Finite Element Analysis
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Lecture No. # 34

In today's lecture, we will be looking at two-dimensional elasticity; and similar to what we have seen in earlier classes, whether it is two-dimensional or three-dimensional elasticity problem, it basically involves solving second order differential equation. So, a 2 and 3D elasticity problems are governed by a system of coupled second order differential equations. So, this governing equation is going to be a coupled equation; coupling is going to be between different primary variables.

The main variables are displacements along the coordinate directions. If it is 2Delasticity problem, it is going to be displacement along x and y directions. If it is 3D elasticity problem, it is going to be displacement along x, y, z directions. Once we solve for these displacements, once the displacements are known, the stresses and strains can easily be calculated.

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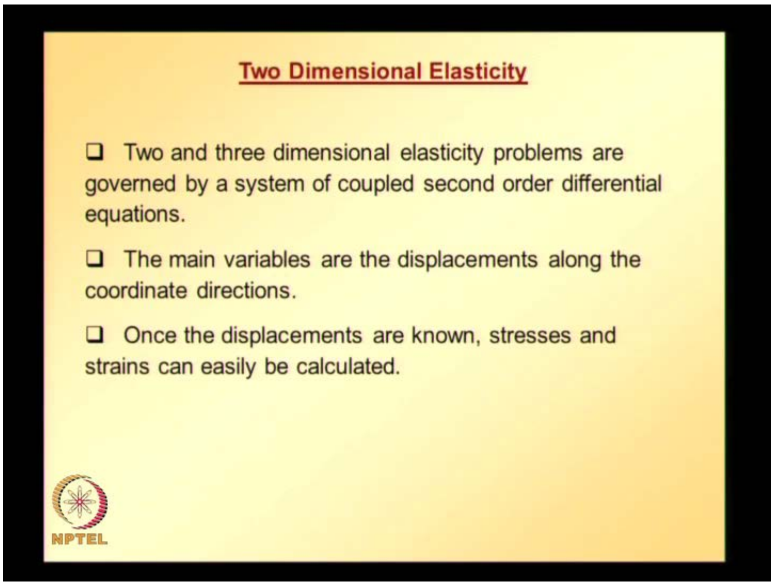

So, what will be doing? In these, in the next few lectures, we look at the governing differential equation for two-dimensional elasticity problems, and then we look at derivation of finite element equations, for a triangular element, four node quadrilateral element and eight node serendipity element. And later, we look at finite element formulation for axisymmetric problems; so now let us get started with this two-dimensional elasticity problem by starting with derivation of governing differential equation.

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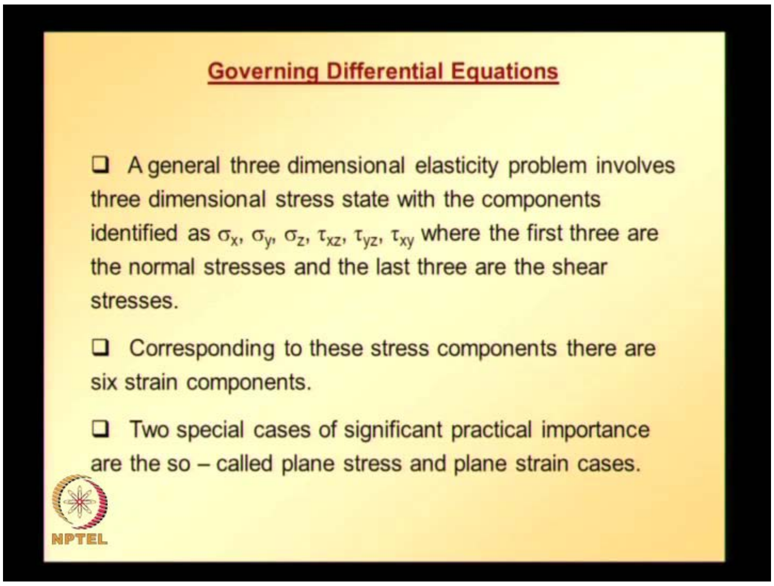

A general three-dimensional elasticity problem involves three-dimensional stress state with components identified as sigma x, sigma y, sigma z, tau x z, tau y z, tau x y where the first three are normal components and the last three are shear components. Corresponding to these stress components, there are six strain components. This 3D elasticity problem can be reduced to two special cases of significant practical importance and they are called plane stress and plane strain cases. So, let us see under what conditions, we can reduce a three-dimensional elasticity problem to a plane stress problem and at what conditions, we can reduce three-dimensional problem to the plane strain case.

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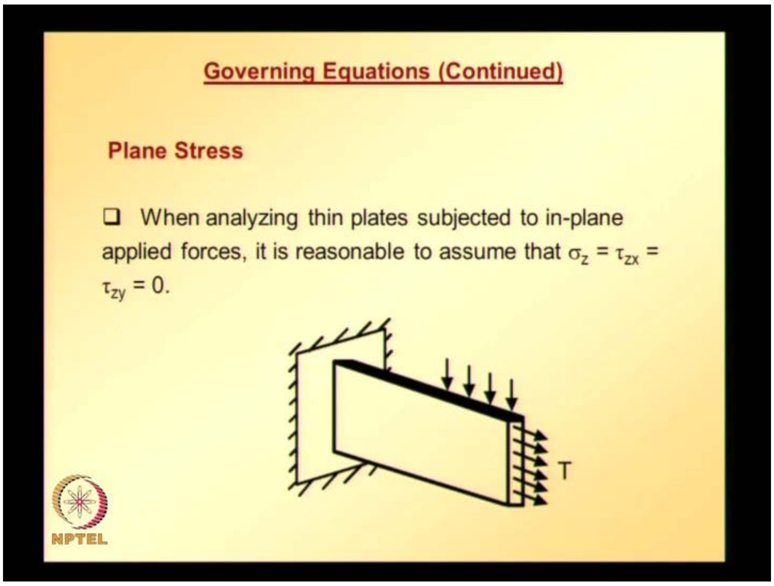

When analyzing thin plates subjected to in-plane forces, it is reasonable to assume the out of plane stress components are 0 like sigma z, tau x z or tau z x is going to be same as tau x z and tau z y. All these three are equal. So, a schematic is shown here; in which, a thin plate is subjected to in-plane applied forces. Under these circumstances, we can assume out of plane stress components are 0 and then, that is what plane stress is so. Out of plane stress components are 0 and the non-zero stress components are in-plane stress components. That is why it is called plane stress condition.

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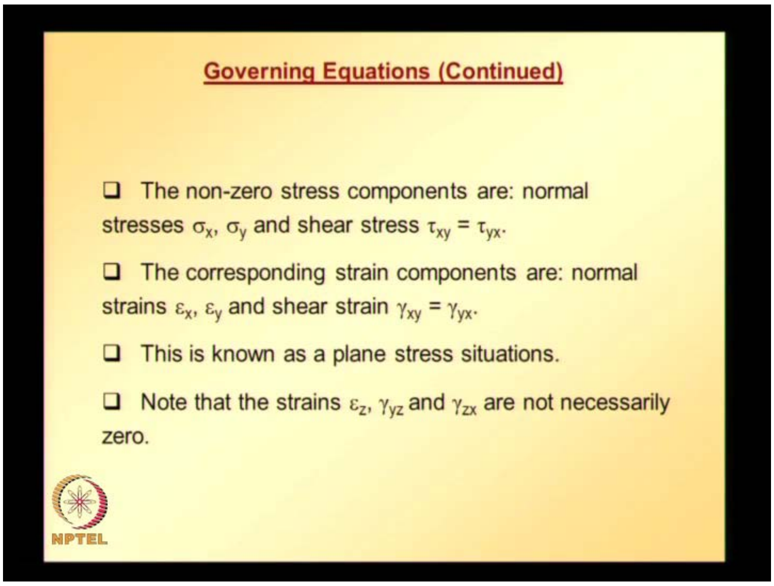

So, we have seen the stress components which are 0 and now, the non 0 stress components are two normal components sigma x, sigma y and one shear component tau x y which is going to be same as tau y x. And the corresponding strain components are two normal strains and one shear strain. This is known as plane stress situation. And please note that, even though out of plane stress components are zero, out of plane strain components are not necessarily 0. Note that, strains strain epsilon x epsilon z, gamma y z and gamma z x are not necessarily 0. So, now let us look at stress components acting on a differential element to develop the governing differential equations.

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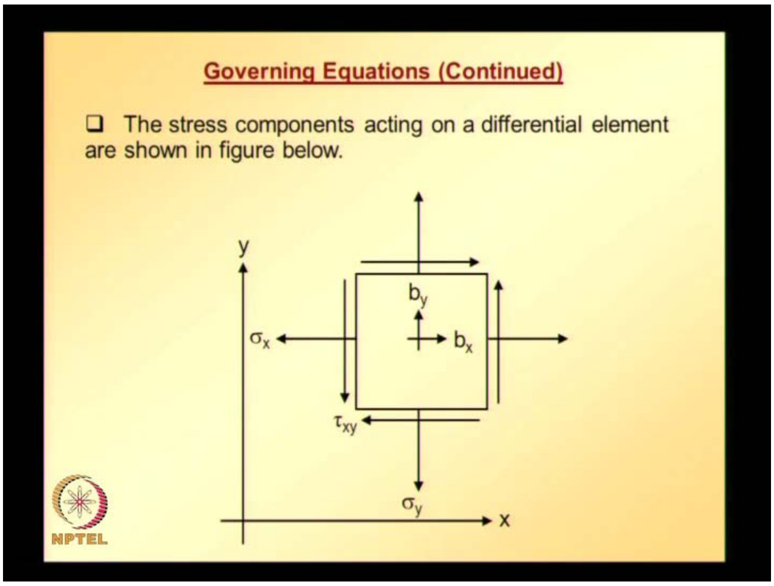

And before that, we also need to be very clear what is going to be our sign convention. So, if you look at this figure on the planes farther from origin, the stress components are positive. If they act along the positive coordinate directions on the near planes or on the planes nearer to the origin, the positive directions are along negative coordinate directions. Physically, this sign convention means tension is positive and compression is negative. So, that is what, we are going to adopt. So, the way the stress components are indicated in the schematic that is going to be our sign convention. And also, in this figure in addition to the stress components, the body force components are also shown.

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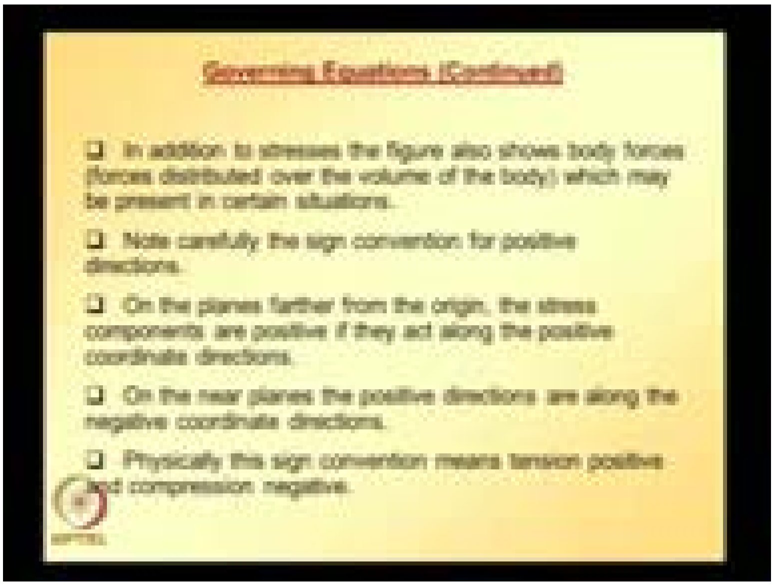

So, now let us look at the equilibrium of differential element.

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So, these are the things, which I just explained.

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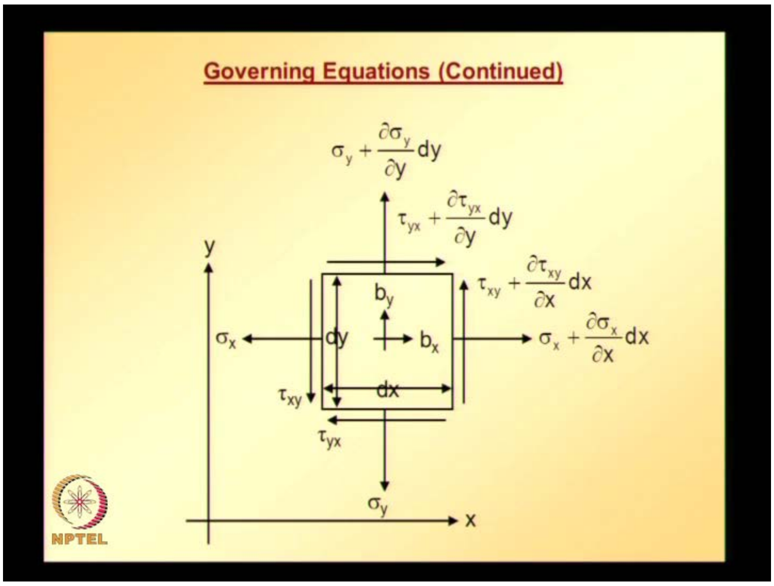

Now, let us take the differential element with all stress components and body force components indicated. By writing equilibrium equations along x direction and y direction, that is considering equilibrium of forces in x and y directions acting on the differential element shown here. That is, sigma f x is equal to zero; sigma f y is equal to zero, applying that condition. And when we apply the condition, moment about any point is equal to zero; that results in what is called complementary property of shear. That is why, we have tau x y is same as tau y x. So, when we apply the two equilibrium conditions, sigma f x is equal to zero; sigma f y is equal to zero; that leads us to two equations.

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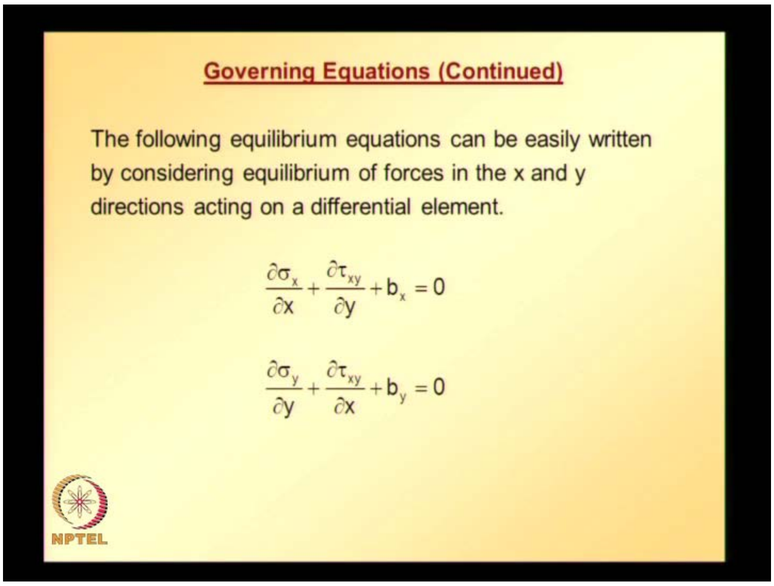

The following equilibrium equations can be easily written by considering equilibrium of forces in the x and y directions acting on the differential element. So, the first equilibrium results in first equilibrium condition results in this equation; second equilibrium condition results in this equation. So, basically the entire governing equation for 2Delasticity problem starts from here. So, now what we need to do is, we need to express these equations in terms of displacements; because displacements are our primary variables, which we need to solve.

So, these equations can be expressed in terms of two displacement components; because we are looking at two-dimensional elasticity problem. So, now we need to express these two differential equations in terms of two displacement components u along x direction and v along y direction. And also to keep the derivation simple, let us assume in the rest part body force is neglected. So, body force is going to be neglected in the derivation that we are going to see in a while and also we assume small displacements and strains. That is we are assuming small deformation conditions.

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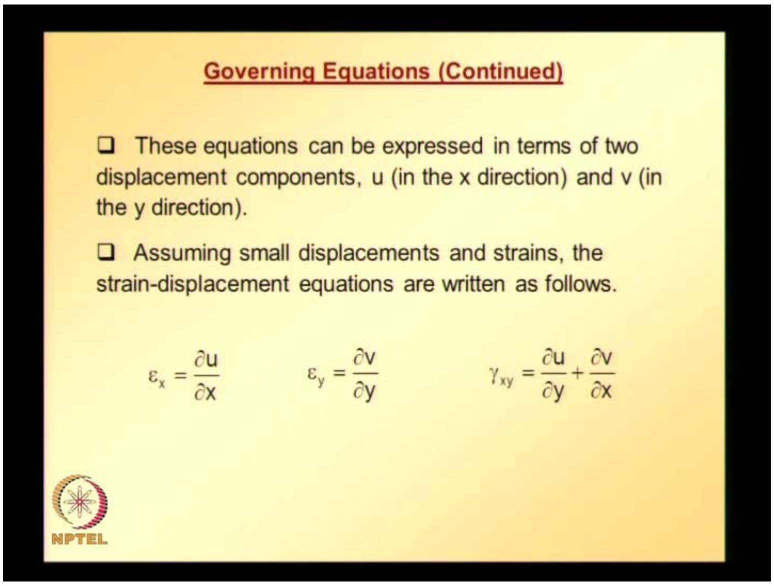

So, when we assume small displacements and strains, strains are related to displacements via these equations. These equations are valid only under small under the assumption of small displacements and strains. So, our job is to express the equilibrium equations which are in terms of stresses. We need to express them in terms of displacements. To do that, we first made this assumption of small displacements and strains and based on that, we got relationship between strains and displacements.

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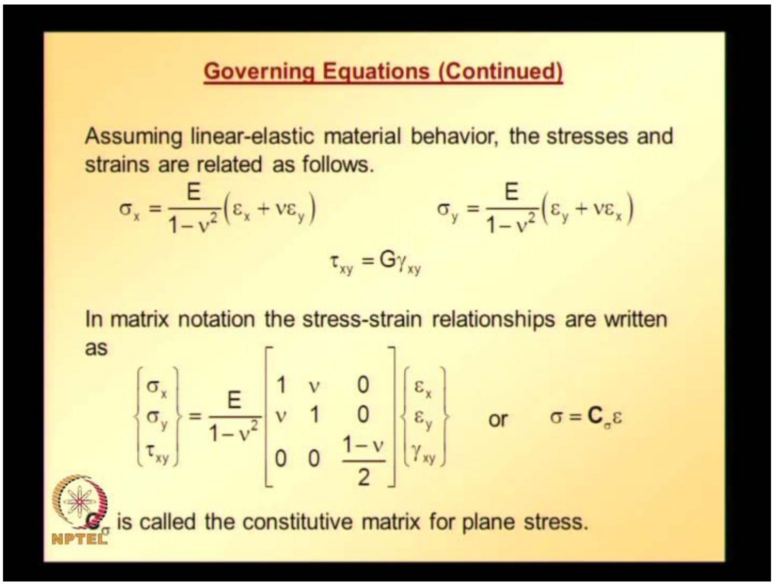

And also under linear-elastic assumptions, assuming linear elastic material behavior, stresses and strains can be related by these three equations. And please note that, we are looking at plane stress case. So, under plane stress conditions and under linear elastic material behavior, stresses are related to strains through these three equations. Three components of stresses are related to three components of strains; through these relations, which we can write these relations in a matrix form.

And here, we require two material constants, which are denoted with E young’s modulus and mu poison’s ratio and also shear modulus. Shear modulus is given by young’s modulus divided by 2 times 1 plus poison’s ratio 1 plus mu. So, when we write these three equations in a matrix form, it looks like this or it can be compactly written as sigma is equal to C subscript sigma epsilon. C subscript sigma denotes constituting matrix corresponding to plane stress condition.

So, we got relations between strains and displacements and also we got relation between stresses and strains for plane stress case under small deformation and linear elastic material behavior assumption. So, now substituting the strain stress strain relations into the first equilibrium equation, the equilibrium equation that we have is in terms of stresses, that we can convert into or that we can express in terms of strains by using the relationship of stresses and strains. So, using the relationship of stresses and strains, the first equilibrium equation can be expressed in terms of strains.

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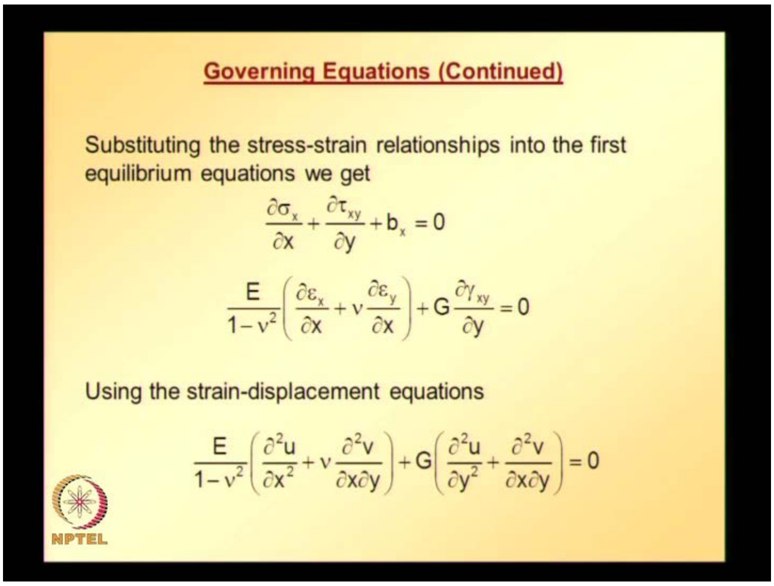

And now, this is how the equilibrium equation looks like; neglecting body force, when it is expressed in terms of strains. And now, when we write the strains in terms of displacements, it looks like this.

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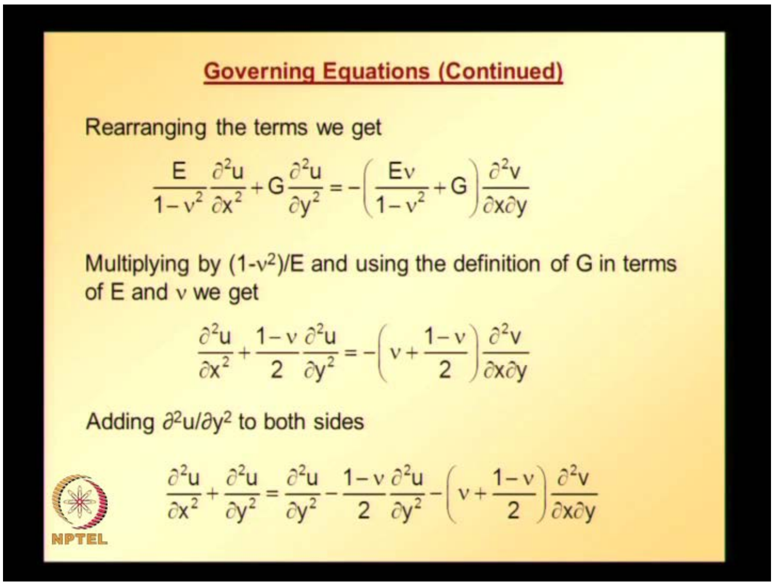

And we can rearrange this equation into this form and do little bit manipulation, multiplying by 1 minus u square over E. And using the definition of shear modulus in terms of young's modulus and mu poison's ratio, we get the relationship G is equal to young's modulus divided by 2 times 1 plus poison's ratio. When we do this manipulation we are going to get this. And then, adding partial derivative second partial derivative of u with respect to y on both sides, we get this equation.

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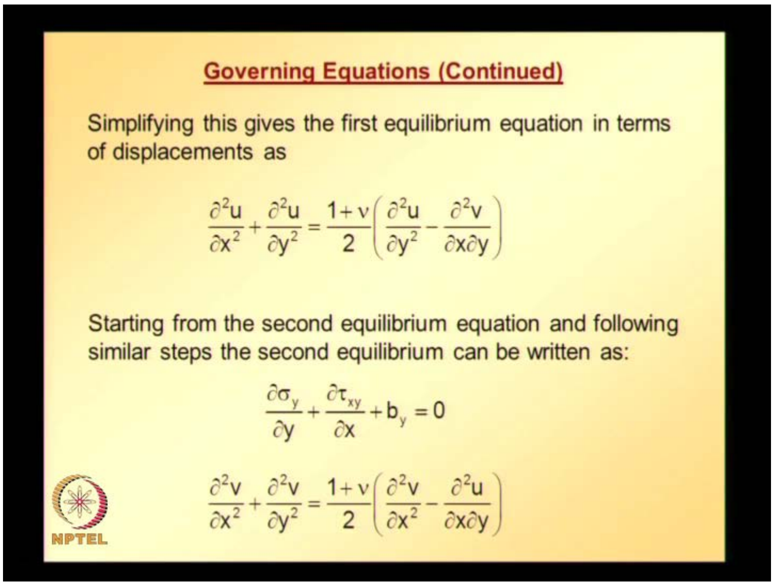

And this can further be simplified and finally, we get this. Starting from governing differential equation, which is in terms of stresses; expressing stresses in terms of strains and strains in terms of displacements and do some kind of manipulations and finally, we arrived at this equation. So, following similar steps and starting from second equilibrium equations, we can get similar equation; except that, it is going to be in terms of displacement component in the y direction. So, solving a 2D elasticity problem involves solving these two differential equations, which are coupled second order differential equations.

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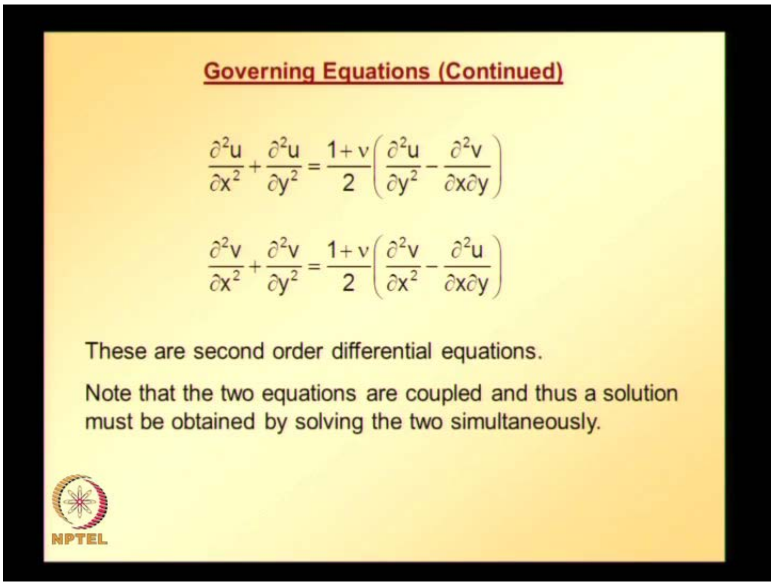

Since these two equations are coupled, solution must be obtained by solving them simultaneously. Since these two equations are second order differential equations, you can guess now. We require two boundary conditions to solve these differential equations. So, since the differential equations is these differential equations are of order two, those boundary conditions of order 0 are essential are going to be essential boundary conditions and those boundary conditions of order one are going to be natural boundary conditions. Since this is a two-dimensional elasticity problem, you can easily guess what are going to be the essential boundary conditions. Displacements are going to be the essential boundary conditions and forces are going to be natural boundary conditions.

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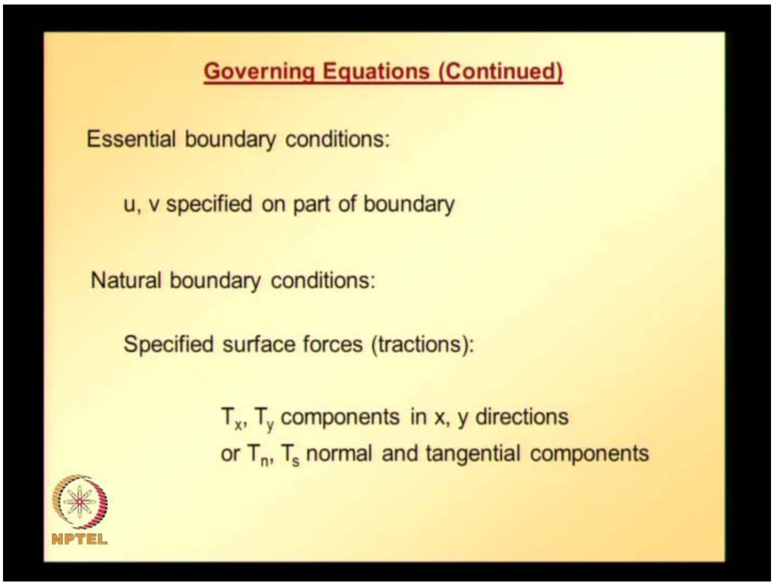

Essential boundary conditions specify the displacements in the x and y directions on a specified part of boundary and natural boundary conditions specified surface forces. And surface forces are nothing but, tractions T x T y in x and y directions or if you have a surface traction can also be specified normal to the surface and tangential to the surface.

So, traction can also be specified in terms of T n and T s normal component and tangential component. Usually instead of specifying surface forces in terms of T x and T y, that is in terms of traction in the x direction and y direction, it is always convenient to express in terms of normal and tangential components. That is why we require to know, what is the relationship between T x, T y and T n, T s.

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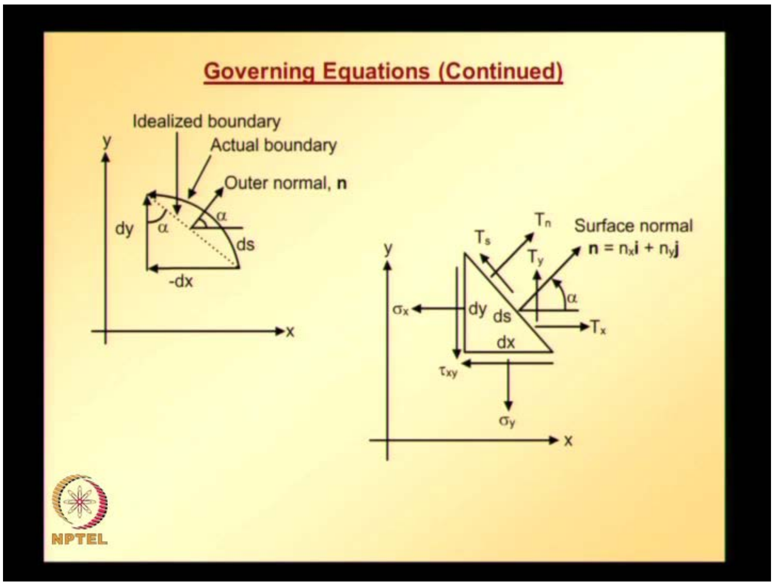

And here, a differential element on the boundary on which, traction is specified is shown. In the first figure, both the actual boundary and idealized boundary both are shown. And we will be doing all calculations on the idealized boundary and normal outer normal is also shown with respect to the idealized boundary. The length of the element is d s, which can be resolved into two parts with respect x and y axis; d x and d y.

And alpha is the angle, which outer normal makes with respect to the x direction positive x direction. And a triangle is shown in the second figure with all forces indicated. Both the applied tractions and also tractions in both x y directions and also normal and tangential to the idealized boundary. And also the internal stress components are also shown in the second figure. So, looking at these two figures, we can easily write the relationship between T x, T y and T n, T s.

And also, we can write the relation between T x traction in the x direction, traction in the y direction and the relationship between that and the stress components sigma x, sigma y and tau x y. The first one that is expressing T x, T y in terms of T n, T s can be obtained using coordinate transformation. By calculating the components of normal in the x direction, y direction whereas, the relationship between T x, T y and the stress components is obtained by writing equilibrium equations of the element, that is shown in the second figure.

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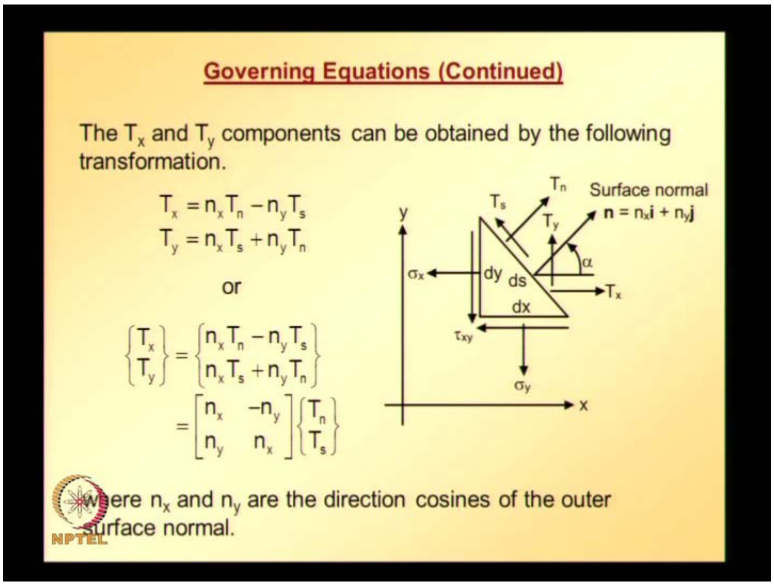

So, now let us start with the first one. T x, T y components can be obtained by following transformation. Once we know tangential or normal and tangential tractions, we can easily calculate T x, T y using these relations, where we need to know what is n x, n y. And that depends on… n x, n y are nothing but, direction cosines of outer surface normal. They can be calculated, once we know what is d x and d y. So, if we know the length of the side of the the length of side of the element on which, tractions are applied. We can easily calculate by knowing the extreme points of that edge. We can easily calculate, what is d s? Once we know d s, we can calculate d x d y.

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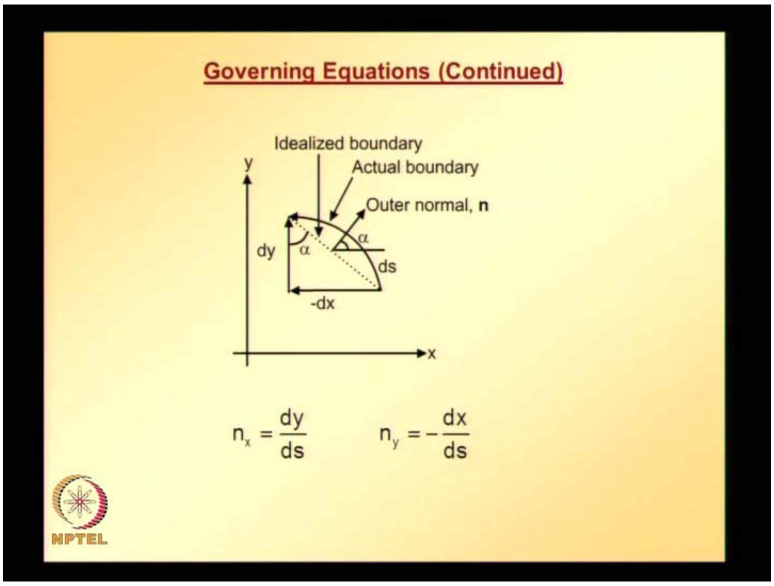

If we know alpha or we can easily calculate n x n y, using these relations. The surface forces can be expressed in terms of primary variables, primary unknowns displacements. By first writing the tractions in terms of stress components and we know the relationship between or how to express stress components in terms of displacement component.

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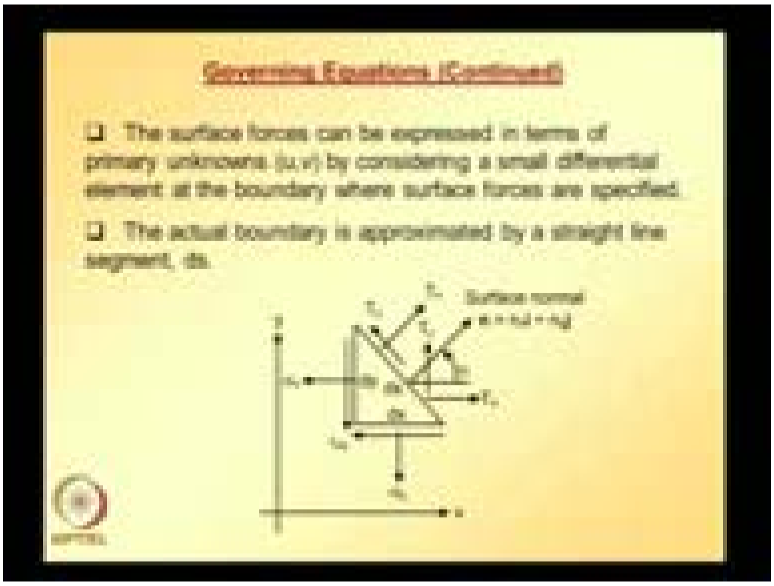

So, first we need to get the relations between tractions and stress components. Surface forces can be expressed in terms of primary unknowns by considering small differential element at the boundary, where surface forces are applied or specified. The actual boundary is approximated by straight line segment d s, idealized boundary and writing equilibrium equations.

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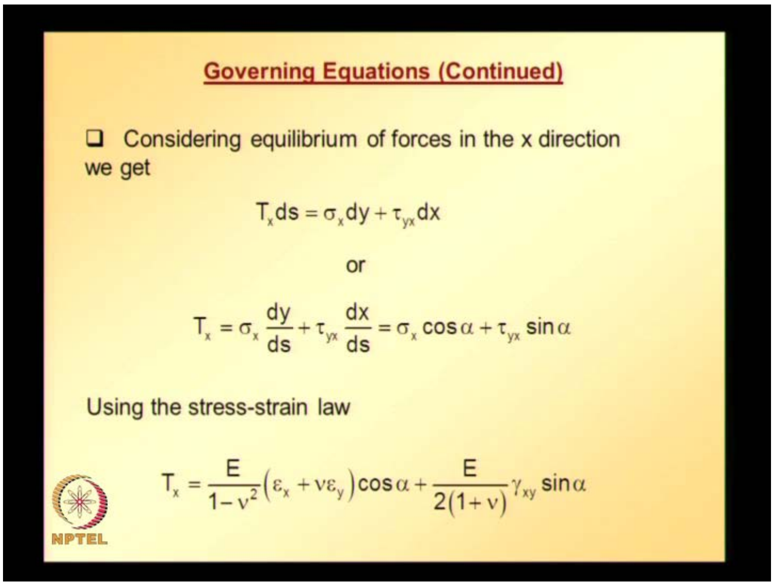

Considering equilibrium forces in x direction, we get this and dividing this equation on both sides with d s, we get this. So, this gives us relationship between traction component in the x direction and internal stress components sigma x, tau x y or tau y x. And now using stress strain law, we can express this in terms of strains and using strain displacement relation, we can express this in terms of displacements.

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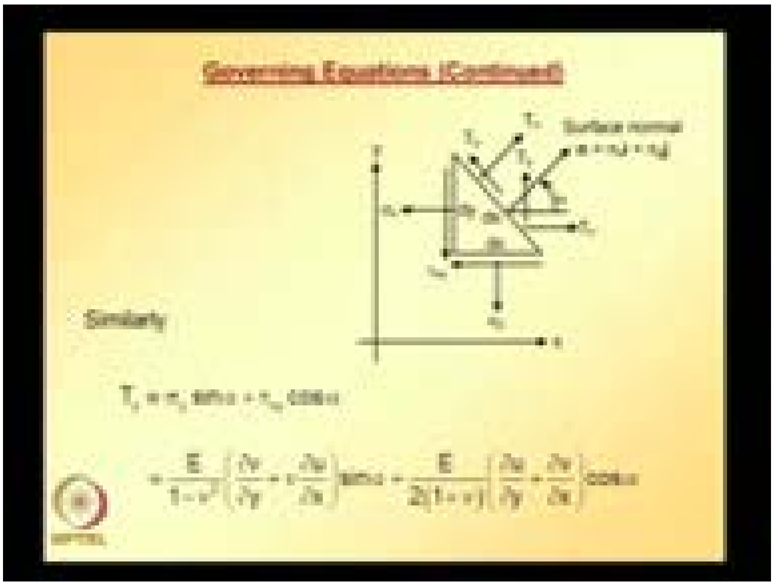

Similarly, starting with the second equilibrium equation or writing equilibrium equation for this differential element in the y direction and again expressing stresses in terms of strain and strain in terms of displacements, we get this equation.

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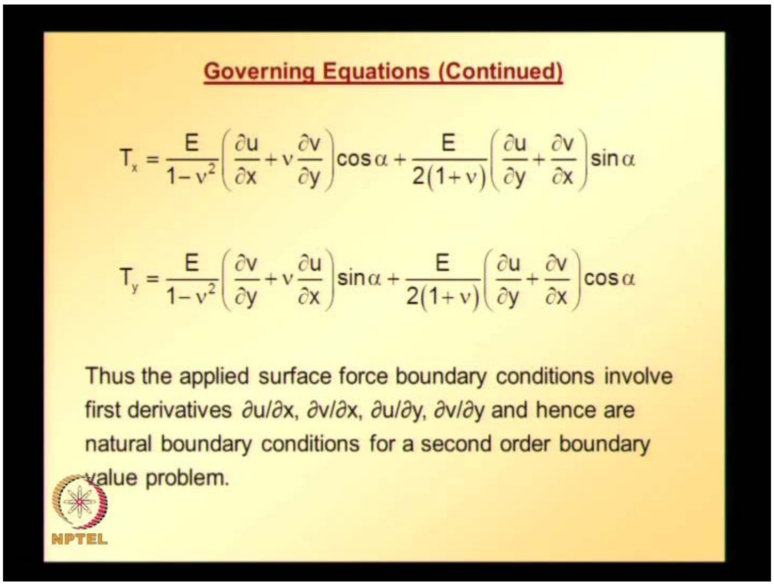

So, T x in terms of displacement is given by this relation and T y in terms of displacements is given by this relation and if you see these equations T x T y, that involves first derivative of displacements. So, tractions are indeed natural boundary condition; because they involve first derivative displacements whereas, displacements are 0 th order equations, which are essential boundary conditions.

So, first order is the equation which comprises of first derivative first order derivatives that is going to be natural boundary condition. So, thus the applied surface force boundary conditions involves first derivative so displacements. Hence, they are natural boundary condition for second order boundary value problem. So, so far we have looked at plane stress case. Now let us see, what is plane strain?

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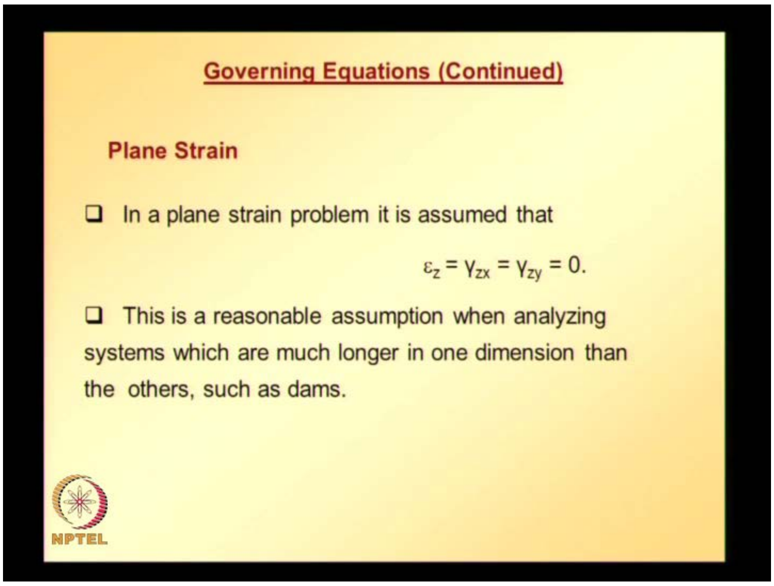

Again, plane strain as the name itself indicates its self explanatory. Thin plane strain components are going to be non 0 and out of plane strain components are going to be 0. In plane strain problem, it is assumed that epsilon z, out of plane normal out of plane normal strain component and also shear components gamma z x, gamma z y or gamma x z is going to be same as gamma z x. Similarly, gamma y z is going to be same as gamma z y. All these out of plane strain components are going to be 0 and this is a reasonable assumption when analyzing systems, which are much longer in one direction or one dimension than the others, such as dams.

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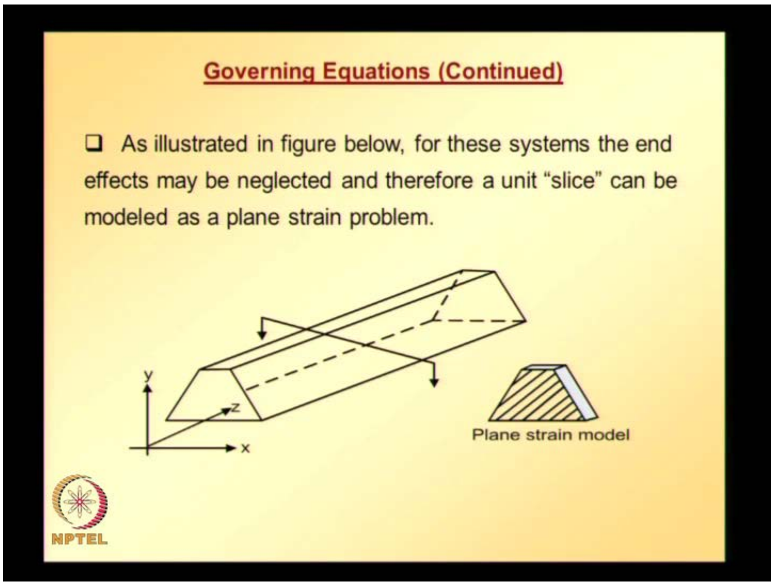

Let us look at a figure, this this plane strain conditions are illustrated in the figure below. The dam is shown here. For these systems end effects may be neglected and therefore, unit slice can be modeled as plane strain problem. Actually, this is a three-dimensional problem. But for computational simplification, we are taking it as a plane strain model; since one of the dimensions is much longer than the other dimensions.

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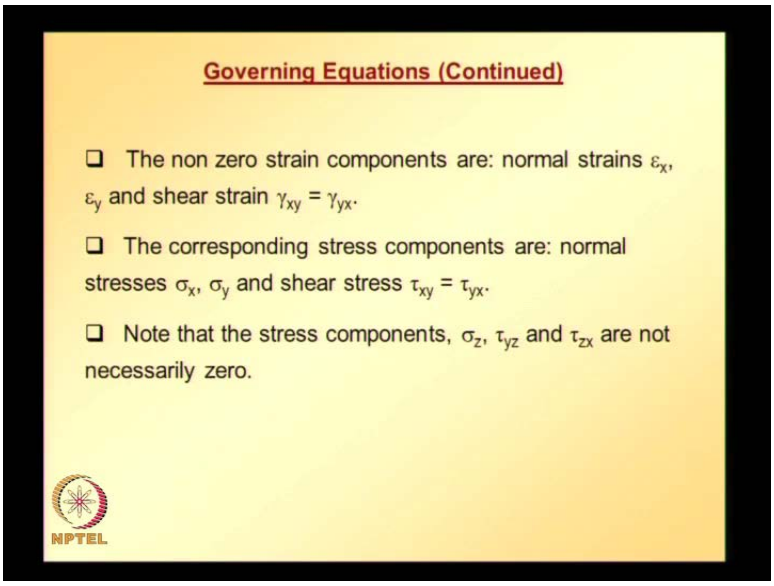

The nonzero strain components are normal strains and shear strains. The corresponding stress components are normal stresses and shear stress. Again out of plane strain components are 0 for plane strain case. But out of plane stress components are not necessarily 0. Note that, stress components sigma z, tau y z and tau z x are not necessarily 0.

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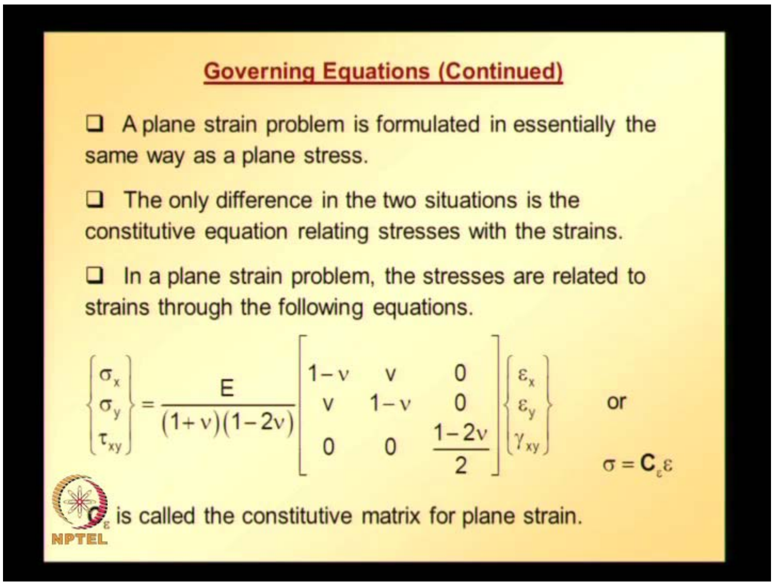

A plane strain problem is formulated essentially the same way as the plane stress problem that we just looked at. The only difference in the two situations is the constitutive equations relating stresses with strains and in plane strain problem, stresses are related to strains through the following equation. And this can be compactly written as sigma is equal to C subscript epsilon times epsilon, where C subscript epsilon is constitutive matrix for plane strain case.

So, whatever equations that we developed or derived for plane stress case, they can be easily transformed into the corresponding equations for plane strain case using simple transformation. So, now let us look at, what is the equivalence between plane stress and plane strain problem. So, once we have set of equations for plane stress case and if you want to quickly convert those equations, for the situation which is applicable to the plane strain case and how to do that?

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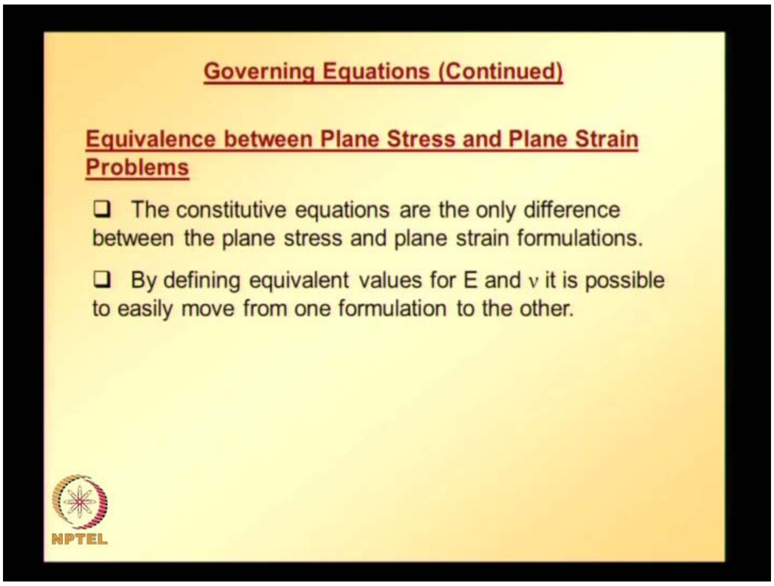

Equivalence between plane stress and plane strain problems. The constitutive equations are only difference between plane stress and plane strain formulations. By defining equivalent values of young's modulus and poison's ratio, it is possible to easily move from one formulation to other.

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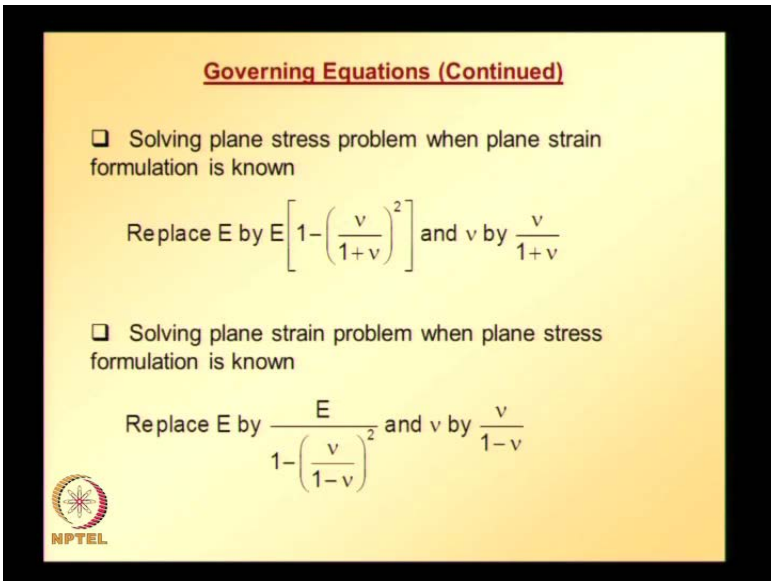

So, solving plane stress problem, when plane strain formulation is known, you apply this small trick. Replace young's modulus E by wherever E is there; replace that with E times in brackets in square brackets 1 minus nu over 1 plus nu square. Similarly, poison's ratio needs to be replaced with nu by nu divided by 1 plus nu. So, these if plane strain formulation is known, and if you want to solve a plane stress case, then do this substitution. And solving plane strain problem, when plane stress formulation is known, do this kind of substitution. So, with this understanding of equivalence, we can use.

If we have any of these formulations, we can easily solve a problem, which is under different assumptions. So, now we looked at the governing differential equations for 2Delasticity problem and also we looked at essential boundary conditions and natural boundary conditions. Now we are ready to derive the finite element equations. But before that, we need to do as we did earlier for other problems. First we need to come up with, if we are using variation or formulation, we need to come up with equivalent functional.

Or if we are using Galerkin criteria, then we need to finally bring the governing differential equation and boundary condition to such a form, where we can substitute approximation of trial solution; derivative of trial solution. So, now let us look at equivalent functional for the 2D elasticity problem. So, what I will do here is, I will directly give the equivalent functional and like we did earlier. That functional can be verified, whether that is the correct functional for this problem or not.

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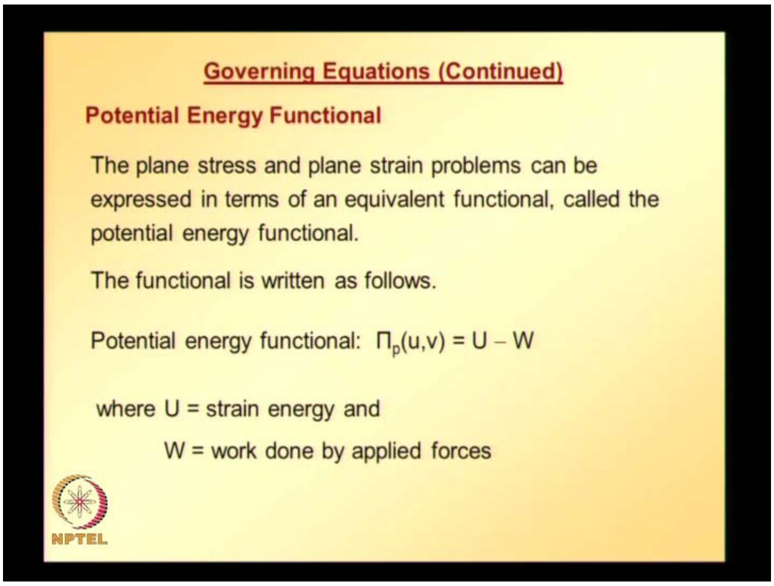

Potential energy functional: The plane stress, plane strain problems can be expressed in terms of equivalent functional called potential energy functional and the functional is written as follows. pi, this functional is going to be function of displacement in the x direction and displacement in the y direction. This functional is denoted with letter pi; potential energy functional is equal to U minus W, where U is nothing but strain energy and W is work done by the applied forces.

So, finally when we start with governing differential equation, the two coupled equations in terms of displacements and apply the essential boundary conditions and natural boundary conditions and do all the process, that we went through in the earlier cases. And finally, we arrive at this functional. So, this functional is directly given to us now. Now, our job is to verify whether this functional is correct functional for 2Delasticity problem or not.

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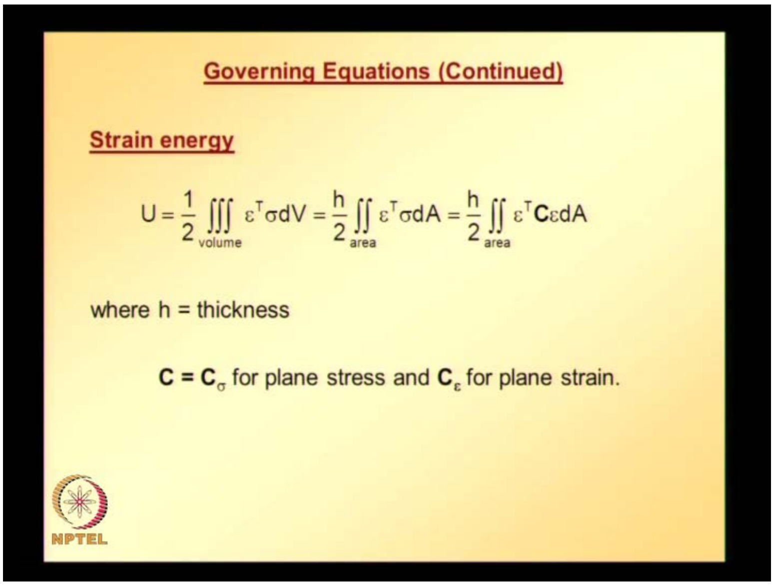

Now, let us look at how strain energy is defined. Strain energy is given by this; half integral over the entire volume of the structure or particular object, that we are looking at epsilon times epsilon transpose sigma. We need to integrate that over the entire volume and if the thickness is constant; if thickness of that particular component is constant, we can pull thickness how; h is thickness here; h is pulled out.

Now, integral becomes area integral and also stress can be expressed in terms of strain. If we know the constitutive matrix corresponding to whether it is plane stress or plane strain, so epsilon sigma is replaced with C times epsilon; where C is the constitutive matrix appropriate constitutive matrix C is equal to C sigma, if it is plane stress. C is equal to C epsilon, if it is plane strain.

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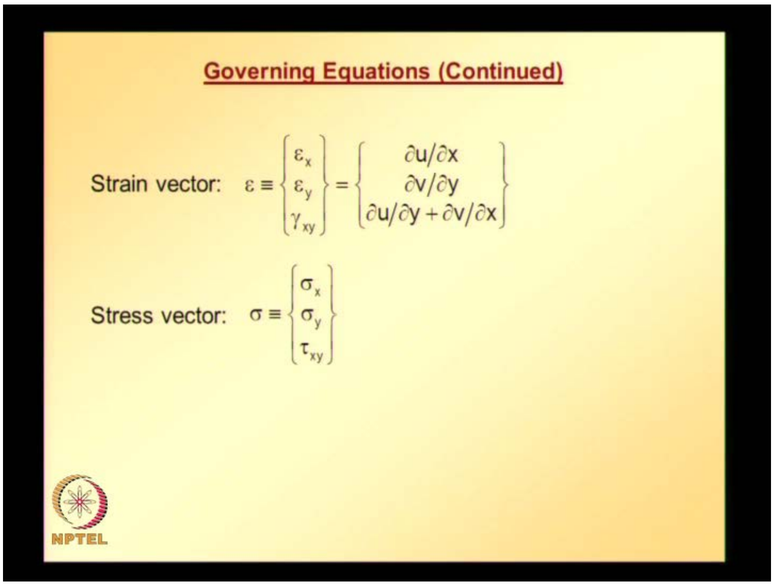

And this is stress and strain, sigma and epsilon involves three components. So, this is how, strain energy U is defined.

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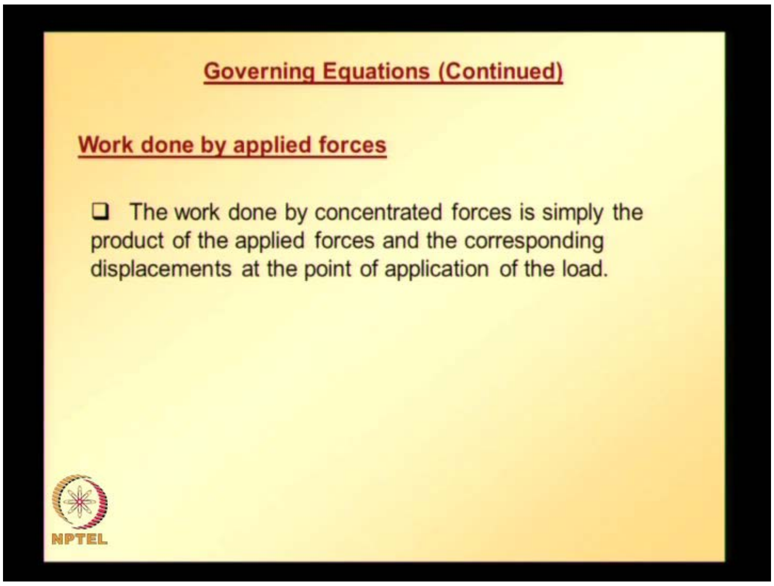

So, now let us look at work done by applied forces, W. If the forces that are applied are concentrated forces, work done by the concentrated forces is simply product of applied forces and corresponding displacements at the point of application of load.

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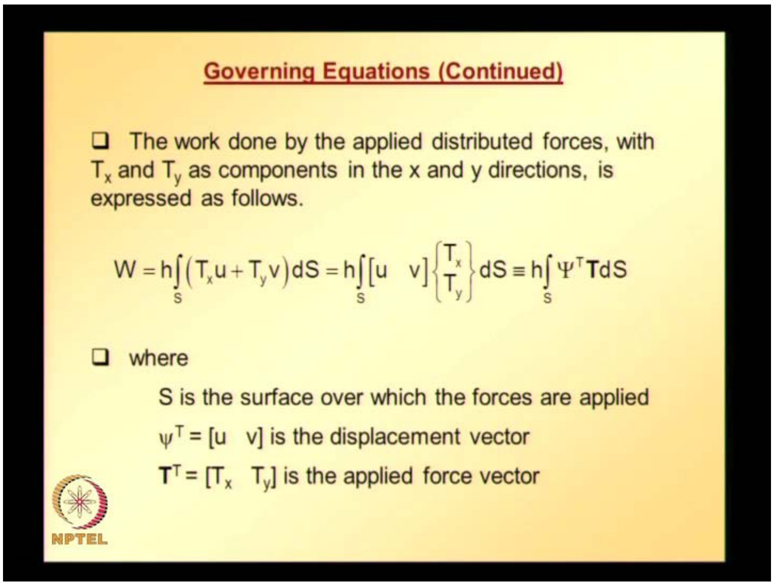

If it is distributed load, the work done by the distributed load is given by traction components in x and y directions multiplied by displacement components in x and y directions integrated over the entire surface over which, tractions are specified. W is given by h times… h is the thickness of that particular structure or object, that we are analyzing and if it is constant h can be taken out, otherwise it needs to be taken inside the integral.

And then, integration needs to be carried out over the surface over which, tractions are specified. And T x times U plus T y times V can be written in a matrix and vector kind of notation, which is shown there. And compactly, it can be written as psi transpose T, where S is the surface over which forces are applied and displacement vector is denoted with psi and applied force vector is denoted with T. But, usually in practical problems, tractions are specified in terms of tangential and normal components. So, it is better we write this equation in terms of T s and T n.

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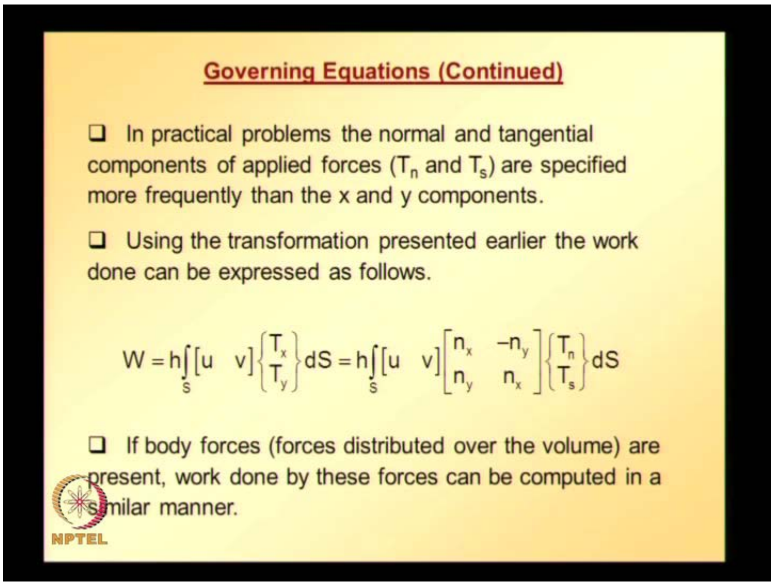

It is very simple. We know the relationship between T x, T y and T n, T s. So, we can use that and rewrite or re express this in terms of T n, T s using direction cosines of outer normal to the outer normal to the surface over which, tractions are specified. If body forces are present, work done by these forces can be computed in a similar manner. So, now this is how, the potential energy functional is defined; strain energy minus work done by forces applied forces.

Work done by the applied forces depends on what kinds of forces are applied, whether they are point forces or concentrated forces or distributed forces. Either way, we can easily calculate, what is going to be the work done. So, now we need to verify the whether the functional that is given, that is strain energy minus work done by the applied forces is correct functional for two-dimensional elasticity problem or not. So, we need to do that verify the functional.

To verify that potential energy functional is appropriate for plane stress, plane strain problems, we must show that governing differential equations are recovered by setting variation of pi p is equal to 0. So, pi is defined as U minus W. So, we need to take variation of it. Finally, we need to see whether we can recover the governing differential equations corresponding to the 2Delasticity problems in terms of displacements.

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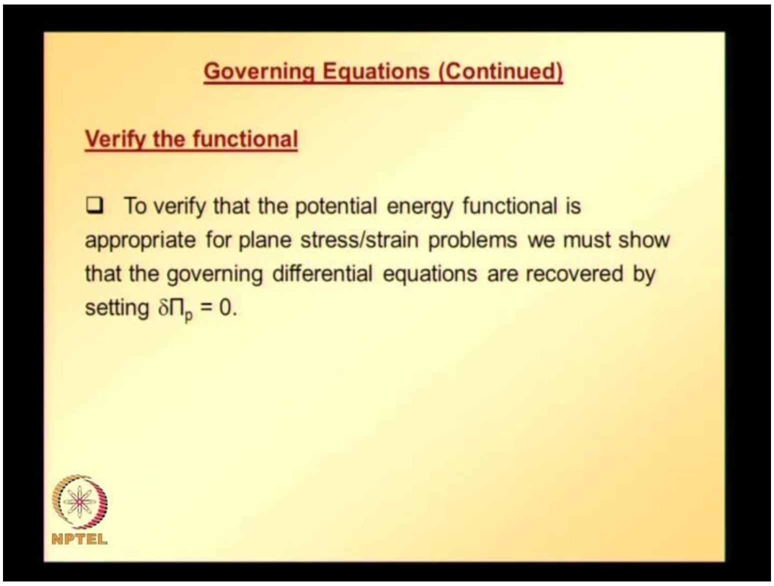

So, that is how we need to do. So, let us proceed.

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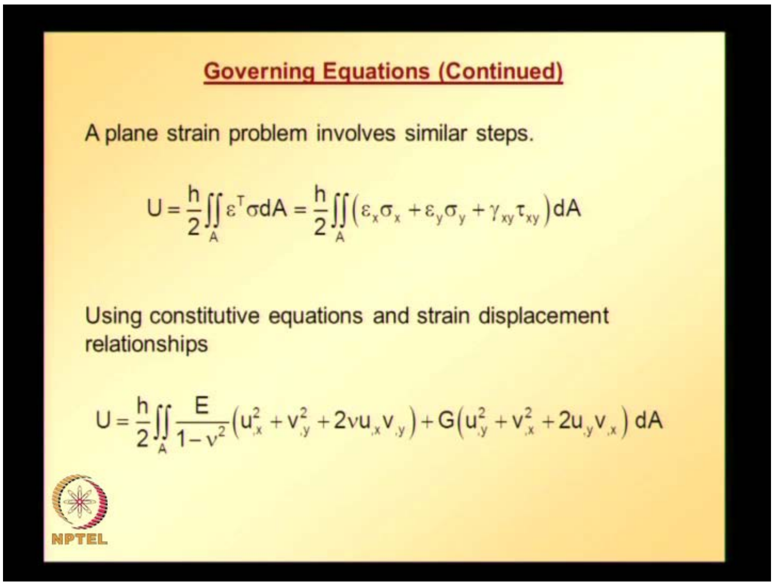

And now, let us say assume that the problem is plane strain problem. And this is how, strain energy is defined. Substituting epsilon and sigma, all the components and performing epsilon transpose sigma, we get this. Now, expressing using constitutive equations and strain displacement relations, we can express this equation in terms of displacements. Finally, strain energy in terms of displacements looks like this. And now, to verify whether pi p variation of pi p is equal to 0 or not, we need to know what is variation of U?

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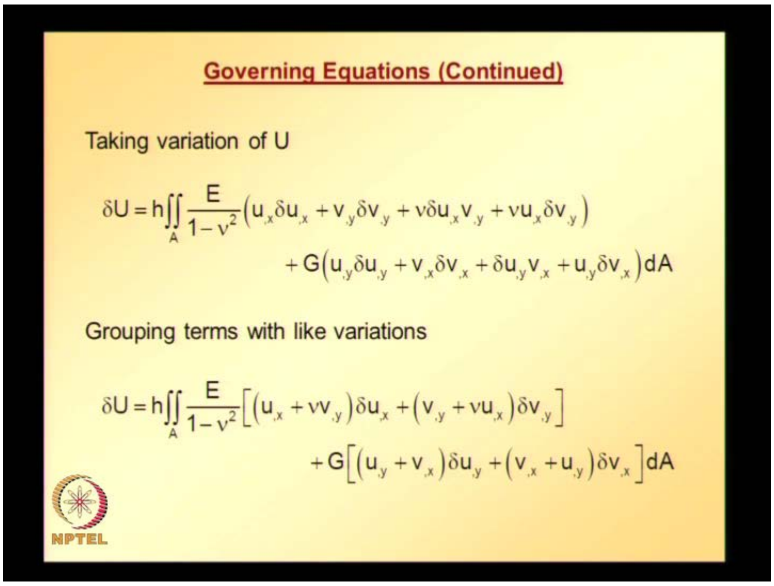

So, taking variation of U; taking variation of previous equation, we get this. Grouping terms with like variations, we get this. And now we need to consider, recall what is Green’s theorem and also, we need to recall that variational operator and differentiation operator are interchangeable. So, using these two conditions so, applying Green’s theorem and applying the condition that, variational operator and differential operator are interchangeable.

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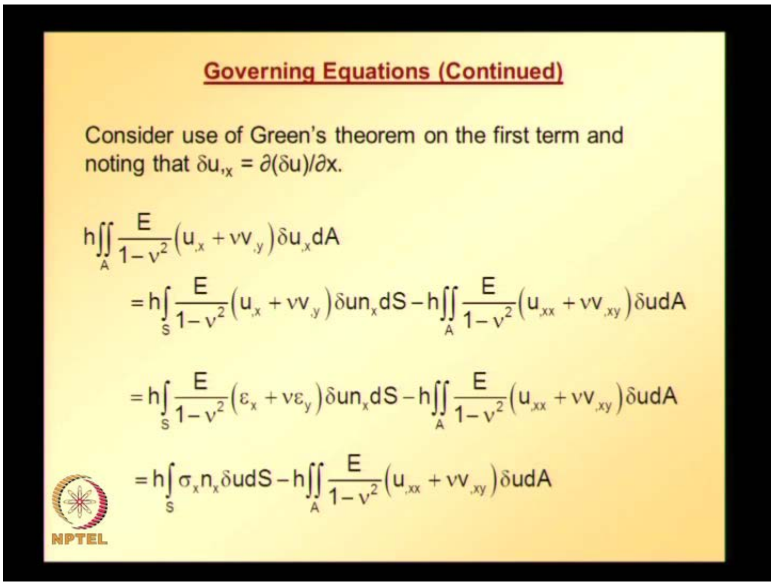

This equation can be written in this manner, which can further be simplified and finally, we get this one.

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So, this is only the first term of U.

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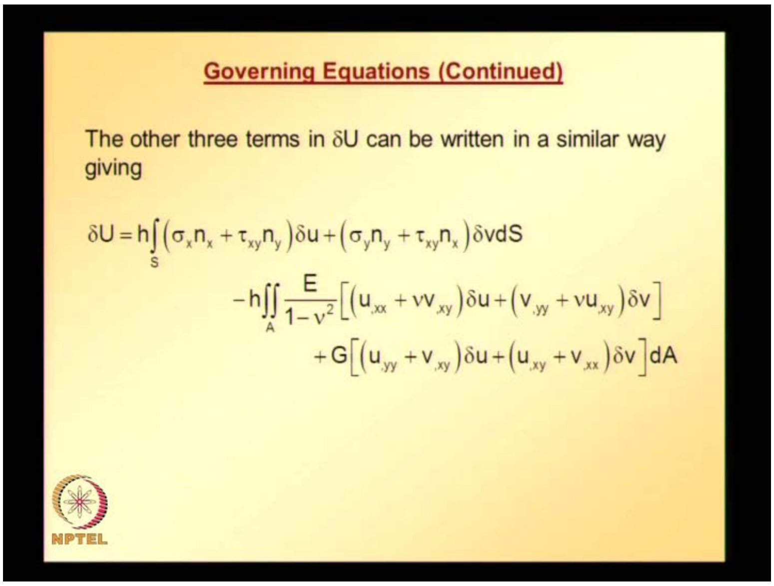

Other terms also can be written in a similar way and then finally, variation of strain energy is given by this one. Now, we need to find variation of work done by the applied forces. As I mentioned, forces can be different kind of forces, concentrated forces, distributed forces and body forces. But for generality, let us assume that the forces are distributed.

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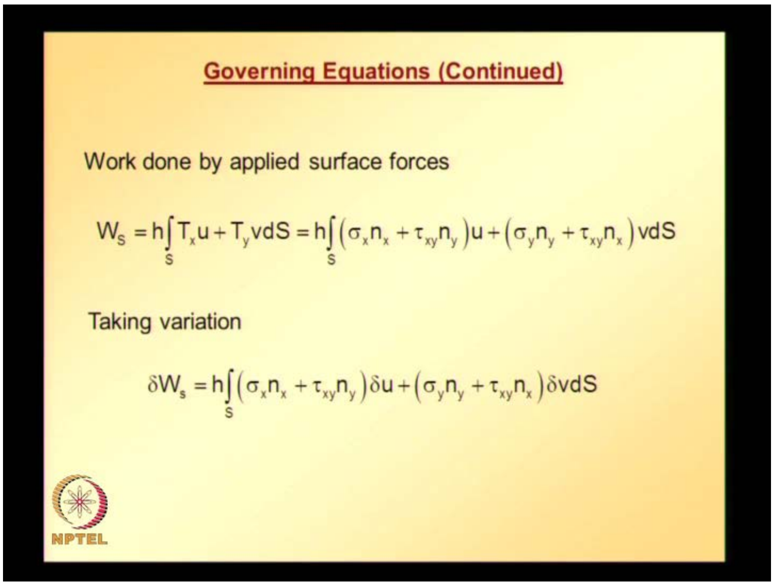

So, work done by the applied surface forces; this is how, W is defined. Again applying the variational operator, because we are interested in knowing, what is variation of W work done by the applied forces. So, this is what variation of W. Now, we got variation U and variation of W.

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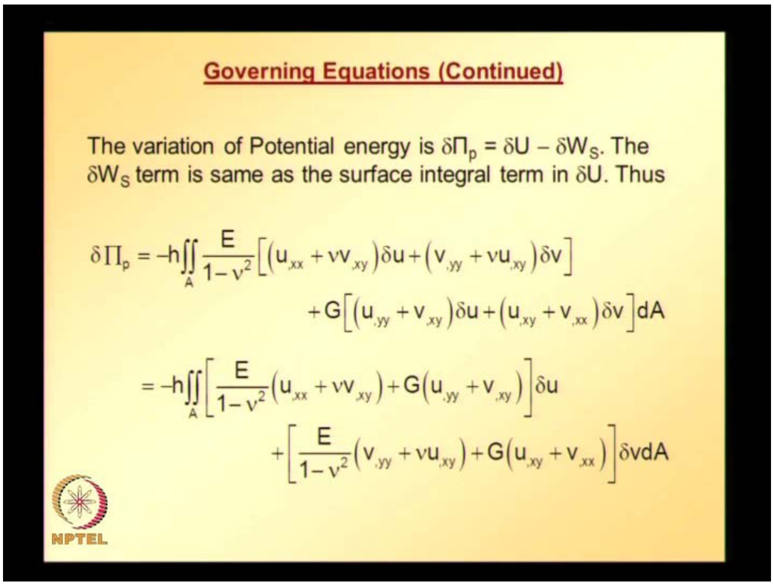

Substituting those two variations of potential energy can be obtained and if you notice variation of W term is same as surface integral term in U, variation of U. So, that term gets cancelled. Finally, we get variation of potential energy as this one and if you see this equation, please notice that variation of U and variation of V are arbitrary and only way that, this equation can be 0. That is, we need to apply or we need to see whether this on what conditions, this variational of pi is going to be 0.

So, variation of pi is going to be zero, only under the condition that the integrant in the first integral is equal to 0 and integrant in the second integral is going to be 0 independently. But variation of U and variation of V are arbitrary and they are not they are non zero. So, only way that this variation of pi is going to be 0 is, integrant in the first integral except variation of U is equal to 0 and integrant in the second integral except variation of V is equal to 0. Under those conditions only, this variation of pi is going to be 0. And indeed the integrant of the term and integrant of the second term are the governing differential equations.

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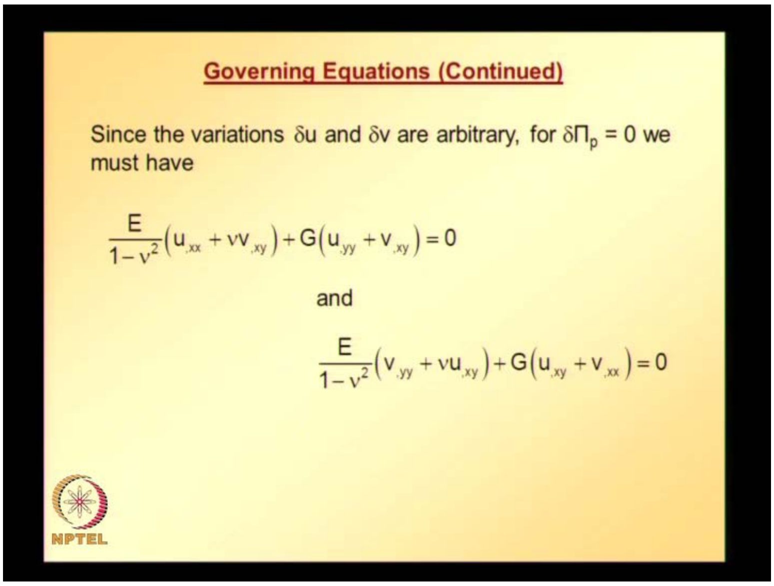

Since variation of U, variation of V are arbitrary, and for variation of pi to be 0, only way that can be 0 is by these two conditions, and which are indeed the governing differential equations. So, pi is the correct functional for this particular problem. So now, we looked at governing differential equations and also, we got the potential energy functional for 2Delasticity problem. So in the next class, we will be looking at deriving the element equations for a triangular element and also four node and eight node elements.