates weights are shown there and using Isoparametric relation r is equal to N 1 r 1 plus N 2 r 2 plus N 3 r 3 plus N 4 r 4. We can easily find, what is the r value at this integration points is by substituting s and t values since, r 1 r 2 r 3 r 4 are already known for this particular element. Similarly, z coordinate for this integration point is given by this and these are required to calculate rest of the quantities, partial derivative of r with respect to s, partial derivative of r with respect to t, similarly, partial derivative of z with respect to s, partial derivative of z with respect to t.

Once, we know all these quantities we can easily calculate what a 1 a 2 and g matrix matrices and also shape function a matrix consisting of all the four node shape functions, which are required for calculating strain displacement matrix, which in turn is used for getting the element stiffness matrix.

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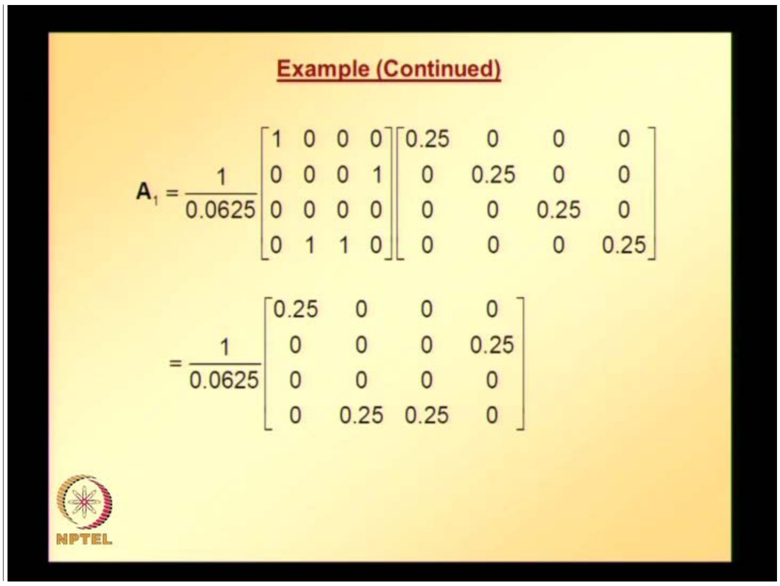

So, with these partial derivatives we can get derivative of Jacobian determinant of Jacobian. A 1 matrix, g matrix consisting of all the four shape function values of all four nodes a 2 matrix.

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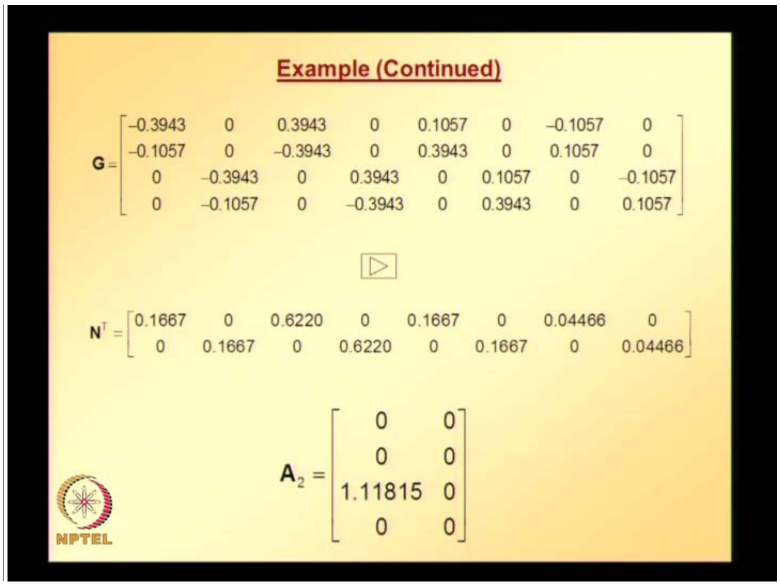

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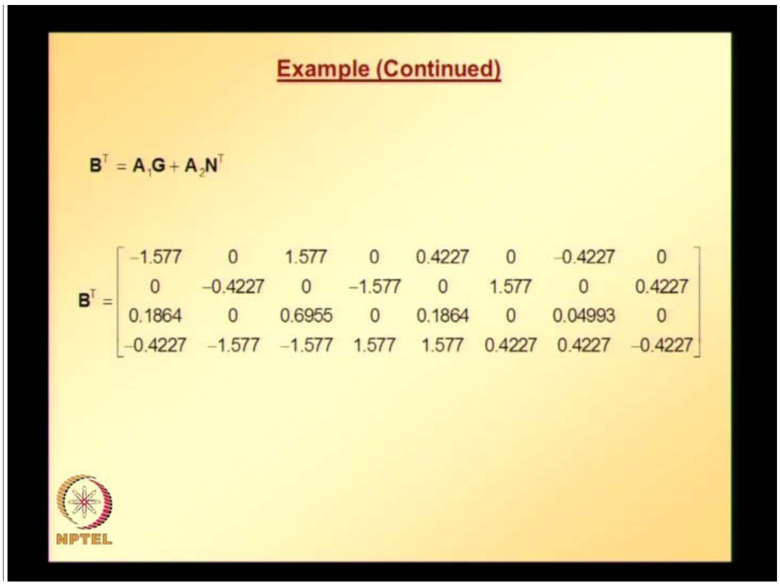

Once, we have all this we can substitute into this equation to find what strain displacement matrix b.

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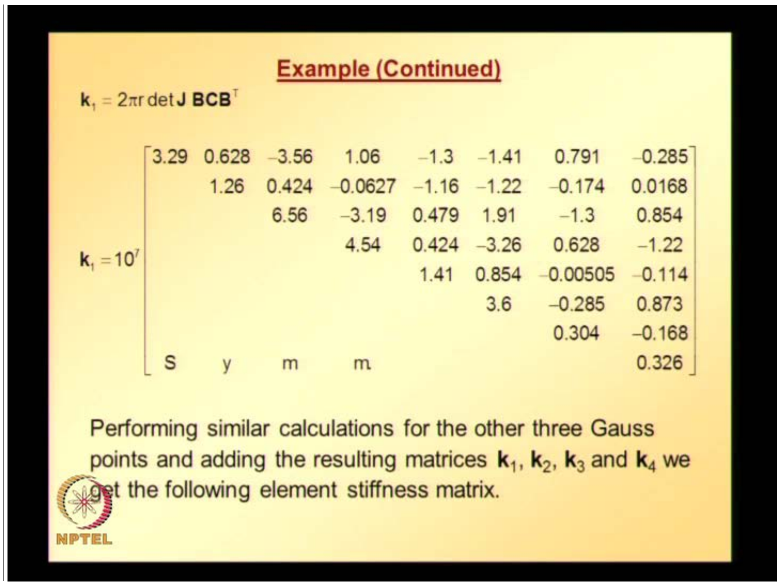

Once, we have this b matrix we can find the contribution to the elements stiffness matrix from this integration point by substituting into this equation. Here, k subscript one is used to denote contribution from first integration point to the element stiffness matrix. So, by plugging in all the values and simplifying us get this matrix, performing similar calculations for the other three integration points and adding the resultant matrices k 1 k 2 k 3 k 4, we get elements stiffness matrix which is shown here.

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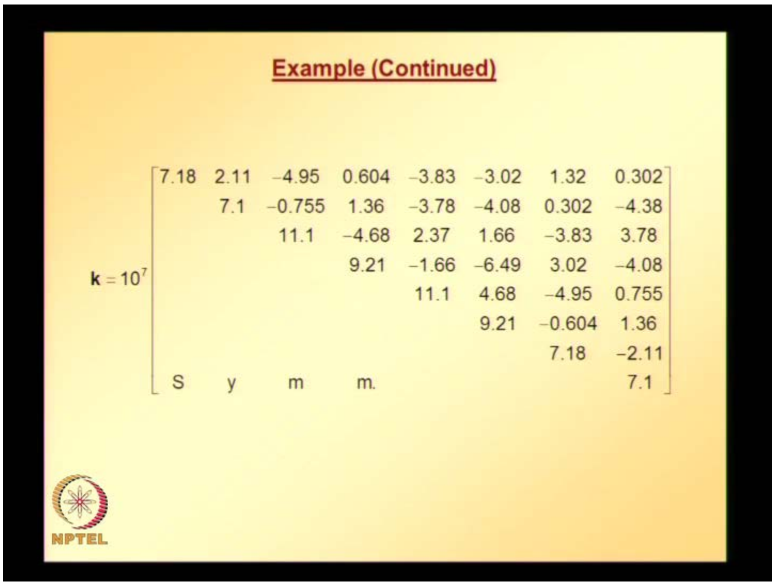

So, we got element stiffness matrix for one element that, we are adopting to solve this problem. Now, our job is to get equivalent nodal force vector.

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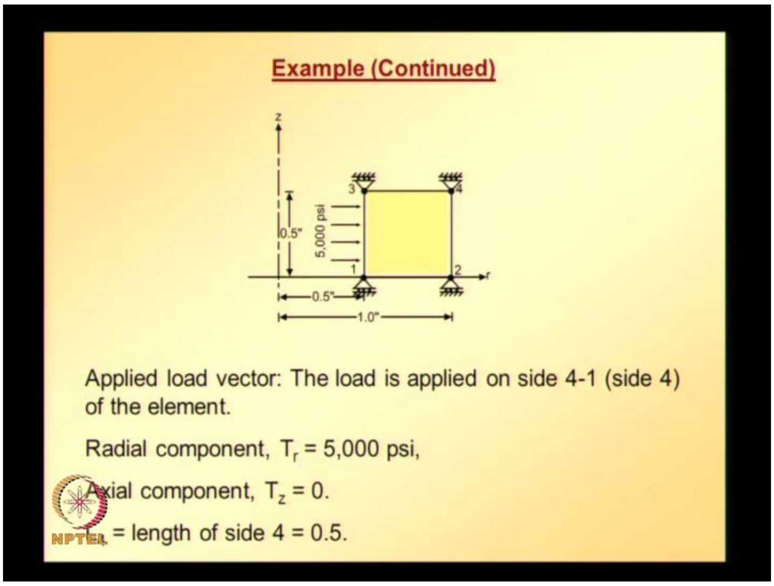

Before, we do that we need to note that traction is applied along only one side joining nodes global nodes three and one and global node three corresponds to local node four and global node one corresponds to global node one corresponds to local node one. So, applied load vector load is applied along side 4 1 here, when a when it is written as 4 1 that is with respect to the local node numbering and that corresponds that is corresponds to side four. So, load is applied only along side 4 1 or side four on the radial component and axial component of tractions are given here and also we need to note what is the length of this side, length of side four is 0.5.

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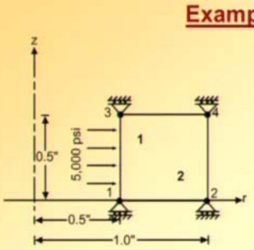

$$Q_{T\ side4} = \frac{\pi L_4}{3}\begin{Bmatrix} (2r_1 + r_4)T_r \\ (2r_1 + r_4)T_z \\ 0 \\ 0 \\ 0 \\ 0 \\ (2r_4 + r_1)T_r \\ (2r_4 + r_1)T_z \end{Bmatrix} = \begin{Bmatrix} 1250\pi \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 1250\pi \\ 0. \end{Bmatrix}$$

So, substituting all these quantities into the equivalent nodal load vector. Equivalent nodal vector formula, we get this. We already discussed how to derive equivalent nodal load vector. So, substituting the coordinates corresponding to all the four nodes for this particular element and traction components in the radial direction and axial direction. We get finally, this equivalent nodal load vector, please note here the non zero components of this equivalent nodal load vector. Since, t z there is no traction applied along axial direction. The component corresponding to axial direction are all zero at all the four nodes and since load is applied in the radial direction at local node one and local node four, the corresponding except those value those locations the other quantities are zero.

The radial directions components at nodes three and two and three are zero only non zero are the radial component at nodes one and four. So, now we got equivalent nodal load vector and we got elements stiffness matrix.

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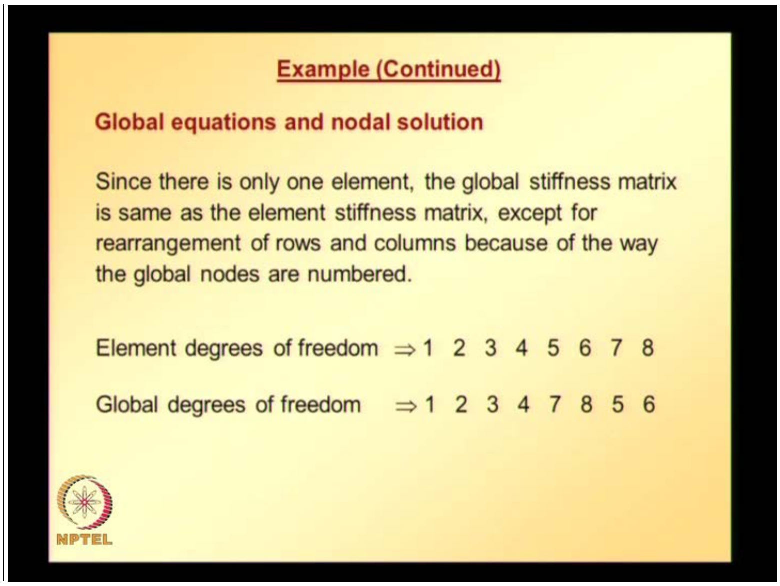

So, we are ready to assemble global equations, and get the nodal solution. Since, there is only one element global stiffness matrix is same as element stiffness matrix, except rearrangements rearrangement of rows and columns, because the way global nodes are numbered, because global node four corresponds to local node three, and global node three corresponds to global node four. So, we need to rearrange rows, and columns corresponding to the element stiffness matrix that, we derive and also equivalent nodal load vector that we derive.

We need to rearrange those two that is element stiffness matrix and equivalent nodal load vector for that, we need to make a note of what are the corresponding degrees of freedom in the local coordinate or local system and what are the corresponding quantities in the global system. So, element degree of freedom corresponds to local degrees of freedom and a global degrees of freedom corresponds to global. So, here with this understanding we need to rearrange the rows and columns and then we need to impose essential boundary conditions.

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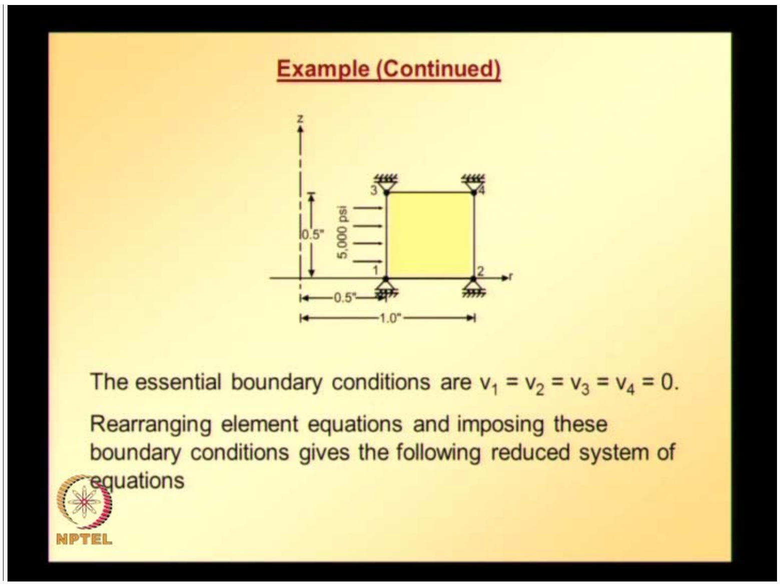

Essential boundary conditions are the displacement components and all the four nodes in the axial direction are zero. So, these are the essential boundary conditions v 1 v 2 v 3 v 4 are zero. So, after rearranging based on global degrees of freedom and local degrees of freedom and imposing the essential boundary condition. Since essential boundary conditions here are zero, we can eliminate the rows and columns corresponding to that location and we get the reduced equation system. So, rearranging element equations and imposing essential boundary conditions gives following reduced system of equations.

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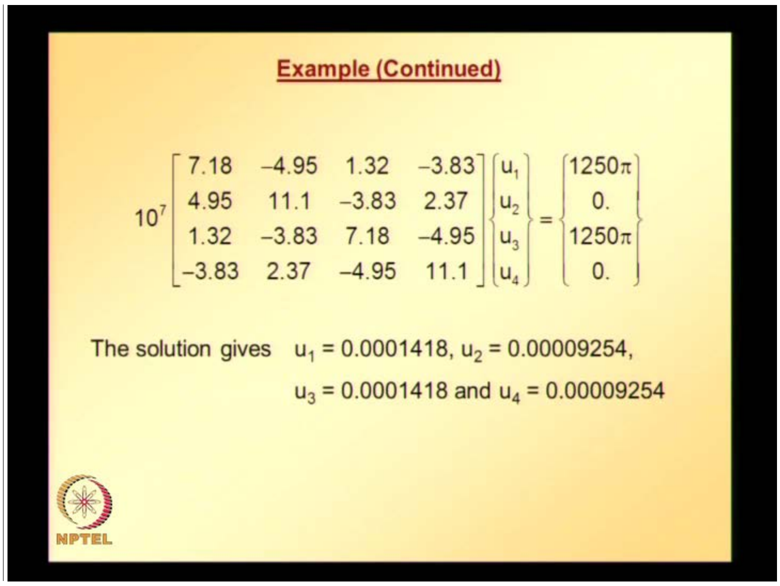

And this is a four by four equation system, because only four degrees of freedom are unknown. So, we can solve this four by four equation system for u 1 u 2 u 3 u 4, which are nothing but, radial displacement components at all the four nodes. So, since we know the displacement components, all the displacement components at all the four nodes we are ready to calculate stresses and before that, we need to calculate strains and then using constitutive matrix we calculate stresses. Similar to that we already did for plane stress plane strain problems and here calculations corresponding to one integration point will be shown. If you require to calculate stresses and strains at any other point other than integration point we need to know, what are the corresponding values of s and t at that point corresponding to r and z coordinates.

So, once we know s and t values corresponding to the point that you are interested you can easily calculate strains and stresses at that point. Once, we find what the strain displacement matrix. So, calculations of stresses for quadrilateral elements strains and stresses vary linearly over element stresses at any point over element can be computed using strain displacement and stress strain relations. Since, the strain displacement matrix b is available with us because, we already assembled the element stiffness matrix since b matrix is available at the integration points stresses at this points can be easily evaluated. At any other point, stress evaluation requires evaluation of mu b matrix at that point for which we require to know, what is s and t coordinates corresponding to r and z coordinates of that particular point.

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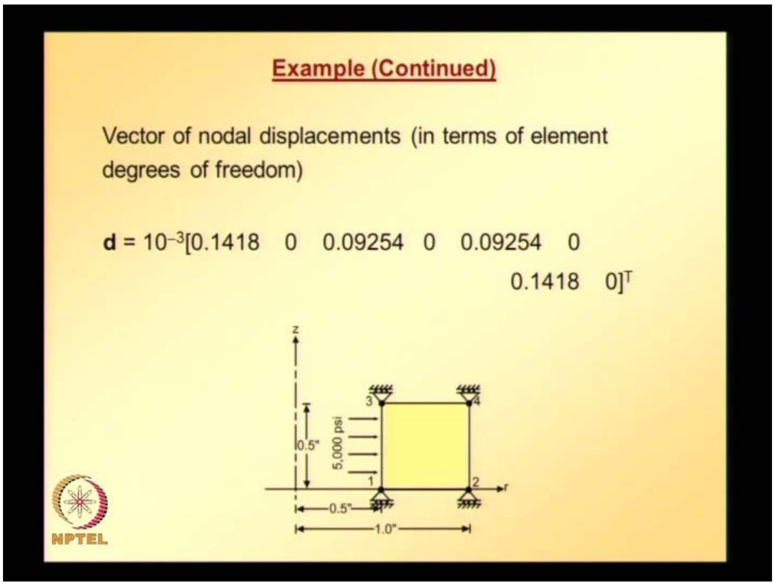

So, now to calculate strains we need to know nodal displacements. Nodal displacements in terms of element degrees of freedom that, these displacement components are arranged with respect to the local node numbering, displacement vectors at integral this is a displacement vector which consists of the components of displacement at all the four nodes.

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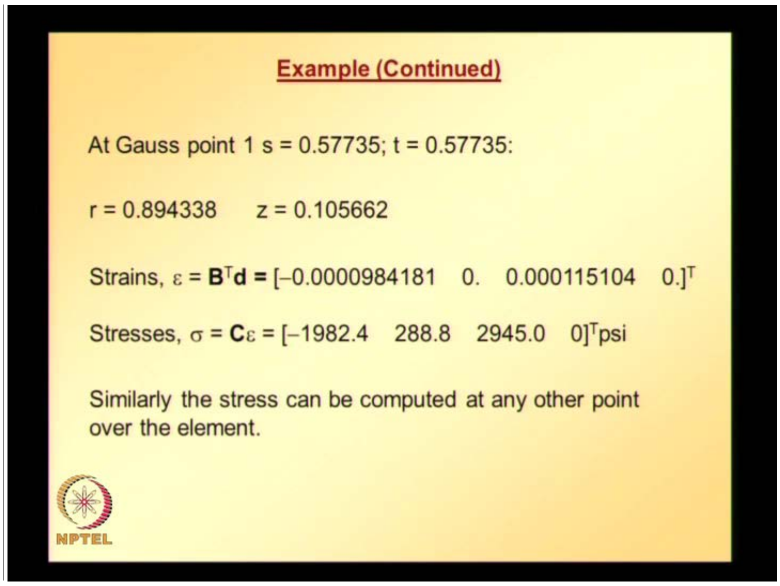

And we are interested in finding, what are the strains and stresses at integration points. So, the first integration point is taken and the corresponding r and z values s and t values are given. So, we can easily find what are r and z values by substituting into the Isoparametric mapping relation. But, sometimes we require knowing what is r and z in that case we need to back calculate what is s and t to calculate strains and stresses. So, once we have this information we can easily plugging into the strain displacement matrix s and t values and we can calculate strains or strain matrix strain vector consisting of all components of strain at this integration point and multiplying the strain vector with constitutive matrix we get a stress vector consisting of all components of stress.

So, similar stress calculations can be made at other integration points or at any point where it is require. So, basically as a part of this axisymmetric elasticity problems, we looked at the governing equations for axisymmetric problems, and also we looked at derivation of element stiffness matrix, and equivalent nodal load vector for linear triangular element, and also four node quadrilateral element.

So, similar equations - similar element stiffness matrix equations, and also equivalent nodal load vector equations can be derived for higher order elements, such as eight node serendipity element, following similar steps. So, in the next class, we will continue with 3D elasticity problems.