Finite Element Analysis

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Lecture No. # 37

In the last few classes, we have seen how to solve three-dimensional elasticity problems taking advantage of geometry and loading, and treating them as plane stress plane strain problems which are essentially two-dimensional problems, but some cases we can also depending on whether the problem is involving axisymmetric geometry, and loading. We can treat them as essentially two-dimensional problems treating them as axisymmetric problems.

So, in today's class or in the next two classes we will be discussing about axisymmetric elastic elasticity problems. So, basically as we did in plane stress plane strain problems will be looking at governing differential equation, and finite element equations for axisymmetric problems using triangular, and quadrilateral elements. So, three-dimensional elasticity problems involving axisymmetric geometry, and loading can be treated as essentially two-dimensional problems. For axisymmetric problems, the structure loading and material properties must be symmetric about axis of revolution.

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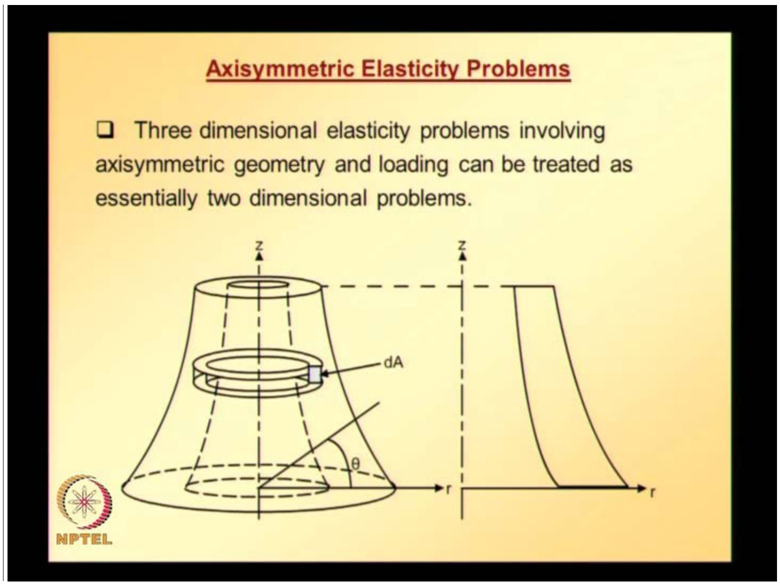

For illustration purpose, a typical situation is shown here with z axis as axis of revolution. So, for three-dimensional elasticity elasticity problem to be modeled as axisymmetric problem. The structure geometry loading, and material properties must be symmetric about axis of revolution. An an Axisymmetric stress analysis problem can be formulated in terms of two displacement components - one in the radial direction r, and the other one is in the axial direction z. So, the displacement component in the radial direction that is in the r direction is denoted in the rest of this lecture, it is denoted using u, and the displacement component in the axial direction or in the z direction is denoted with w.

This is similar to u and v that we used for denoting the displacements in x and y directions for plane stress plane strain problems.

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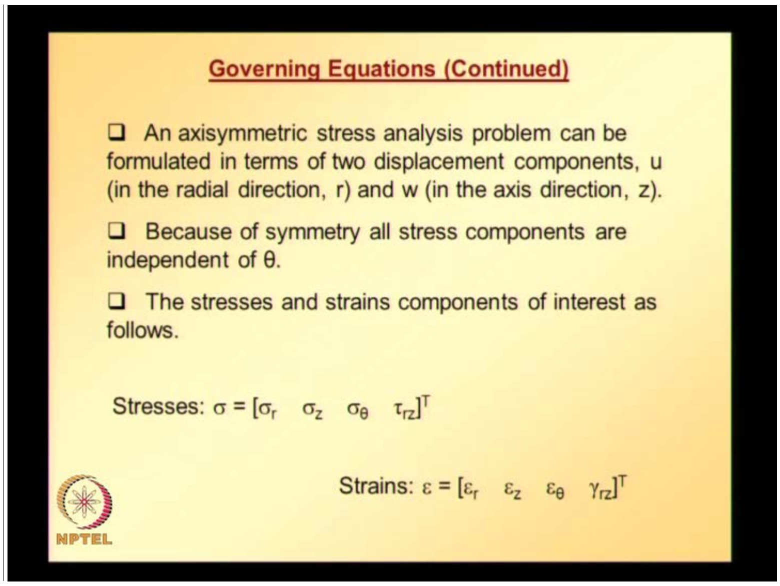

And because of symmetry, all stress components are independent of theta. The stress and strain components of interest are as follows sigma r sigma z sigma theta tau r z and the corresponding strain components are epsilon r epsilon z epsilon theta gamma r z. Assuming small displacements and strains the strain displacement relations, similar to that of plane stress plane strain problems. We require first to identify, what are the non zero stress components and non zero strain components and then we need to also know, what is the relationship between strains and displacements and also how the various stress components are related to various strain components.

So, we require all these equations for us to develop the finite element equations based on potential energy functional similar to what we did for plane stress plane strain problems.

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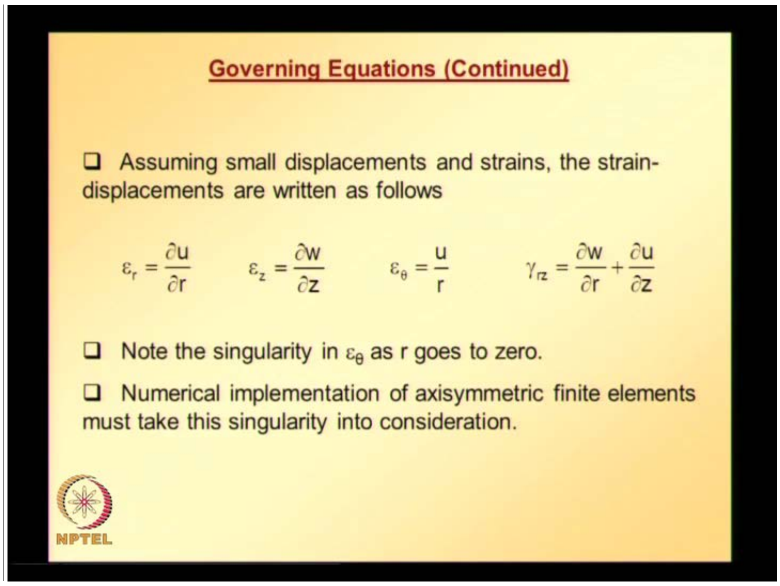

So, assuming small displacements and strains are related to or displacements are related to strains via these equations and if you observe these equations singularity, there is singularity in epsilon theta. That is as r goes to zero epsilon theta tends to infinity numerical simulation numerical implementation of axisymmetric finite elements must take this singularity into consideration. So, these four equations give us relation between strains and displacements and similar to that, what we did for plane stress plane strain problems assuming linear elastic material behavior.

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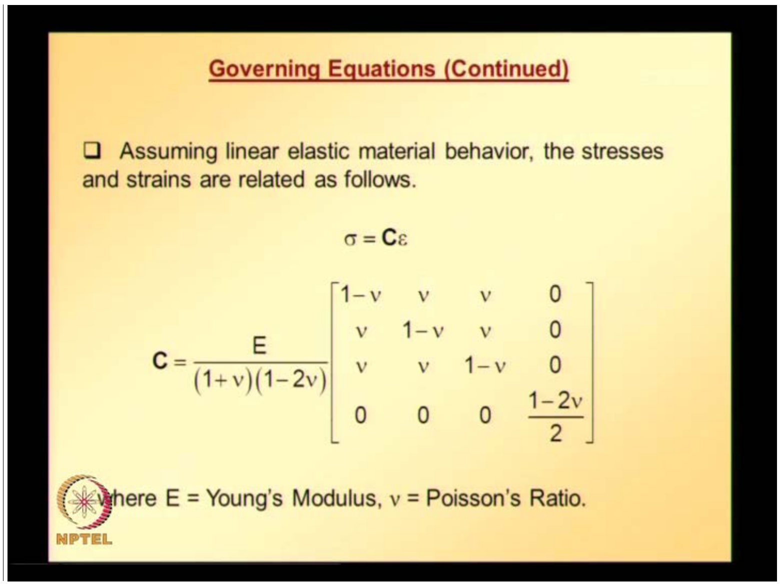

The strains and stresses or stresses and strains can be related via this equation, sigma is equal to constitutive matrix denoted with capital c times epsilon, where constitutive matrix is defined like this which depends on two material constants, Youngs modulus and Poissons ratio and if you compare these equations with corresponding plane stress plane strain equations. It can be seen that the two problems are very similar, only difference is the primary unknowns of Axisymmetric problems are u and w, whereas if you recall it is u and v in case of plane stress plane strain problem for the discussion that we had.

If somebody is interested they can use different notation but, we used u and v for displacement components in x and y directions for plane stress plane strain problems. So, now to develop finite element equations, we require potential energy functional for axisymmetric problems can be written like this.

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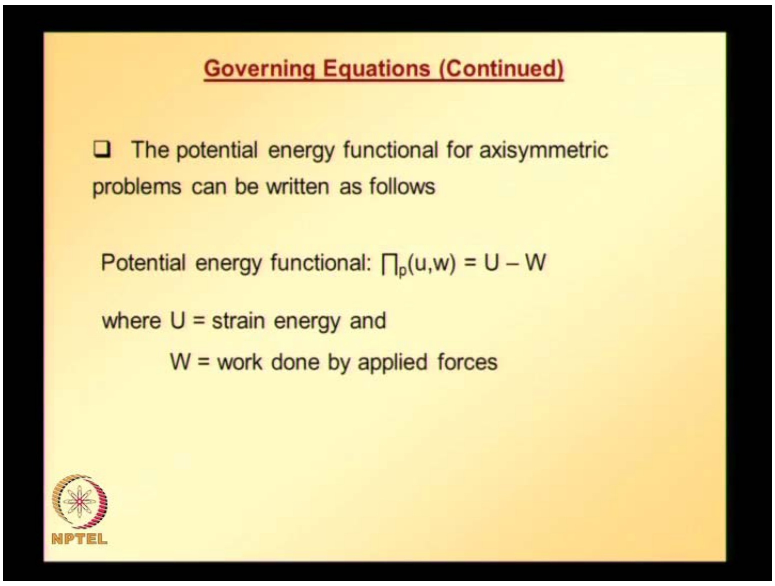

Similar to plane stress plane strain problems except that potential energy functional. Now is going to be function of u and w displacement in the r direction and displacement in the z direction that axial direction so, potential energy functional is defined like this u minus w. I guess by this time you can easily understand, what u stands for and what w stands for but, for completeness it is given here u is strain energy, w is work done by the applied forces. So, now we need to see how to calculate this strain energy and worked done by the applied forces for axisymmetric problems.

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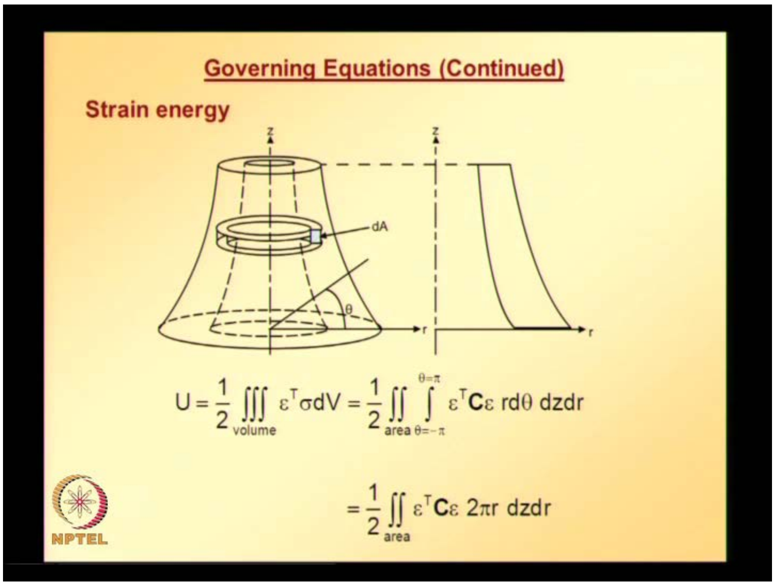

So strain energy, we are dealing with Axisymmetric problem typical problem is shown in the figure and u is defined as integral volume integral epsilon transpose sigma. Since the problem is Axisymmetric, we can integrate between theta going from minus pi to pi and simplify this as shown in the slide. So, finally strain energy is given by half area integral of epsilon transpose c times epsilon times 2 pi r.

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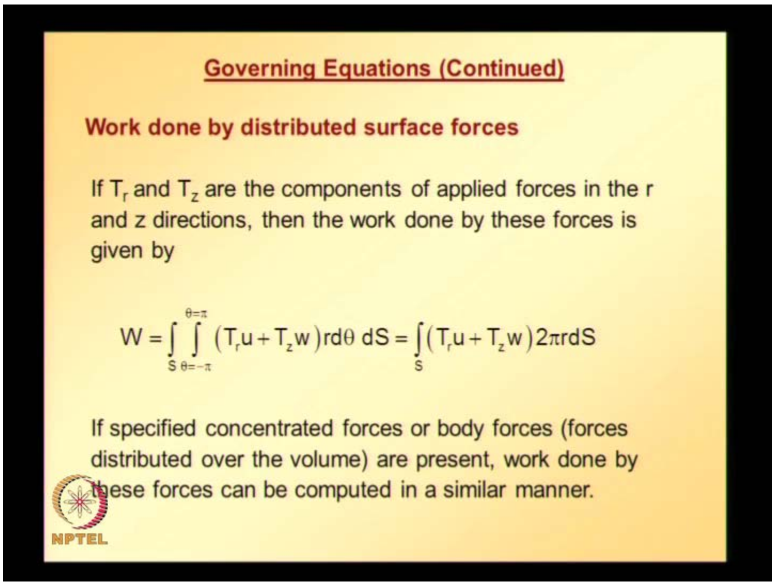

Now, work done by the distributed surface forces if t r t z are the components of applied forces in r and z directions, then work done by these forces is given by this. Evaluation of this strain energy and work done by the applied forces is very much similar to that of plane stress plane strain problem except that, we need to take care of geometry that is why it should pay little bitter attention to that. Here we are integrating between theta minus pi to pi because; this axisymmetric problem and we can by doing this we can actually eliminate theta in expression for work done by the distributed forces.

So, w finally is integral over this the line along which or the side along which the traction is applied or distributed force is apply we need to evaluate the integral that is given in this equation, which is integral displacement in the r direction times traction in the r direction plus displacement in the z direction times traction in the z direction and entire thing times 2 pi r over, we need to evaluate this over the line or edge over which distributed force is applied.

If specified concentrated forces or body forces are present work done by these forces can be calculated in a similar manner. So, with these definitions let us develop equations for a triangular element axisymmetric linear triangular element.

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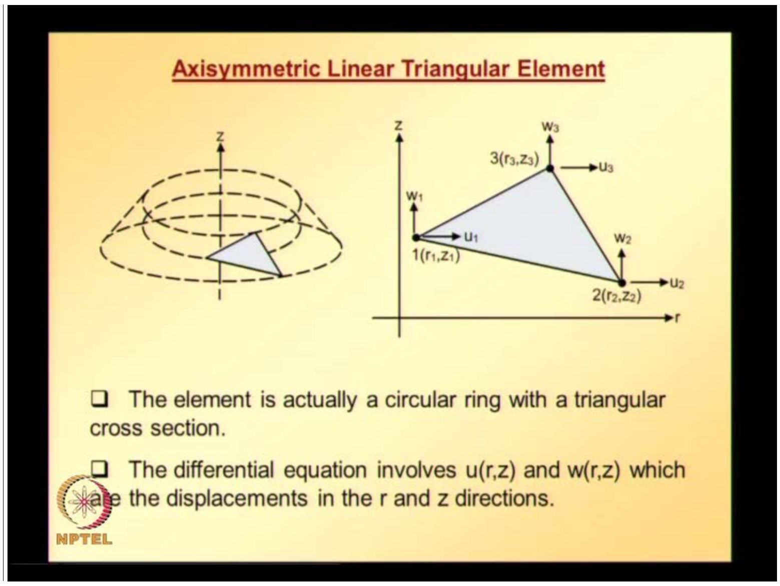

A typical triangular element is shown here, the element is actually a circular ring with triangular cross section and the differential equation involves displacements in the r direction and displacement in the z direction and similar to that of plane stress plane strain problems, two different trial solutions are required, one for displacement component in the r direction and another for displacement component in the z direction.

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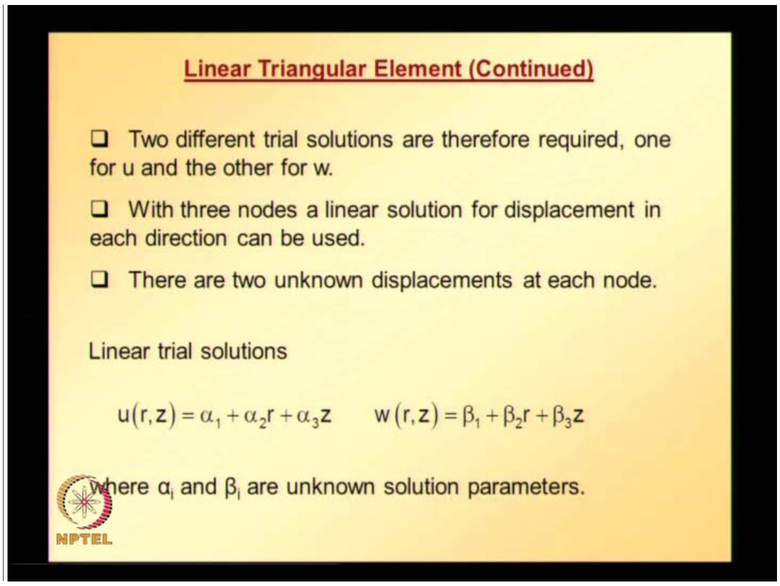

With three nodes, a linear solution for displacement in each direction can be used this is similar to what we did for plane stress plane strain problems. Because we are trying to develop element equations for three node triangular element. So, a linear solution for displacement in each direction can be used, because there are three nodes we need to start with a polynomial having three coefficients and since we are dealing with two dimensional problems. So, the trial solution will be something like u is equal to alpha one plus alpha two times r plus alpha three times z something like that.

So, since there are also keep in mind there are two unknown displacements at each node. The trial solutions for the displacement component in the r direction and displacement component in the z direction looks like this and both these displacement components are going to be a functions of are they are going to be function of r and z. Here alphas and betas are unknown solution parameters the solution the trial solution.

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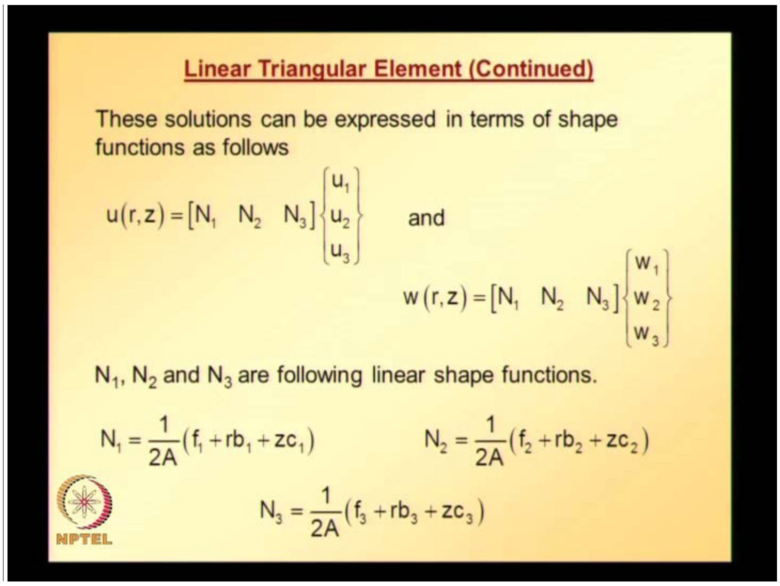

This two trial solutions can be expressed in terms of shape functions like this u and w. This is similar to what we did for plane stress plane strain problems and where N 1, N 2, N 3 are the shape functions or linear shape functions for triangular three node triangular element, where N 1, N 2, N 3 depends on the geometrical coordinates of the three node triangular element. So, once the coordinates of three node triangular element are given then, we can easily figure out what are this N 1, N 2 and N 3. So, N 1, N 2, N 3 are defined like this as you can see, N 1 is linear in with linear or N 1, N 2, N 3 are all three are linear with respect to r and z f 1 fs bs cs are some coefficients, which are functions of special coordinates of triangular element.

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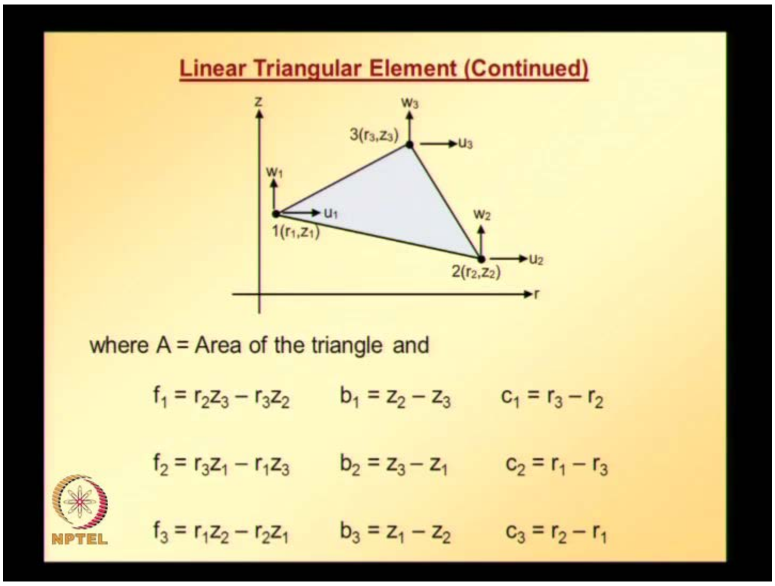

Special coordinates of nodes of triangular element so, if the coordinate information of all three nodes is given. We can easily find what is area of triangle fs and bs fs bs and cs f 1 f 2 f 3 b 1 b 2 b 3 c 1 c 2 c 3. Area of triangle can also be easily computed, using the relation that area is half times determinant of matrix consisting of one ones in the first row and the coordinates of r in the second row and coordinates of z in the third row of all the three nodes. This formula is already there with you, which we used for calculating area of triangles when we are dealing with plane stress plane strain problems. So, far we have seen how to express trial solutions for axisymmetric linear triangle triangular element trial solutions, in terms of finite element shape functions. So, now we are ready to actually derive the element stiffness matrix and these two trial solutions, which we just seen.

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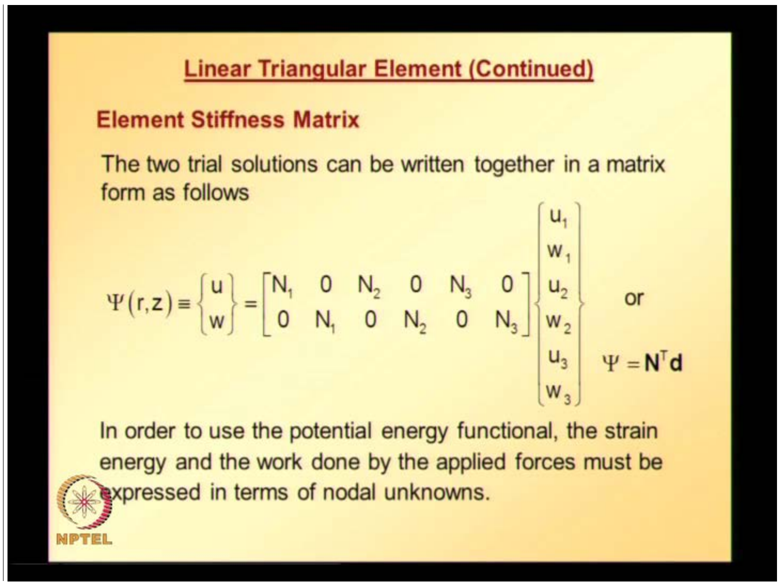

They can be written together in matrix form as follows, which can be compactly written as psi is equal to n transpose d and psi is a vector consisting of the displacement components in the r direction and displacement component in the z direction. In order to use potential energy functional strain energy and work done by the applied forces must be expressed, in terms of nodal unknowns or nodal parameters that is u 1, v 1, w 1, u 1, u 2, u 3, w 1, w 2, w 3. Strain energy, in terms of nodal unknowns can be expressed once we know the relationship between strains and displacements, because you know potential energy functional is function of strain and stress. So, or finally we have seen potential energy or expressed potential energy as a function of strain.

So, we need to know what is the relationship when the trial solutions that we just obtained and the strains so, that we can plug in this information into the definition of strain energy.

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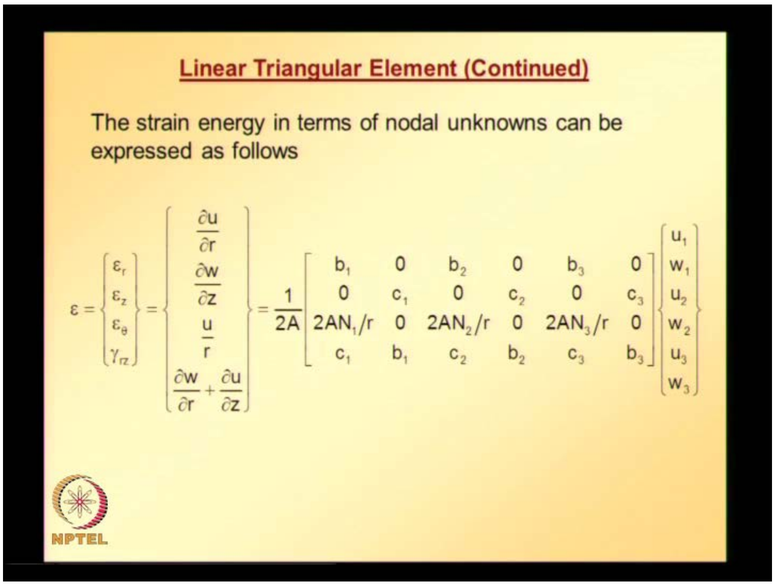

Strain energy, in terms of nodal unknowns can be expressed as follows. First we need to write what is strain vector strain? Vector consists of three four components here and the four components are defined and now we know, u in terms of finite element shape functions of the three nodes and the nodal parameters and also we know w in terms of three finite element shape functions and nodal parameters w 1, w 2, w 3. So, using that information we can further write this vector of strains likes what is shown there and if you carefully see the last part of equation. You can realize that epsilon r is constant over the entire element.

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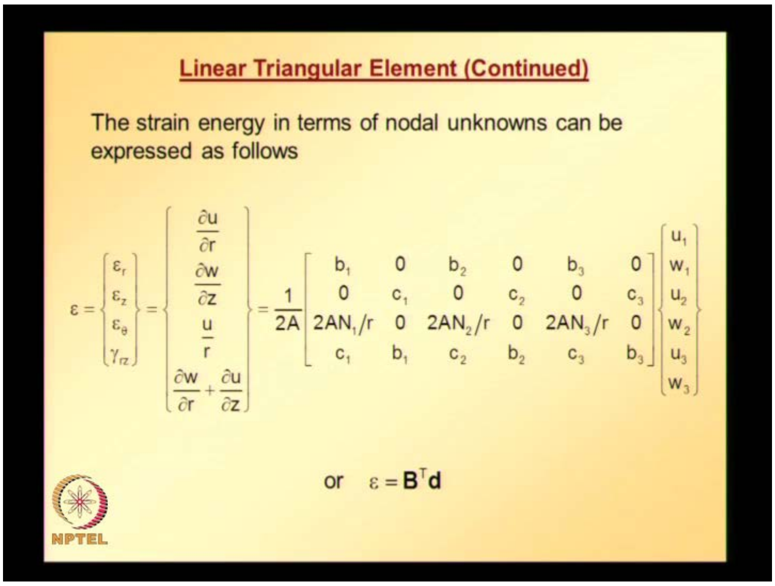

Because, bs all are functions of special coordinates, which are going to be constant for a particular element. So, epsilon r is going to be constant; similarly, epsilon z is constant and gamma r z is also constant. Only quantity which is variable is epsilon r, which is function of r as shown the last part of equation. So, epsilon this vector or this relation can be compactly written as epsilon is equal to b transpose d and we need to plug in this information into the strain energy definition.

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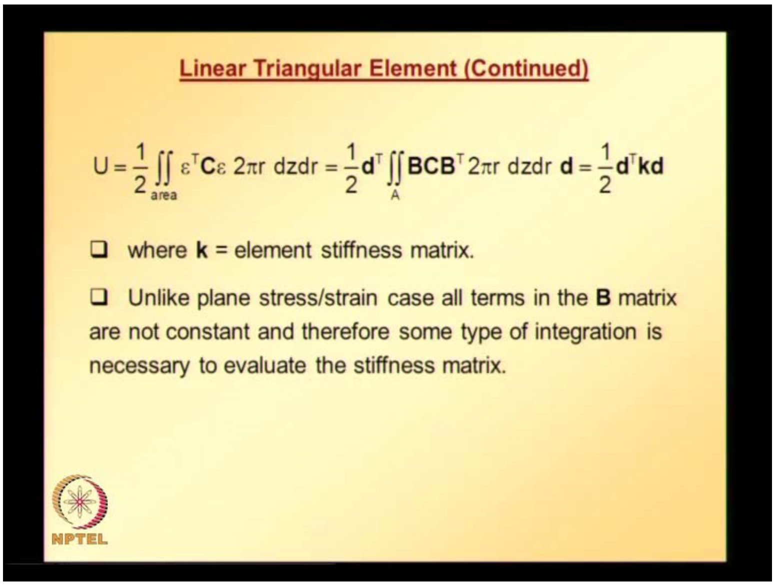

This is by definition strain energy for axisymmetric problem and substituting epsilon is b transpose d substituting epsilon is equal to be b transpose d. As we can further simplify this like the way it is shown there, where k is element stiffness matrix. As, I just mentioned unlike plane stress plane strain case all terms in b matrix are not constant and therefore, some type of numerical integration is necessary to evaluate stiffness matrix k.

One of the simplest integration that you can adopt is one point integration that is, evaluating all the quantities which are functions of x of u, which are functions of special coordinates r and z at the centroid of triangle where, the coordinates of centroid are given by this r bar and z bar, r bar is nothing but, average of all the r coordinates and z bar is nothing but, average of all z coordinates.

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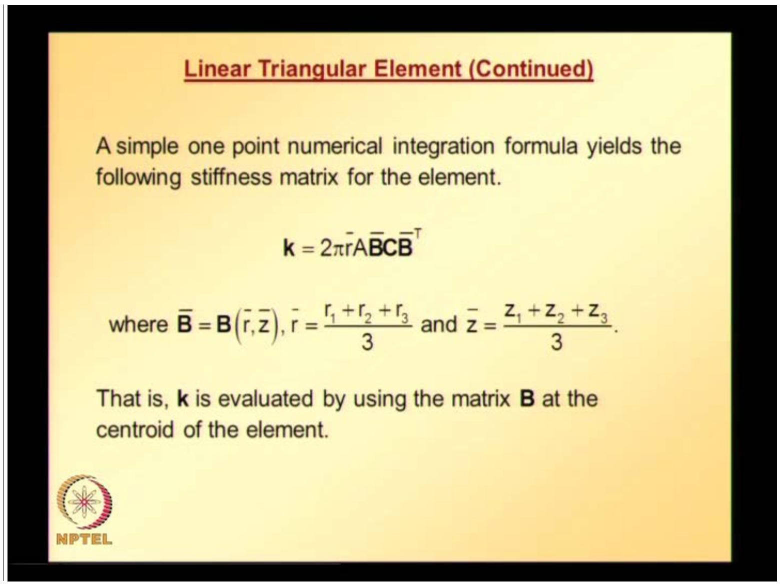

So, instead of evaluating k at every point one can evaluate k at the centroid, if we use this one point integration formula that is k is evaluated by using matrix b at the element centroid or centroid of the element or if somebody is interested in evaluating this more accurately, then we they can adopt the numerical integrations can that, we already looked at by selecting the points and weights from the table that is already supplied to you. So, now let us look at the other quantity that is work done by the applied forces or how to evaluate equivalent nodal load vector?

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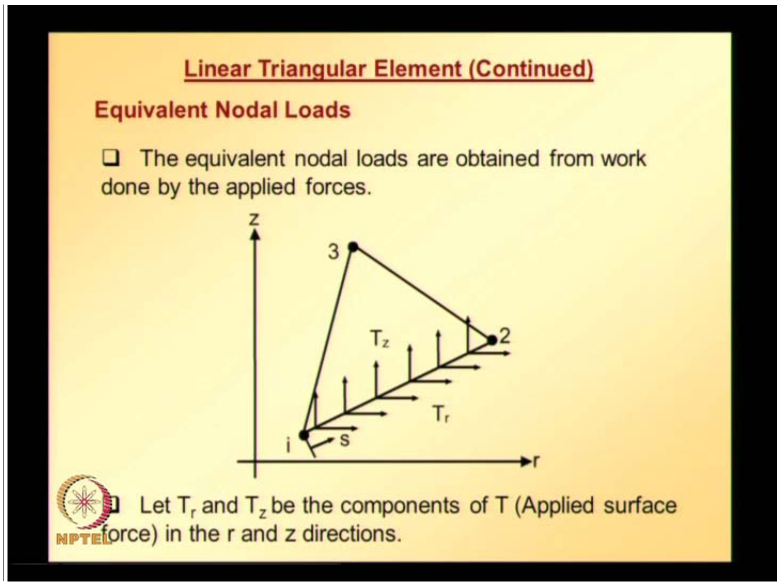

Equivalent nodal loads are obtained from work done by the applied forces. For Illustration purpose, considered uniformly distributed forces which are applied along element edges. Let t r t z be the components of applied traction or applied surface force in r and z directions, then work done by the applied forces is given by this.

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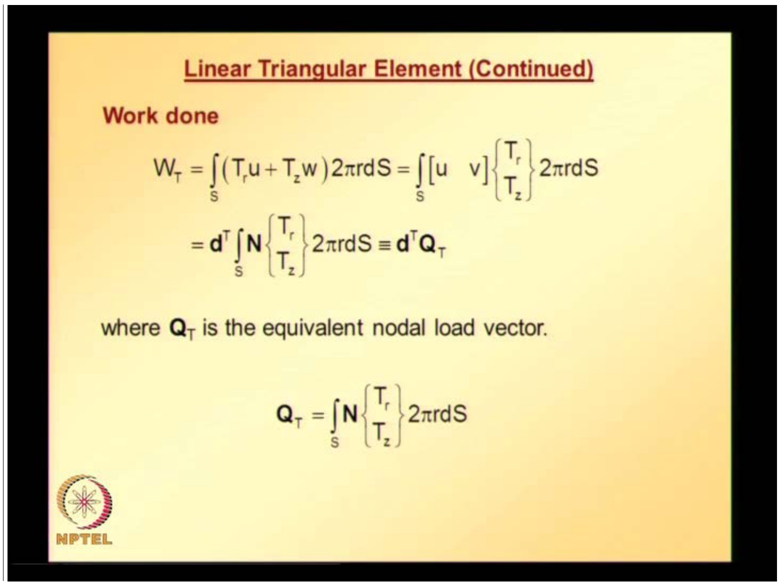

One traction in the r direction times displacement in the r direction plus traction in the z direction times displacement in the z direction, whole thing multiplied by 2 pi r and integrated over the element edge along which this traction is applied, where t r t z are the components of t applied surface traction in r and z directions. So, w t are work done by work done by the applied traction can be compactly written as d transpose Q ,T where Q T is equivalent nodal load vector and Q T is defined like this. So, to evaluate Q T we require to know, what is n is nothing but, a matrix consisting of shape functions, along the side or element edge over, which tractions are specified or along the side or edge over, which we require to or we are interested in evaluated assembling this equivalent nodal load vector.

And this integration can be performed in closed form, if the specified surface tractions or simple functions of r and z. Similar to that we discussed earlier the simplest case is when r and z are specified r and z are constant or uniform traction is specified along one or more sides of an element.

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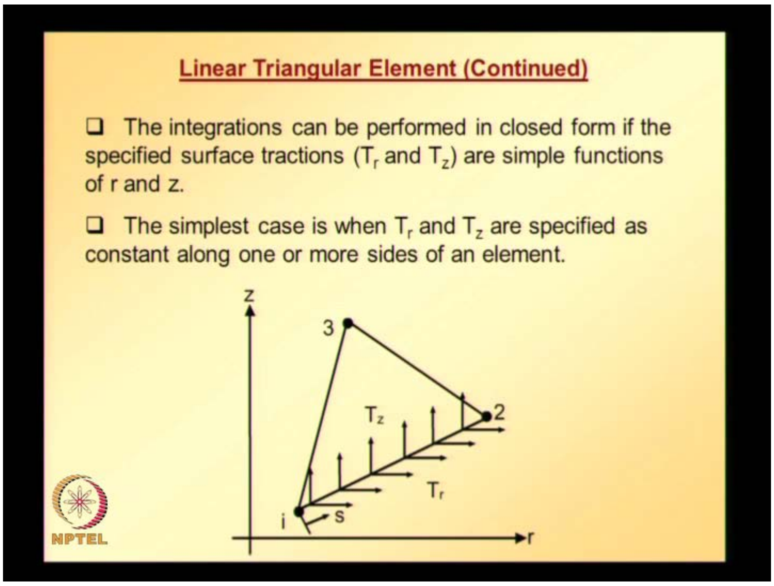

As an illustration, consider uniform pressure applied along side 1 2, that is t r t z are constant integrations can be performed easily by defining local coordinate system as shown in figure along side 1 2 and along this side, the shape function matrix are to get the shape function matrix, we require to know what are the shape functions of N 1, N 2 and N 3 and the shape functions of N 1, N 2 can easily be written using Lagrange interpolation formula. Once we define local coordinate system as shown in the figure, along side 1 2 shape functions N 1, N 2 are linear functions of s. Because, alongside 1 2 we have only two nodes so, N 1 N 2 are going to be linear functions of local coordinate system s, which is defined along side 1 2 and N 3 is going to be zero along element edge 1 2.

So, writing the shape function, shape functions of N 1, N 2 using Lagrange interpolation formula and also with respect to the local coordinate system defined alongside 1 2. We can finally, get these to N 1, N 2, where l 1 2 is length of side 1 2 so, with this we can write shape function matrix consisting of N 1, N 2, N 3.

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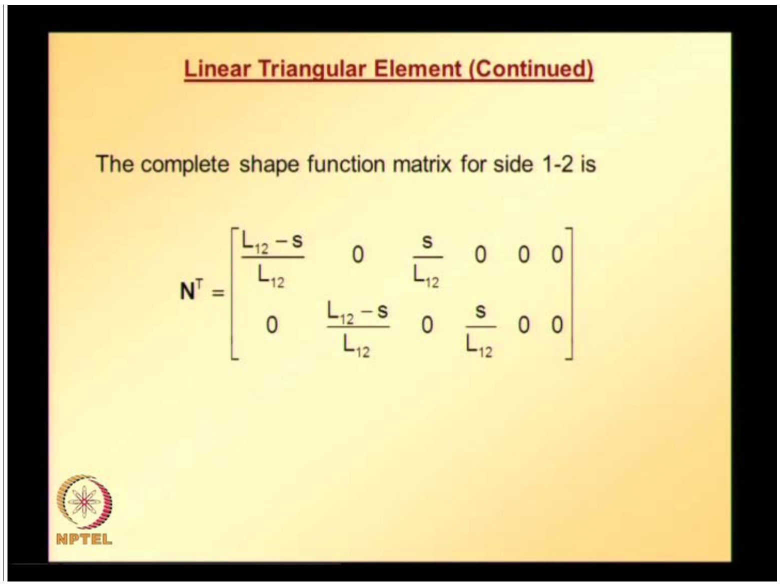

Which is required for evaluating or which is required for computing equivalent nodal load vector? So, this is the shape function matrix along side 1 2.

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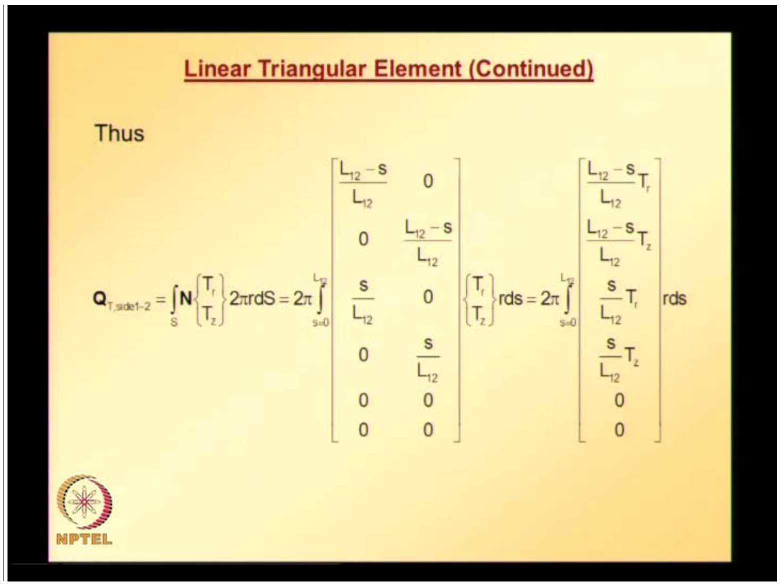

And then we need to substitute this into the definition of equivalent nodal load vector and if you see the last the final part of the equation. We have r, please note that r can also be interpolated using finite element shape functions or the similar that is what Isoparametric mapping that we discussed earlier. So, r can also be expressed as a function of N 1, N 2 along side 1 2, before we simplify this equation further.

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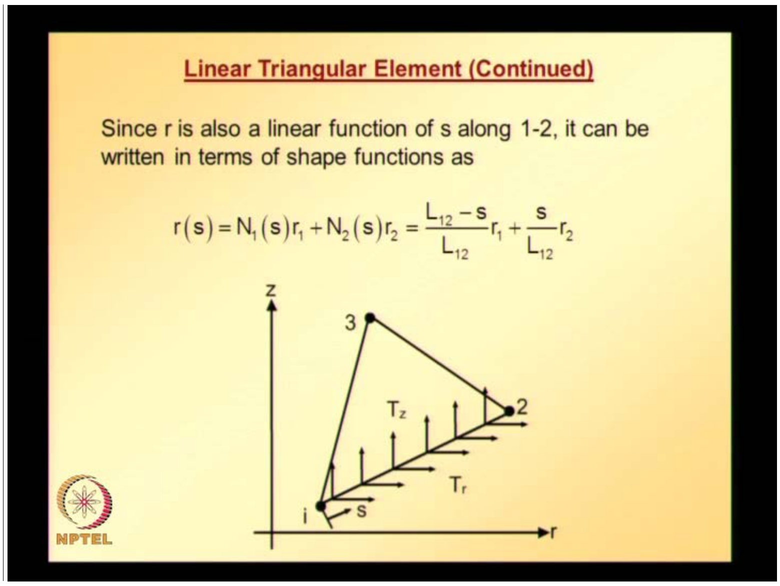

So, since r is also a linear function of s along side 1 2, it can be written in terms of shape functions like this. So, now substituting r into the previous into the previous equation of Q T and by integrating each of the terms here, integration of one of the terms is shown, the details of integration of one of the terms is shown.

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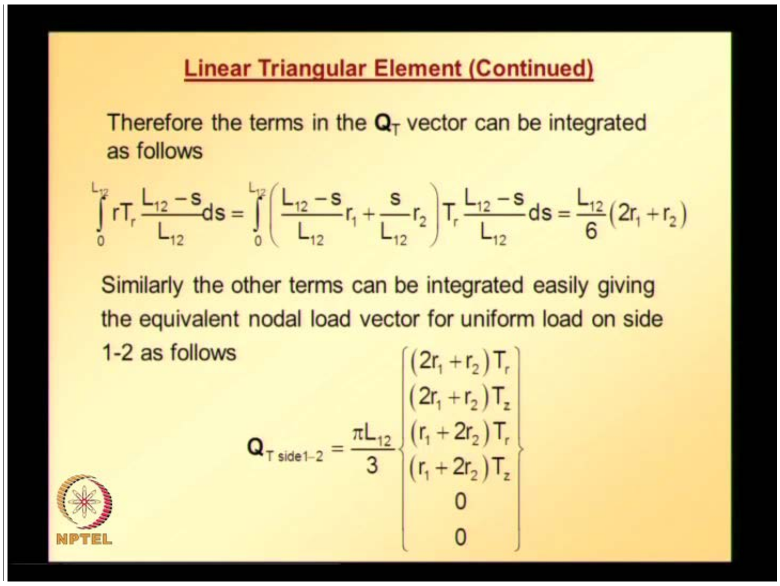

Similarly, other terms can be integrated to get the complete equivalent nodal load vector for uniform load along side 1 2. And once we carry out integration for the other terms also, Q T looks like this for side 1 2. Please note that is only applicable in case in the case of uniform load r t or t z are constant along the applied side or edge. And similar expressions for equivalent load can be written if pressure is applied along sides two three and three one.

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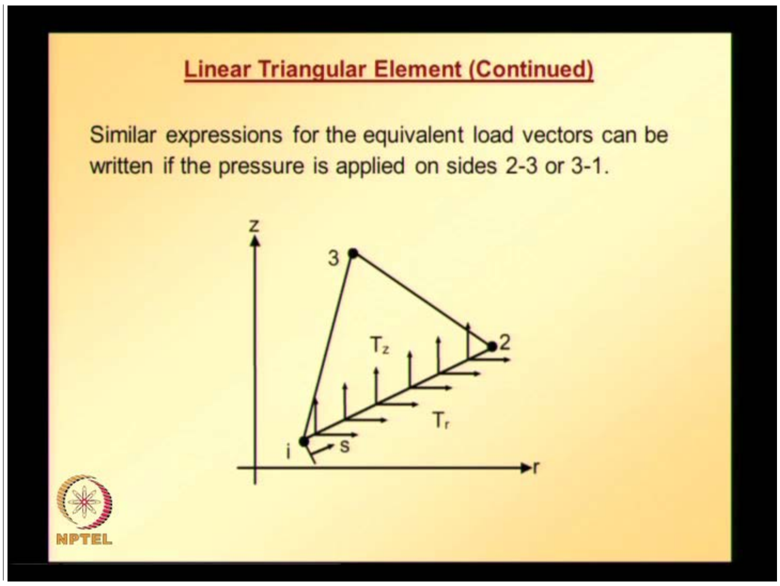

If the components of tractions t r t z are not constant, then we need to take care of that while doing integration or we need to use numerical integration scheme to simplify the integrate, once T r T z becomes complicate. So, far we discussed element stiffness matrix and how to assemble equivalent nodal load vector and rest of the things like assembly and solution procedure are standard, which are similar to that of plane stress plane strain problems, that we already discussed or the earlier problems that we already seen.

So, assuming that the solution is obtained once the nodal displacements are known the strains and stresses for each element can be obtained similar to that we did for plane stress plane strain problem.

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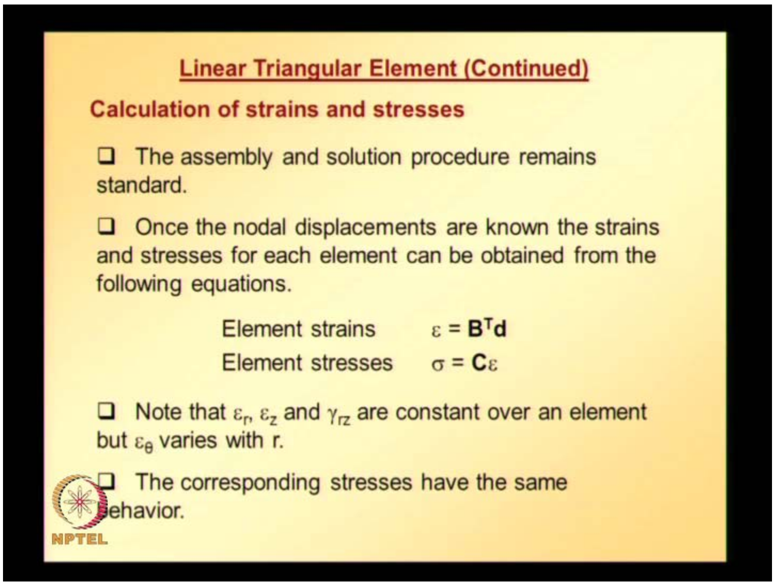

But, for completeness it is again repeated here, calculations of strains and stresses assembly procedure assembly and solution procedure remains standard. Once the nodal displacements are known strains and stresses for each element can be obtained, except that there are four components of strain. There are four components of stresses and the constitutive matrix is of dimension four by four and b matrix strain displacement matrix is of dimension four by six, except that these equations are similar to that of plane stress plane strain problems and note that epsilon r epsilon z gamma r z are constant over element but, epsilon theta varies with r.

This is what, I discussed when we are looking at the b matrix strain displacement matrix for axisymmetric problems and similarly, the corresponding stresses will have the same behavior. The corresponding stresses have the same behavior that is, sigma r sigma z tau r z is going to be constant, whereas sigma theta is going to be function of r that is, it is going to vary with r. So, to illustrate all the things that we discussed, so far related to axisymmetric problems. Let us take an example and go through all the steps to understand this well and again in this example tractions are assumed to be uniform for simplicity.

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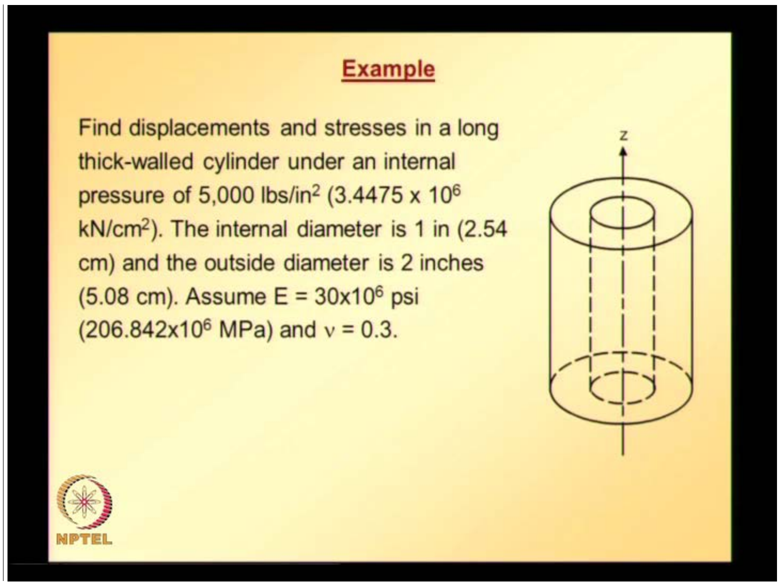

So, this is the problem statement. Find displacements and stresses in a long thick cylinder under an internal pressure value is given, internal diameter, outer diameter outside diameter and also material property details are given both in FPS units and SI units. Since, FPS units the values of given in FPS units are appearing to be a round figures will be looking at the details of work out in FPS units. But, as I repeatedly mentioned earlier as long as we use consistent units, the procedure wise it is not much different. So, this is the thick cylinder that, we are going to solve for displacements and stresses and as you can easily see, it satisfies all the condition that the structure geometry loading and also material properties are symmetric with respect to the axis of revolution.

So, we can take symmetry axisymmetric into advantage and we can solve this as a two dimensional problem. But, before we do that since this is a long thick cylinder, we need to also decide how many or how we are going to model the end effects or how we are going to model this long cylinder.

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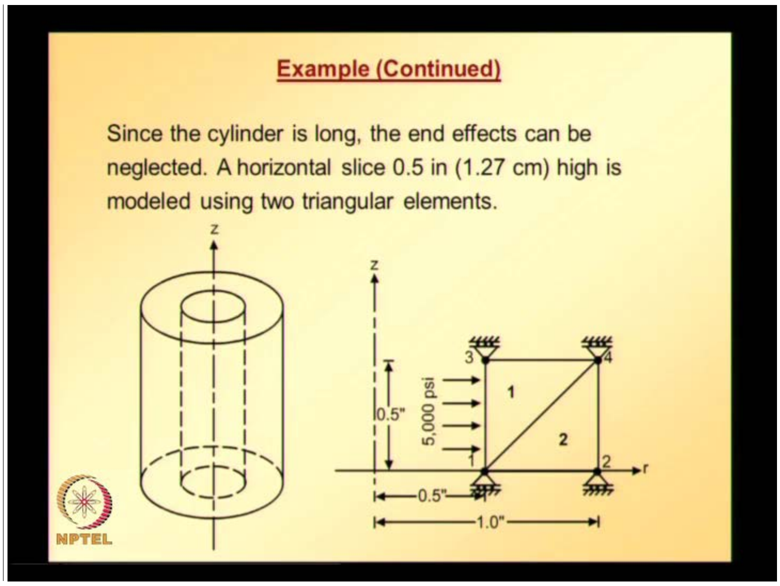

Since, the cylinder is long the end effects are neglected can be neglected and this is how it can be model? A horizontal slice of 0.5 inch or 1.27 centimeters height is modeled using two triangular elements. So, this is how this thick long thick walled cylinder is modeled, since the length is long, the end effects can be neglected in the entire cylinder analysis is reduce to solving this model, which consists of two triangular elements. And the model is shown with two triangular elements and also the boundary conditions at each of the nodes are shown. Since, the pressure is applied from inside internal pressure is applied, the cylinder is going to expand in the radial direction and since the cylinder is long the displacement in the z direction is going to be neglected.

So, the boundary conditions at the four nodes are as shown, the displacement in the z direction is constrained, whereas each of these nodes is allowed to have displacement component in the radial direction. And element one consists of or comprises of nodes one four three and element two constrains comprises of nodes one two four. At each node there are two degrees of freedom one in the r direction another one is in the z direction. So, element one contribution goes into the rows and columns, corresponding to node one four three into the global equation system. Similarly, element two contribution goes into one two four are the rows and columns, corresponding to nodes one two four in the global equation system.

And global equation system is going to be of dimension eight by eight. So, the contribution from element one goes into one five six seven eight rows and columns, element two contribution goes into one two three dour seven eight rows and columns of the final global equation system. So, with this understanding and also one more thing, we need to assemble the equivalent nodal load vector only along side three one or one three. Because, only along that element edge traction is specified and traction value that is specified with respect to the coordinate system that is defined is acting in the positive direction. So, it is going to be t r and its value is going to be 5000 psi.

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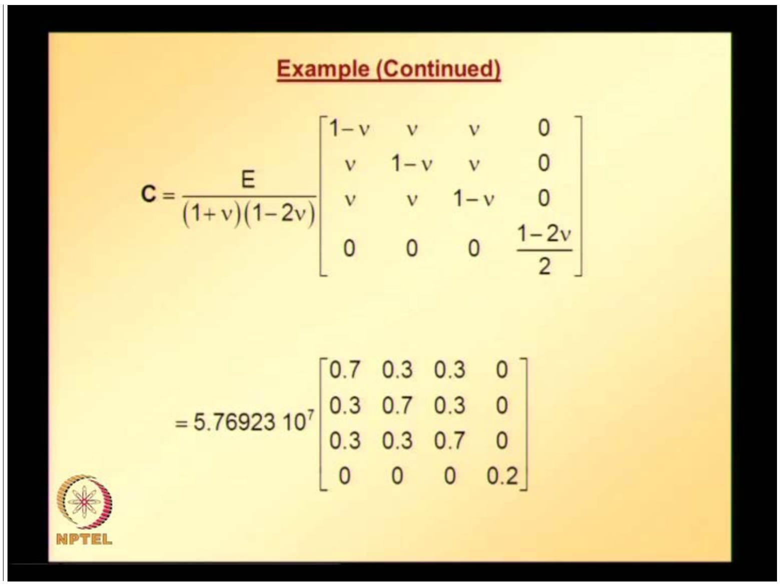

So, with these understanding let us gets started and this is constitutive matrix definition and substituting Youngs modulus, Poissons ratio values that are given for this problem we get c to be this one.

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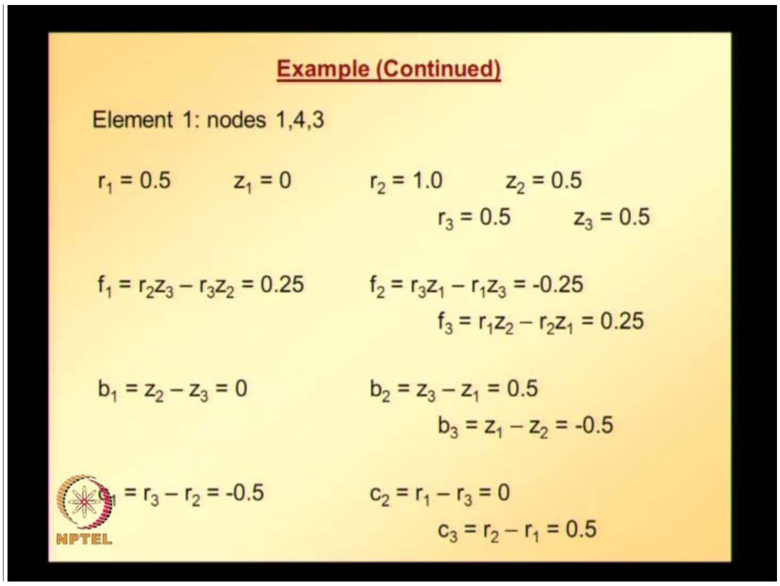

And looking at the geometry of these two triangular elements, we can easily figure out what are the coordinates of all the nodes. So, element one comprises of nodes one four three and the corresponding coordinates geometry coordinates are noted and all the coefficients are calculated.

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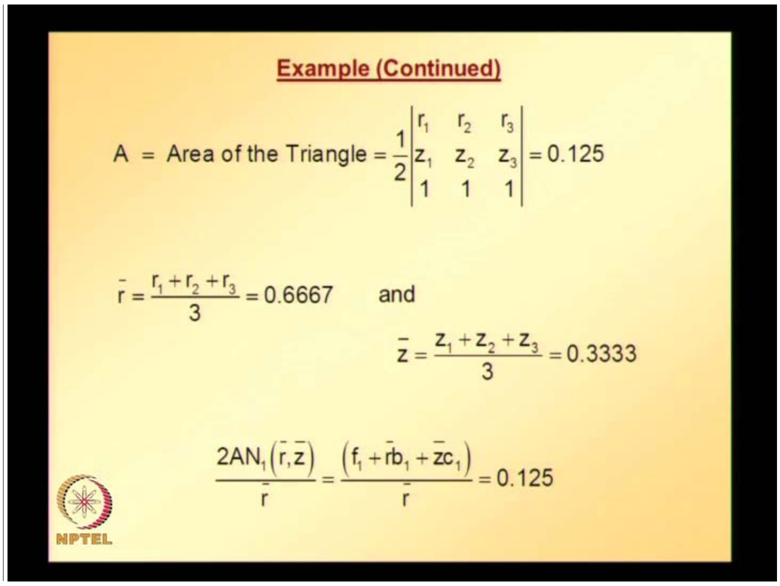

Once we have this information, one more thing that we require is we need to find what is the area of element one, this can also be obtained from the nodal coordinate information and the centroid coordinates and to evaluate epsilon theta we require this quantity.

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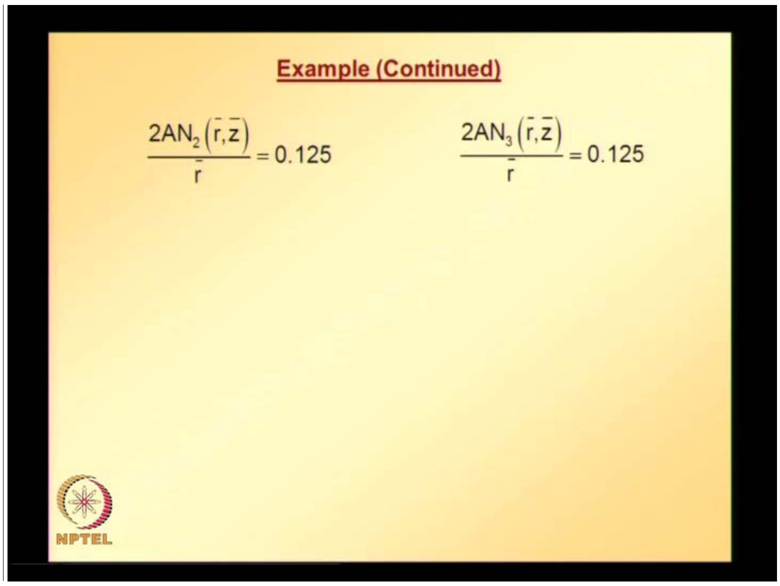

Similarly, these two quantities are required to evaluate epsilon theta are to get the third row of the strain displacement matrix of Axisymmetric problem, when we are using linear triangular elements.

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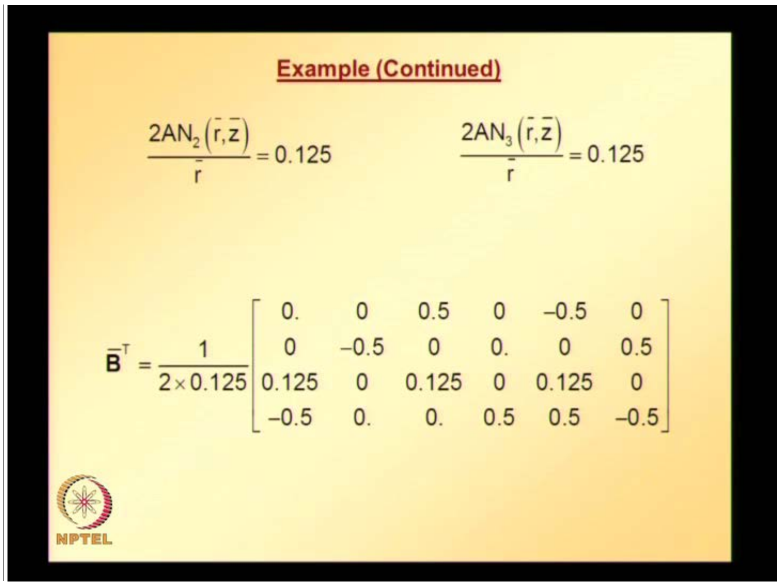

So, b matrix finally, b matrix evaluated at the centroid is given is obtained like this. So, using this we can easily get the element stiffness matrix.

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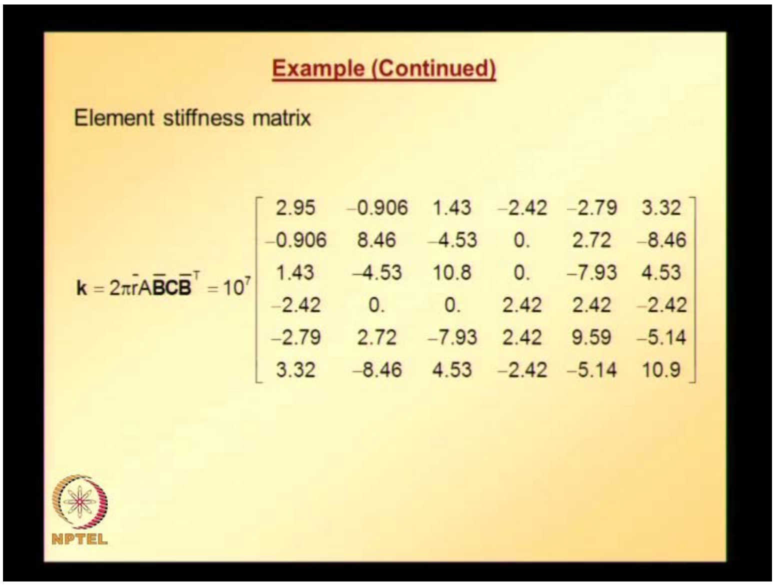

So this we did because, the traction applied is uniform traction and also if you want more accurate evaluation of the stiffness matrix. As I mentioned earlier, one can use numerical integration scheme that we discussed earlier instead of using one point rule, where we evaluated the stiffness matrix only at the centroid. So, now if you see the model load is applied along the edge three one or one three, which actually is a part of element one.

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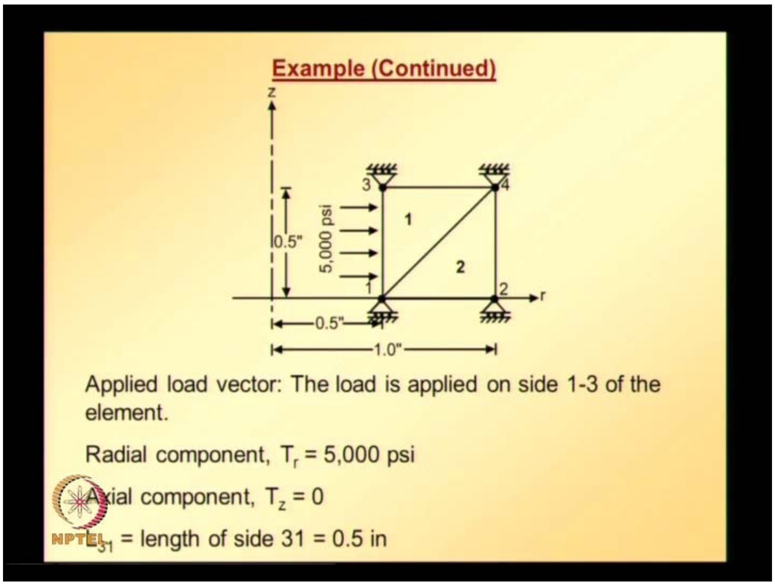

So, applied load vector or equivalent nodal load vector, we need to assemble for side one three or three one and to do that, we need to note down what are the various traction components t r t z and also length of side three one.

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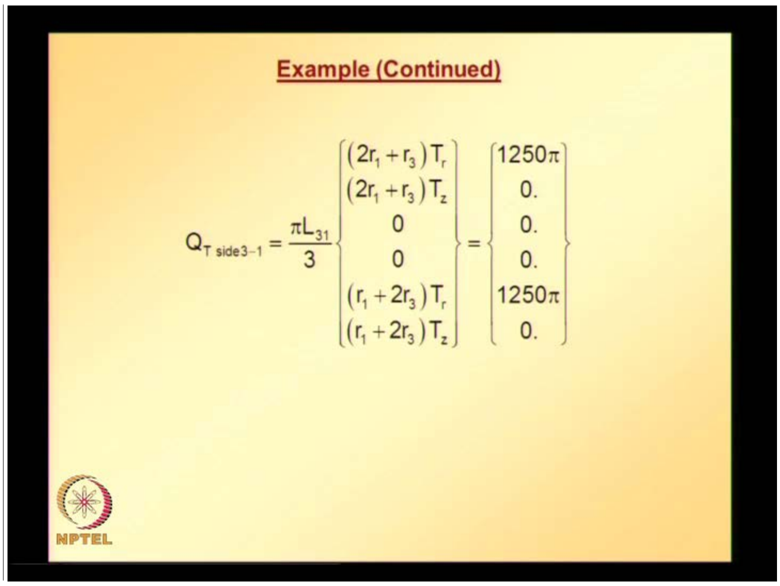

So, once we have this information, we can plug in into the formula that we already derive for uniform traction components, equivalent nodal load vector for three nodes axisymmetric linear triangular element, and we please note that we do not need to assemble this equivalent nodal load vector for the other edges or sides because, no traction is specified over the rest of the model.

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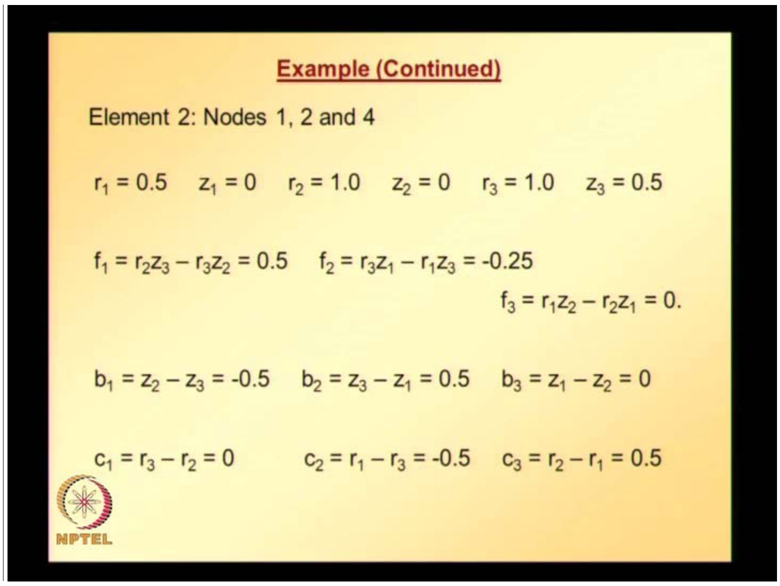

Now, let us go to element to and assemble the element stiffness matrix, noting down the special coordinates of all the three nodes. We can calculate the coefficients fs bs and cs.

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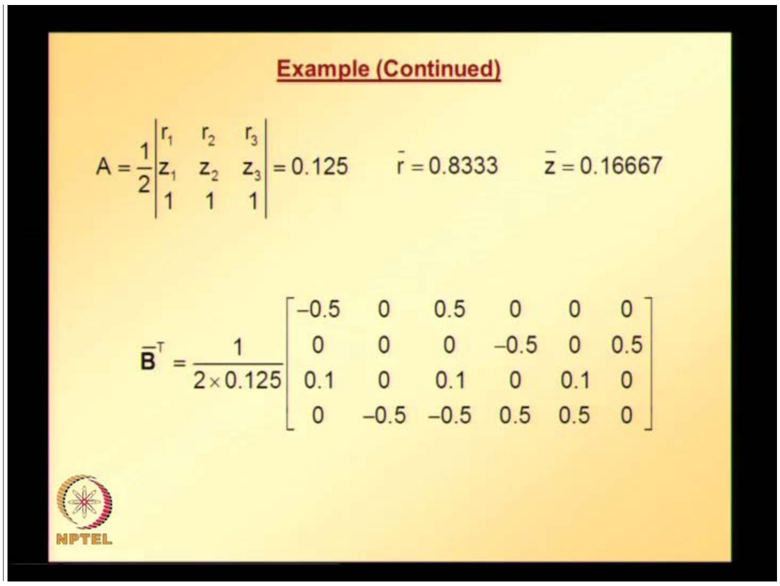

Also, we can calculate what is the area of element two and the centroid of element two with respect to the coordinate system that is, defined and then strain displacement matrix evaluated at centroid of element. Once we have this we can get element stiffness matrix for element two.

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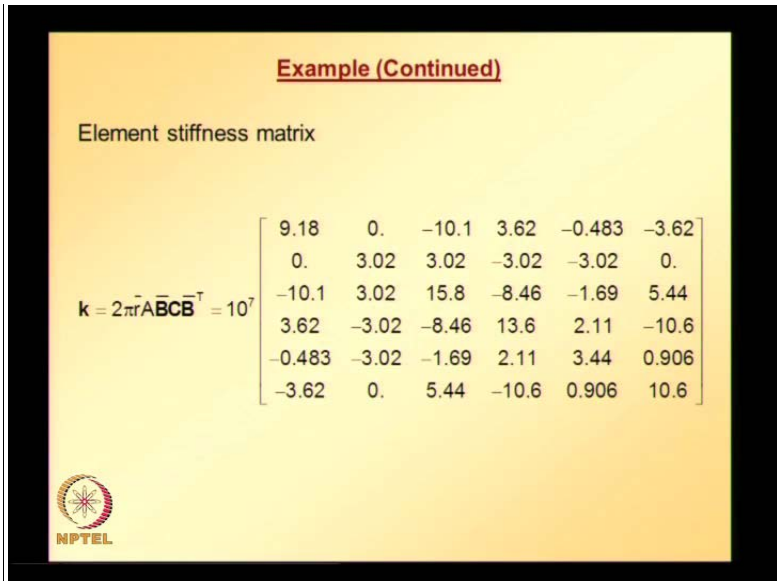

So, we obtained element stiffness matrix for element one and two, we need to before we assemble the global equation system. We need to know where the contribution from element one goes in, where the contribution from element two goes a into the global equation system that, information is noted here for clarity.

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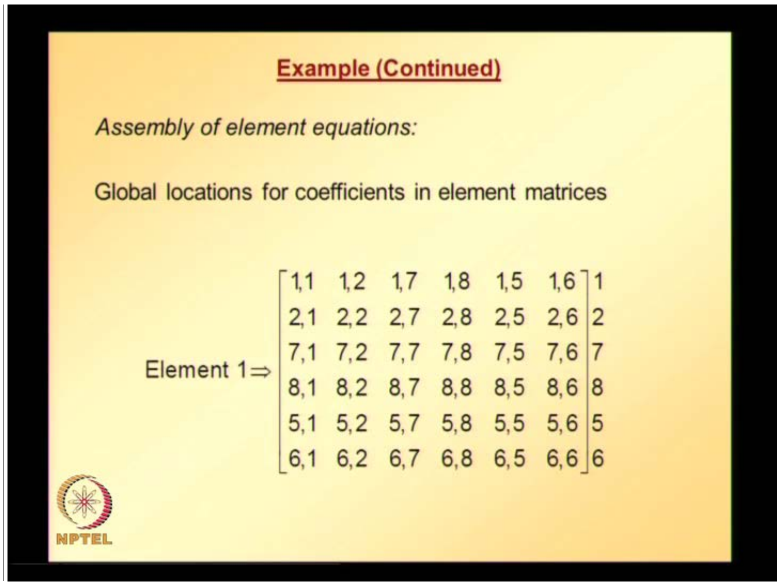

So, element one contribution goes into one two seven eight five six rows and columns and the locations are given in the matrix.

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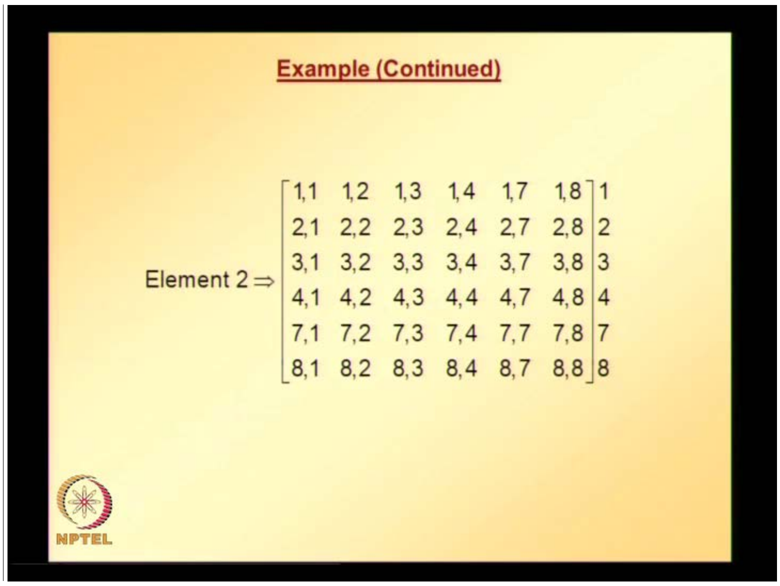

Element two contributions go into one two three four seven eight rows and columns and the corresponding global locations are also given in the matrix.

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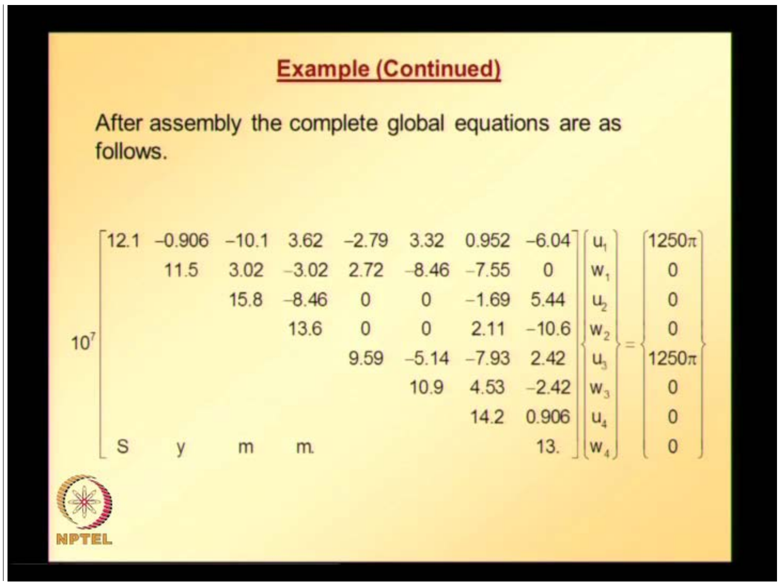

So, with this information we can easily assemble the final global equation system or if somebody is smart enough, they can directly write the reduced equation system, because w 1, w 2, w 3, w 4 are all 0. We can eliminate those rows and columns and write the reduced equation system directly. So, this is full complete global equation and the boundaries conditions are essential boundary conditions are w 1 is equal to 0, w 2 is equal to 0, w 3 and w 4 are 0.

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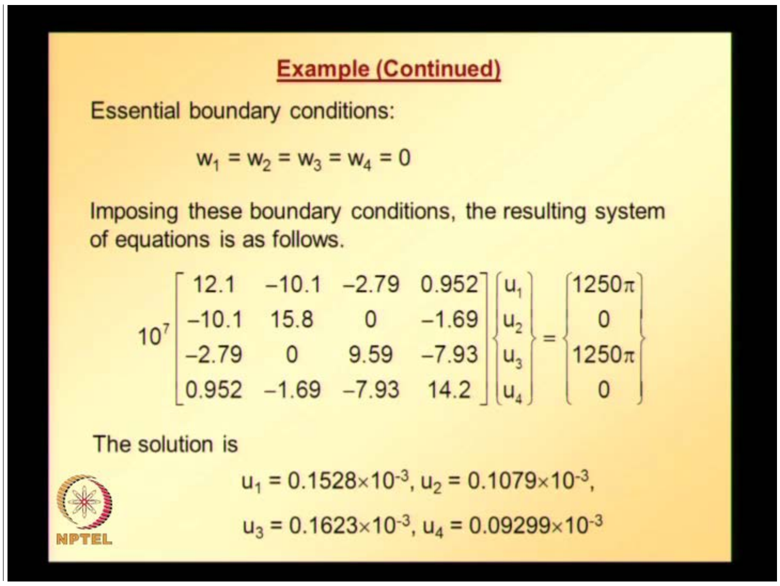

So, eliminating the rows and columns corresponding to these degrees of freedom, it is w 1 to w 4. We get the reduced equation system, which we can solve for u 1 to u 4 radial displacement components at all the four nodes. So, once we have this nodal solution, we can calculate we can do post processing like calculating stresses and strains.

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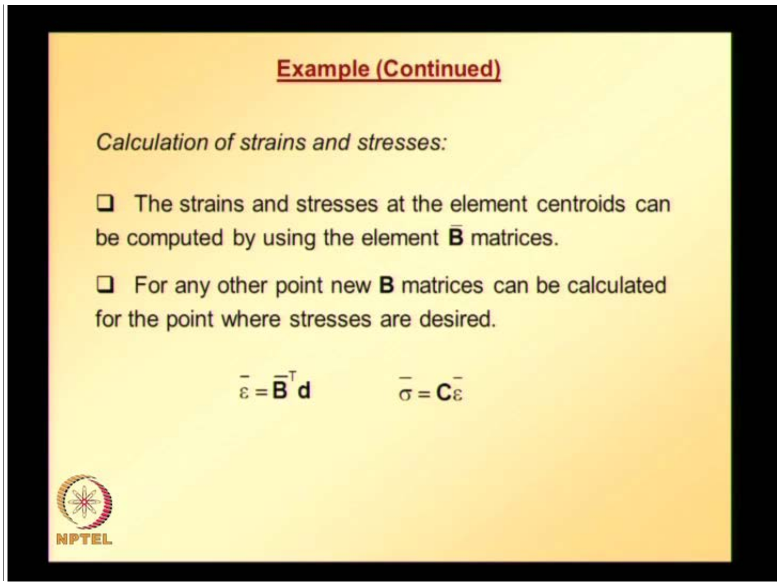

Strains and stresses at element centroid can be computed using the element b bar matrix that, we already have because we evaluated strain displacement matrix at the element centroid. But if somebody is interested at some other point mu b matrix needs to be evaluated first before we calculate stresses and strains. So, for illustration purpose calculations of stresses and strains at the element centroid are shown here. So, strain at the element centroid is given by b bar transpose times d and stress at element centroid is given by a sigma bar is equal to c times epsilon bar.

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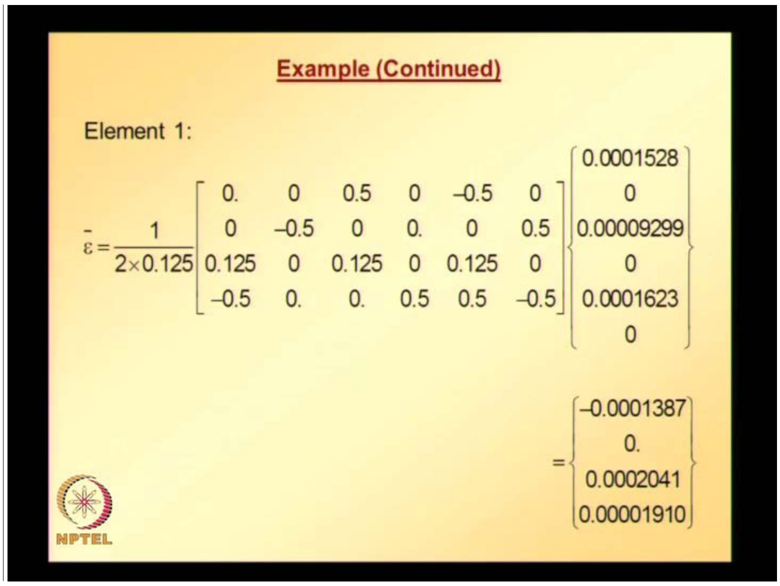

So, for element one strain is given by this and using because, for element one and element two, centroid is different. We need to calculate this separately for element one and element two.

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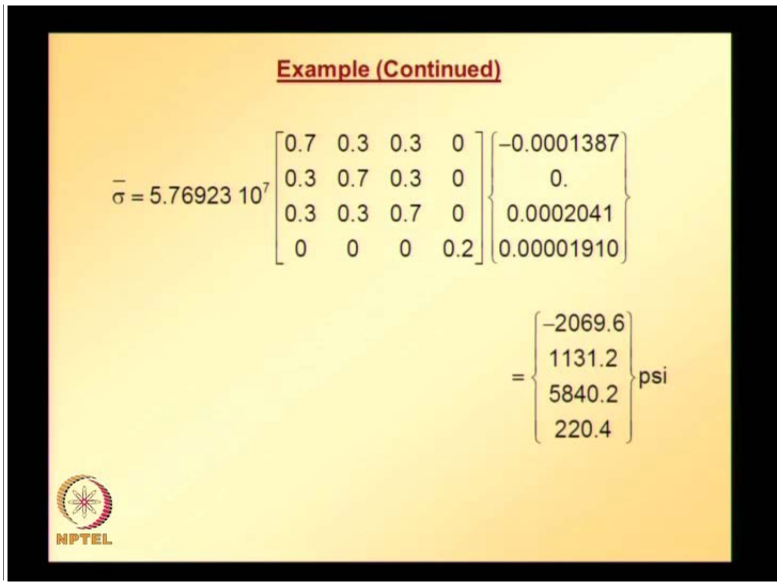

Strain, once we know the strain for element one, we can calculate stress for element one all the stress components for element one.

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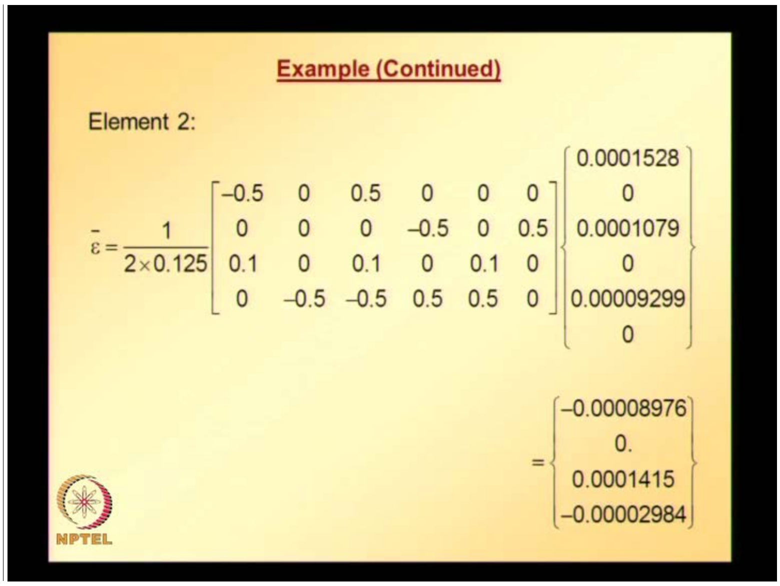

Similar exercise, we can repeat for element two and also we can calculate stresses for element two.

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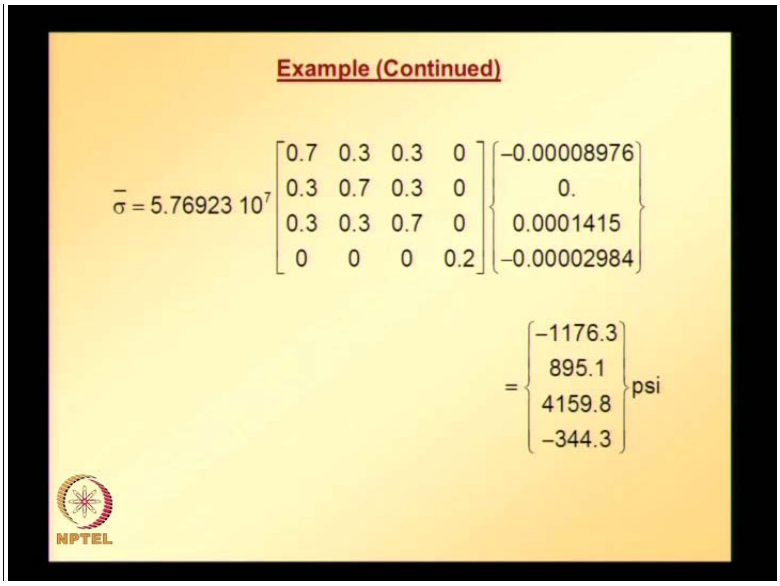

So, in this lecture we have seen the governing differential equation for axisymmetric problems, and also finite element equations for three node linear triangular element. In the next class, we will see quadrilateral element for solving axisymmetric problems.