Finite Element Analysis

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Lecture No. # 32

In the next two lectures, let us see some practical applications, whose governing differential equations are of the form of the general two-dimensional boundary value problems. For which, we already derived finite element equations. And for each of these applications that we are going to look at, the basic variables and equations. Let us look at first to the basic variables and associated equations, corresponding to that application.

These variables are then interpretate in the context of variables in the general two-dimensional boundary value problems. And then we will derive the finite element equations, and finite element solutions, we will try to obtain using the derived finite element equations. So, as a part of this, let us start with torsion of prismatic bars.

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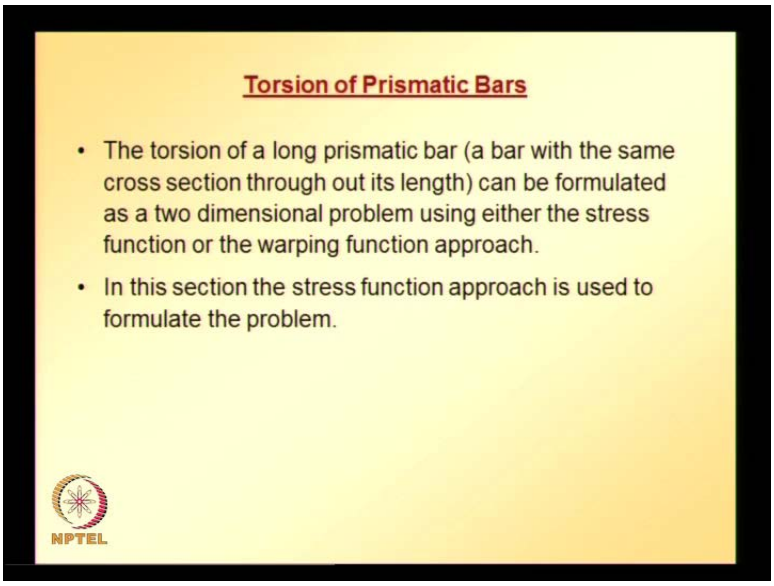

Torsion of a long prismatic bar that is bar with same cross section throughout its length can be formulated as two-dimensional problem. Using two approaches, one is either stress function one is stress function approach, and other is warping function approach.

But whatever equations that we are going to derive will stick with stress function approach.

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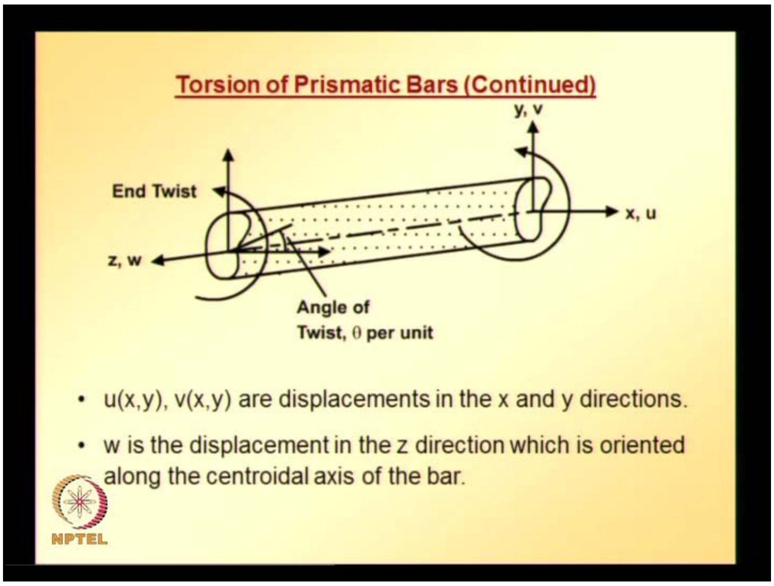

So, this figure shows variables associated with the torsion problem; u as a function of x y, v as a function of x y are the displacements in x and y directions; x y directions, the axis x y axis and z axis are clearly marked in the figure. And u and v are the displacements in x and y directions; w is the displacement in z direction, which is oriented along centroidal axis of the bar. For deriving the governing equations, linear elastic behavior and small displacements are assumed.

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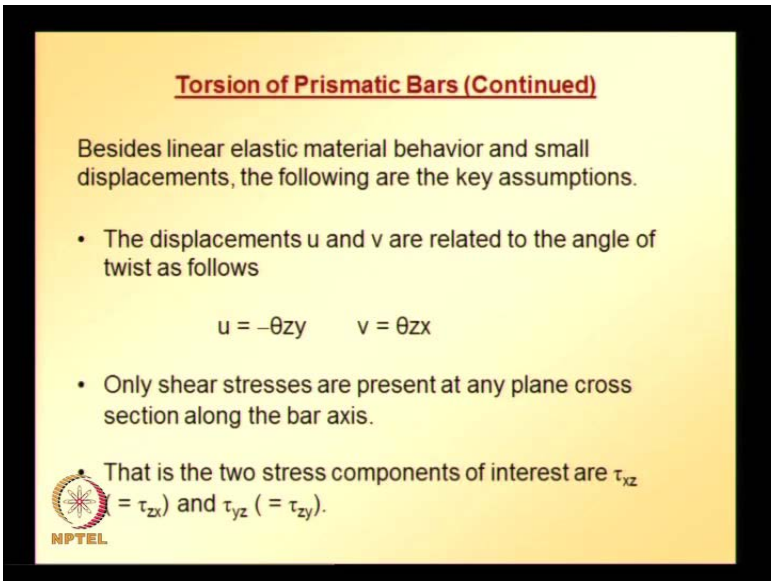

Besides linear elastic behavior small displacements, the following are the key assumptions. Displacements u and v are related to the angle of twist; angle of twist is indicated in the figure. And the displacement along x direction and displacement along y direction are related to angle of twist, theta. Via these two equations, only shear stresses are present at any plane cross section along the bar axis. That means two stress components of interest are tau x z, which is going to be same as tau z x and tau y z, which is going to be same as tau z y.

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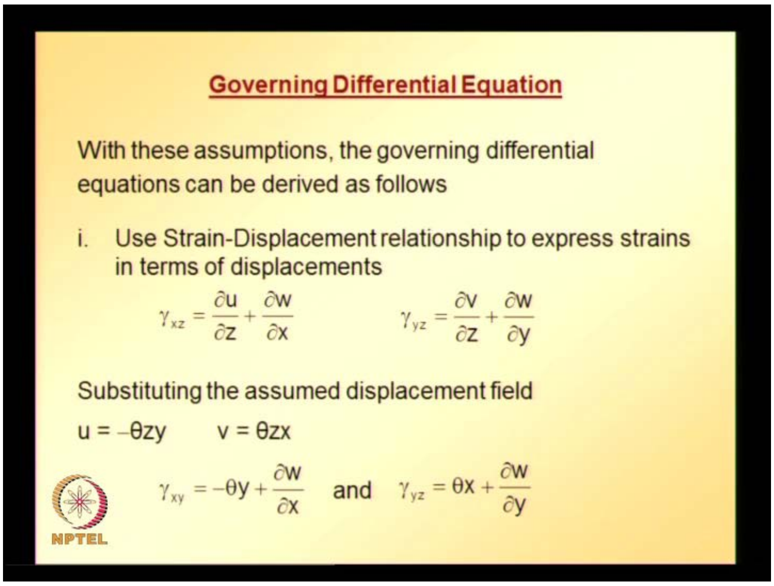

With these assumptions, the governing differential equations can be derived as follows. Using strain displacement relation shape, we can get the strains in terms of displacements and these are the relations. Substituting, we already know displacements along x and y directions that is u and v in terms of theta z or theta. Substituting the assumed displacement field, u is equal to minus theta z y, v is equal to minus sorry v is equal to theta z x. Substituting these, the previous strains can be expressed in terms of angle of twist, theta.(No audio from 05:30 to 05:40) Once we have strains, we can calculate stresses using Hook's law.

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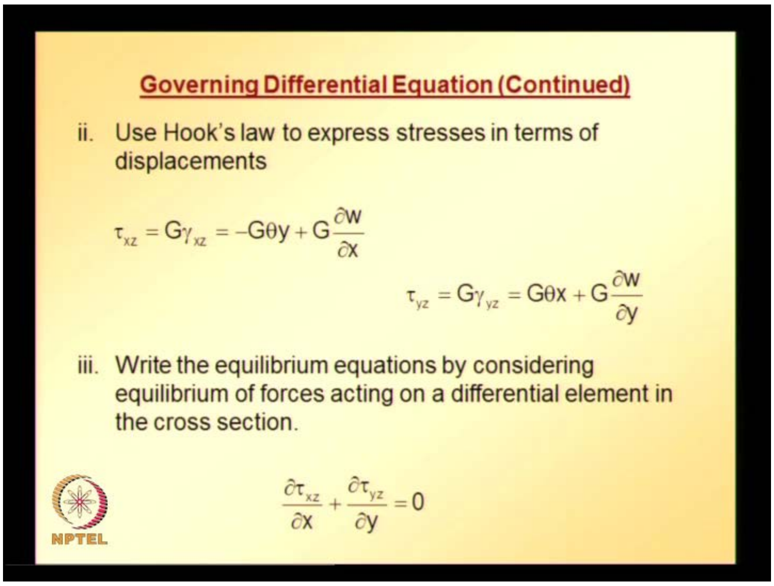

Use Hook's law to express stresses in terms of displacements. Based on Hook's law, we get this relation. If it is in shear stress and shear strain and substituting the gamma x z that we just obtained in to this equation, we get in terms of angle of twist. The other stress component is this one. Write the equilibrium equations by considering equilibrium of forces acting on a differential element in the cross section, we get this equation.

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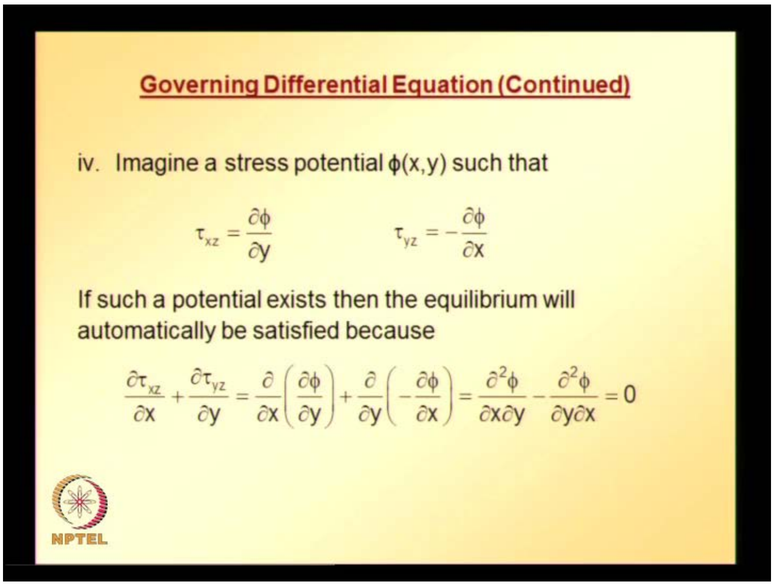

Now, imagine a stress potential such that, it satisfies these equations. And if you can or if you are able to find such a stress potential, which satisfies these two conditions. Then, equilibrium will automatically satisfied, which is shown here. Because when we substitute the stress components, we notice that equilibrium equation is automatically satisfied.

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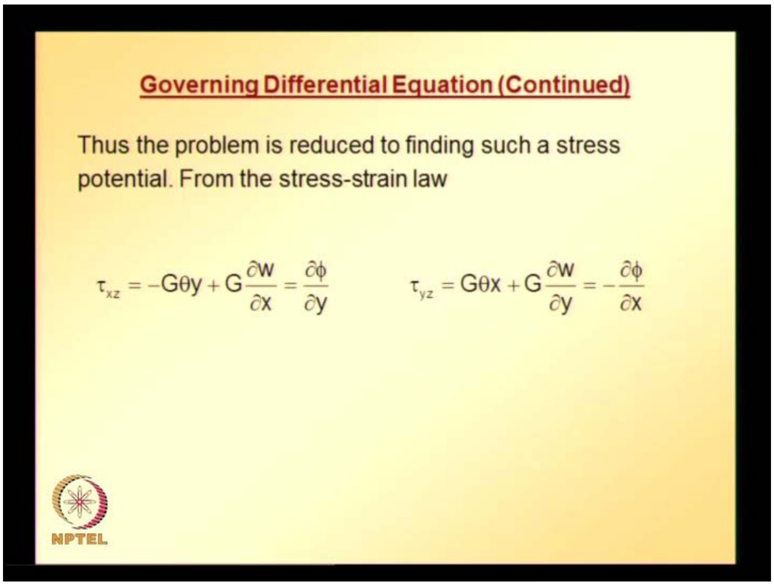

So, basically the problem is reduced to finding a stress function. From stress-strain law, we know that stresses are given by these two equations, which in turn can be related to the stress potential. To eliminate w, differentiate the first equation with respect to y, the second equation with respect to x and subtract two equations from each other.

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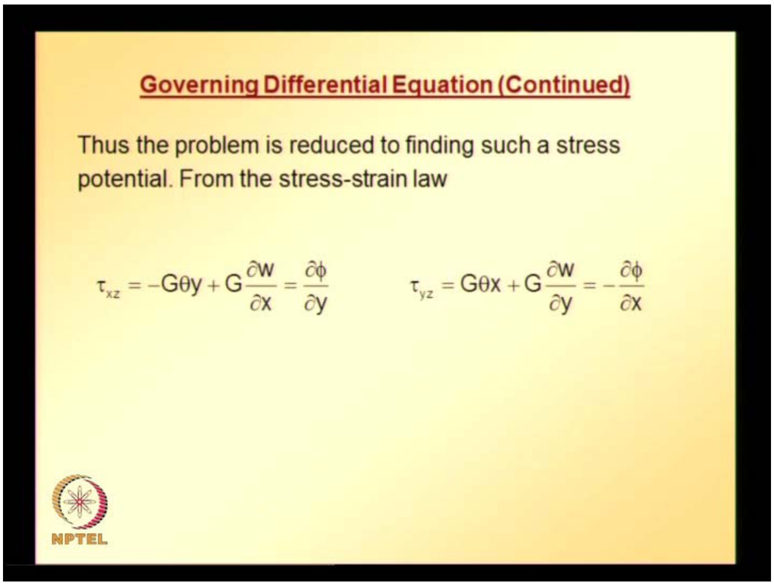

(No audio from 08:05 to 08:15) Subtracting the second equation from the first, we get the governing differential equation in terms of stress functions.

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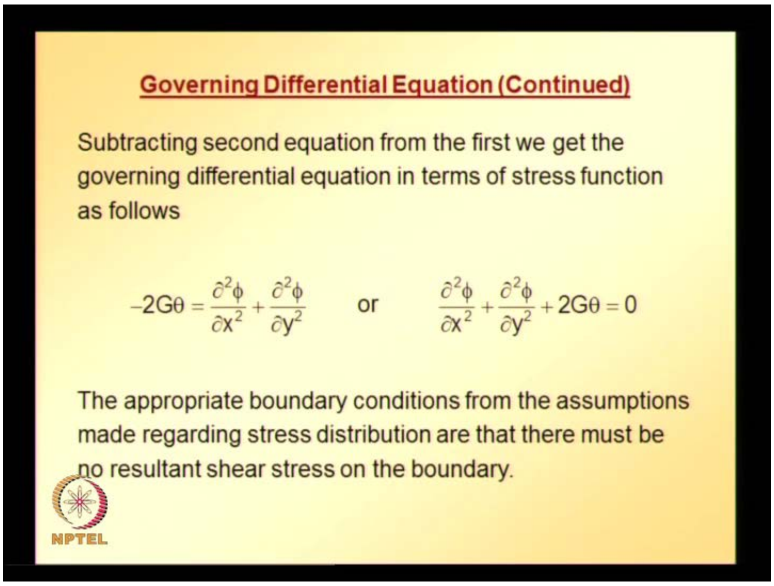

(No audio from 08:22 to 08:31)The appropriate boundary conditions from the assumptions made regarding stress distribution are that there must be no resultant shear stress on the boundary. So, this is the governing differential equation. So, now we need to look at the boundary conditions. So, boundary conditions will be derived based on the condition, no resultant shear stress there is no resultant shear stress on the boundary.

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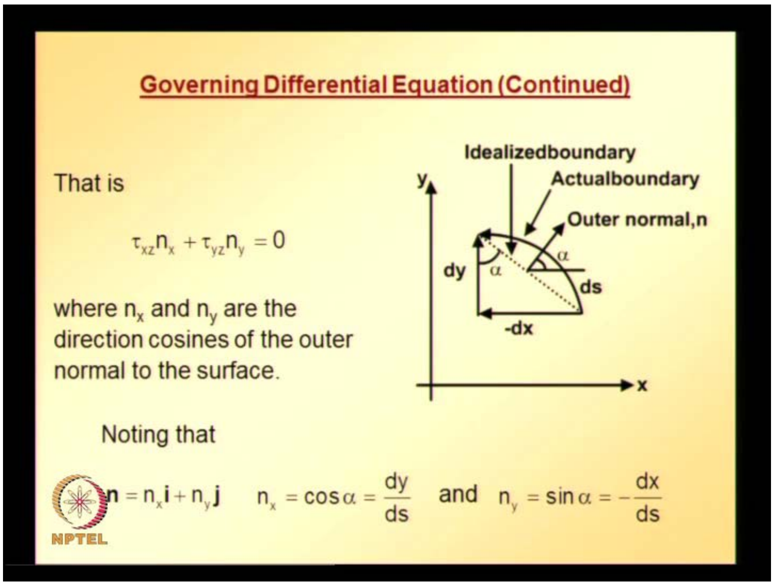

That is, tau x z times n x plus tau y z times n y is equal to 0, where n x, n y are nothing but the components of outer normal, which is shown in the figure along x and y directions. Or in other words, n x n y are the direction cosines of outer normal to the surface. Since we know the relation between the shear stress components in the stress potential, we can express this equation in terms of stress potential.

For that, noting that outer normal has components n x, n y and n x is given by cos alpha; cos alpha can be calculated based on the figure that is shown there. That is d y over d s and n y is equal to sin alpha, which is given by minus d x over d s, where d s is a small element of idealized boundary; So, substituting the stress components in terms of stress shear stress components in terms of stress potential and also substituting n x, n y in terms of d y, d s and d x, d s.

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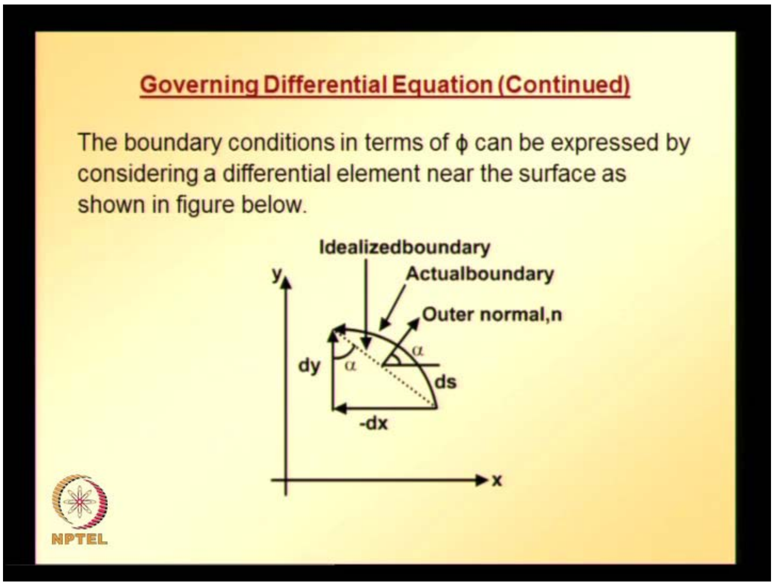

The boundary condition in terms of stress potential, phi can be expressed by considering differential element along the surface as shown in the figure.

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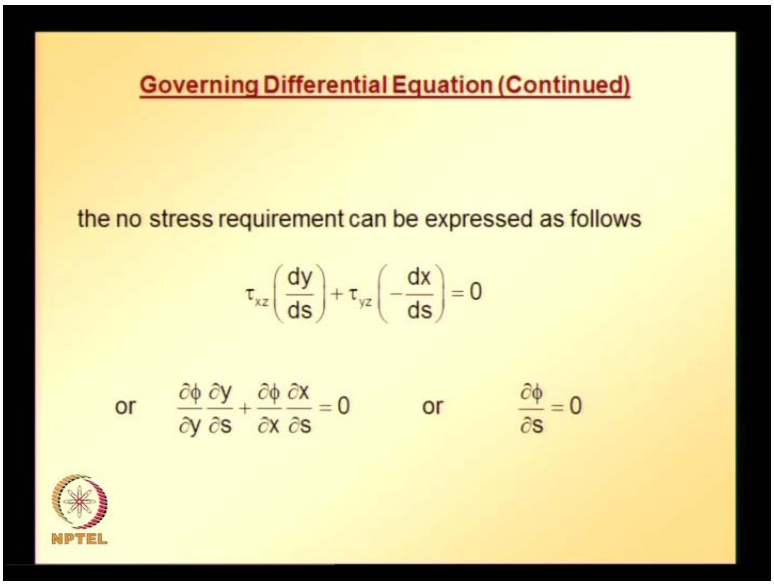

No stress requirement can be expressed as here basically n x, n y are substituted and substituting stress components in terms of stress potential. Finally, we get this equation; that is partial derivative of stress potential with respect to s is equal to 0. (No audio from 11:54 to 12:07)

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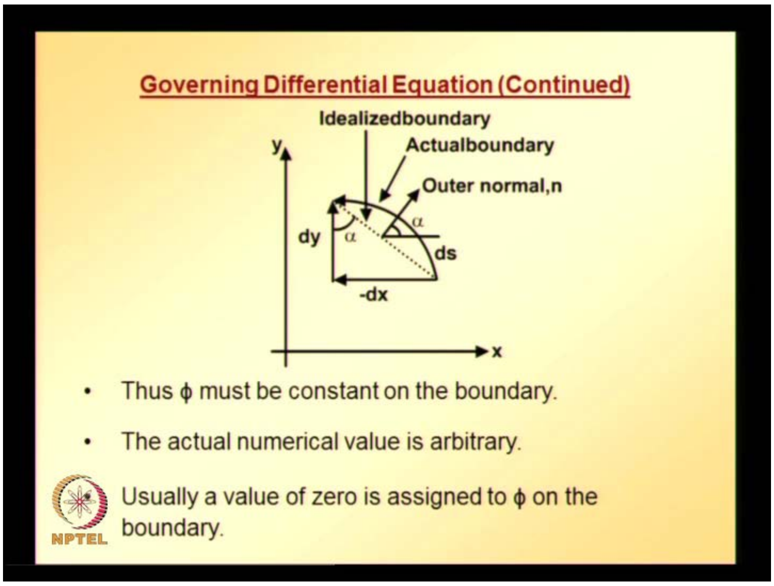

Basically, the meaning of partial derivative of stress potential with respect to s is equal to 0 means stress potential phi must be constant on the boundary. The actual numerical value can be arbitrary; usually a value of 0 is assigned to phi on the boundary.

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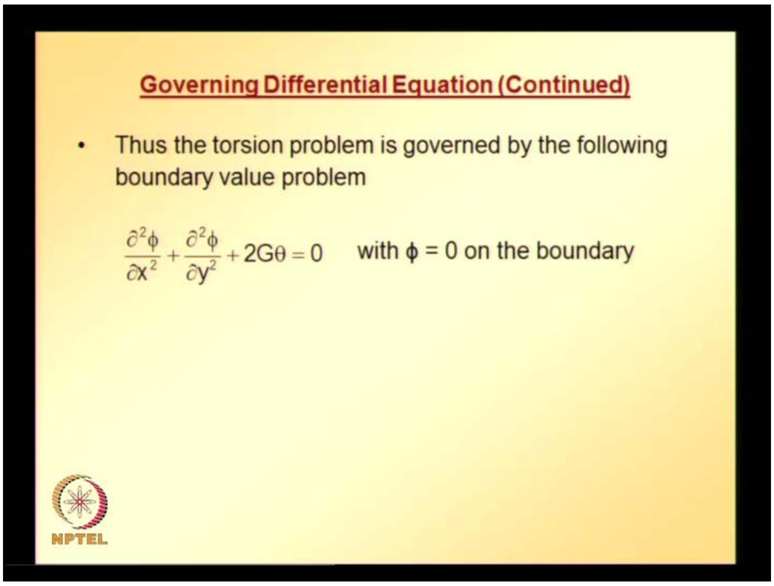

So, basically the torsion problem is governed by the following boundary value problem. The governing differential equation with that is second derivative of phi with respect x plus second derivative of phi with respect to y plus two times shear modulus times angle of twist is equal to 0 with shear with stress potential sorry with stress potential phi is equal to 0 on the boundary.

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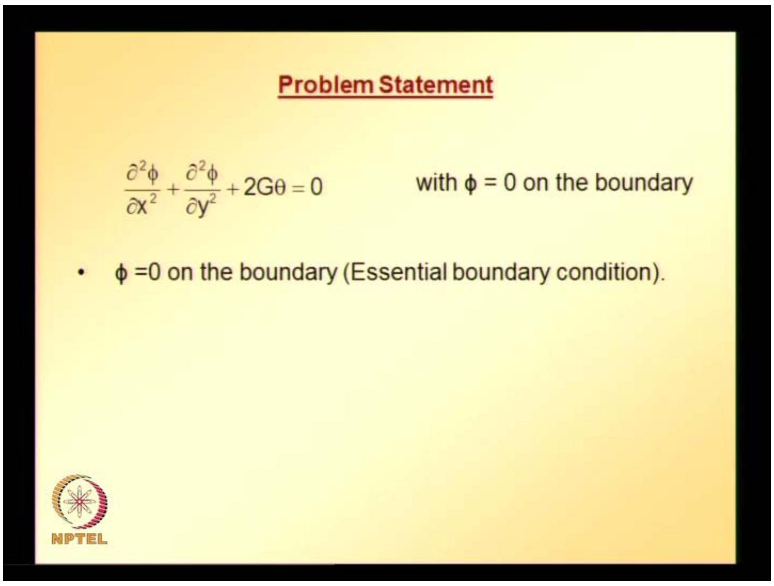

So, this is the problem statement. So, basically to repeat that, this is the problem statement.

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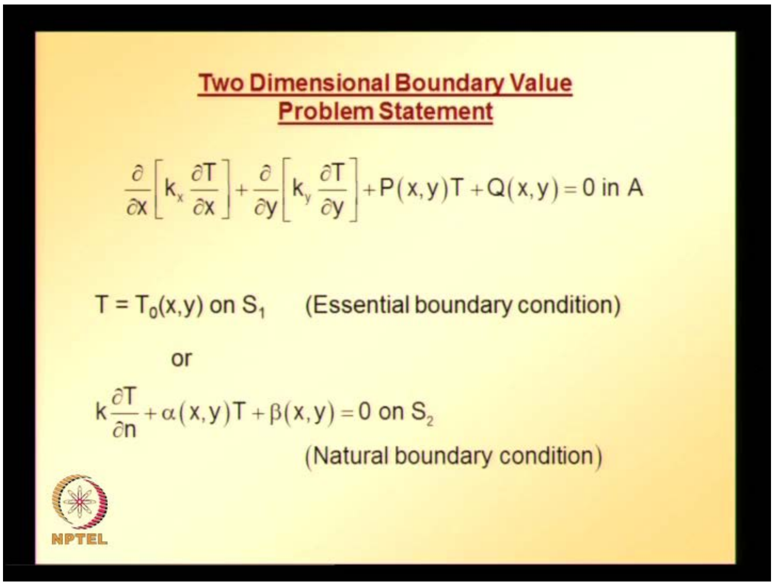

So, we can derive the finite element equations for the torsion problem by comparing with a general two dimensional boundary value problem. The equations corresponding to general two dimensional boundary value problem. So, basically let us look let us go back and see what we have done for two dimensional boundary value problem. The two dimensional boundary value problem statement is like this. Partial derivative of k x times dou t over dou x with respect to x plus partial derivative of k y times dou t over dou y with respect to y plus p times t plus q is equal to 0 over a domain A subjected to do these boundary conditions; essential boundary condition and natural boundary condition can be specified.

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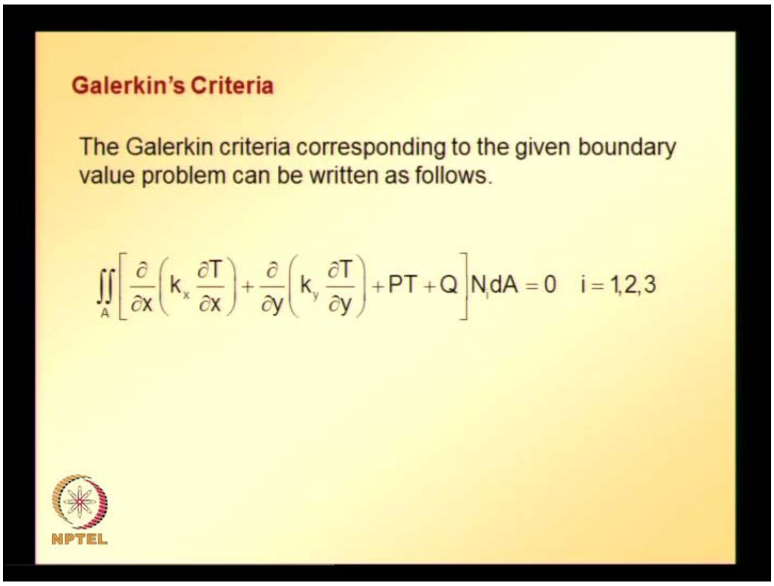

So, we used if you recall we use Galerkin criteria and we derived the finite element equations. The Galerkin criteria corresponding to the general two dimensional boundary value problem can be written like this and we can use Green's theorem on the first two terms.

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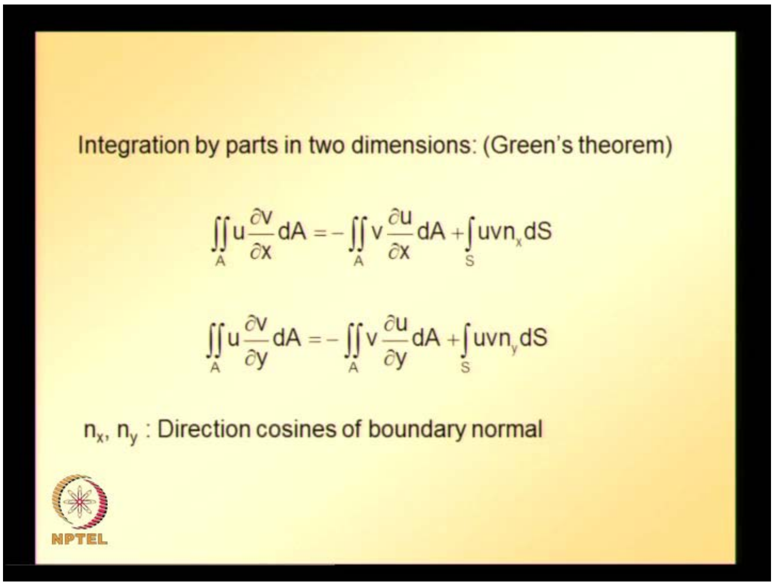

Green's theorem is summarized here; integration by parts in two dimensions. These are the two formulas that we can use to simplify the previous equation. So, using these two formulas, the previous equation; here n x, n y have same meaning as what we discussed; direction cosines of boundary normal.

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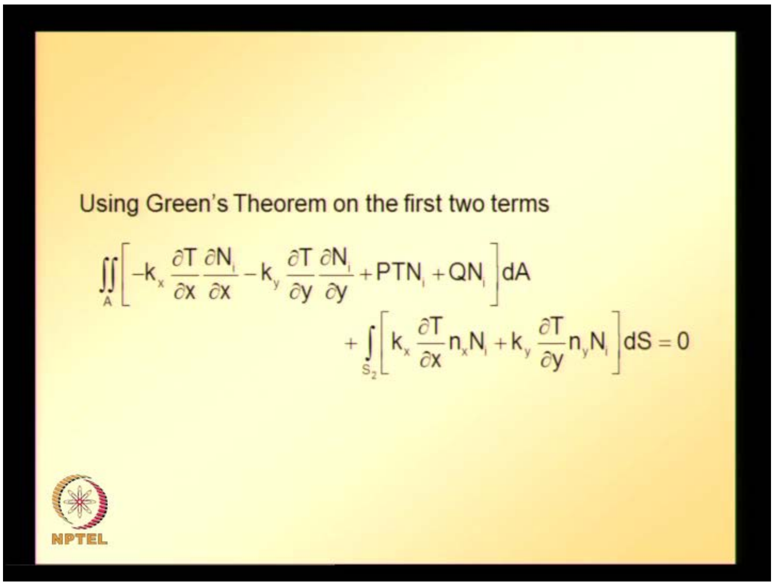

Using these two equations, the first two terms of the earlier equation can be written in this manner. If you see the second integral, integral over s 2 based on the explanation that is provided earlier, when we are looking at the derivation of the finite element equations for general two dimensional boundary value problem.

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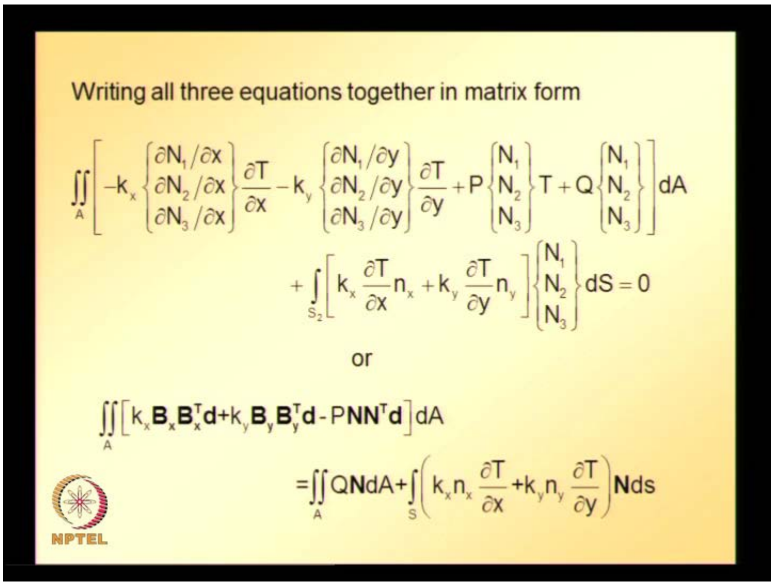

This equation can be further written in matrix form like this and the integral the term with integral over s 2 can be replaced with the term with integral over s, as shown in the second equation; for the reasons that, I explained you earlier. So, basically using Galerkin criteria, we get this in terms of derivatives of shape functions. B x, B y are the matrices consisting of derivative shape functions and N is matrix consisting of finite element shape functions and this equation the last equation can further be simplified by substituting the natural boundary condition that is provided.

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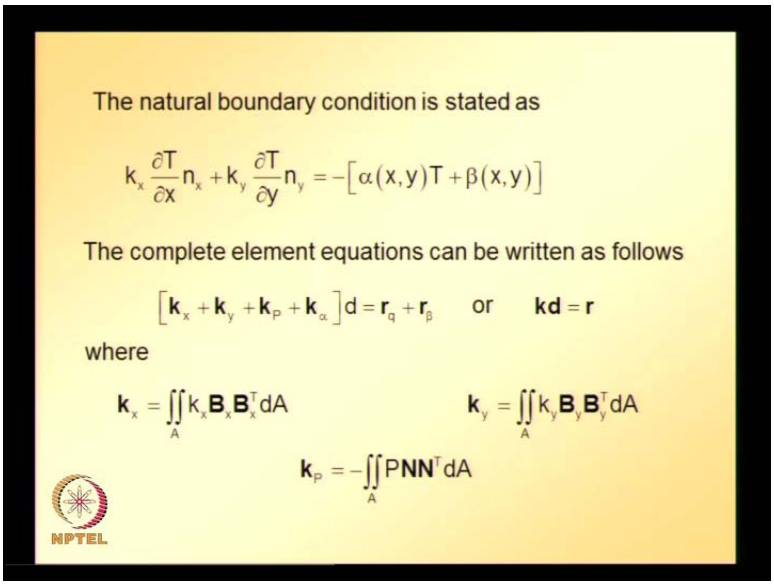

So, substituting this natural boundary condition into the previous equation, the previous equation can be further written or further simplified and the complete element equations can be written as follows, where each of these components k x, k y, k p, k alpha, r beta, r q, are as defined here.

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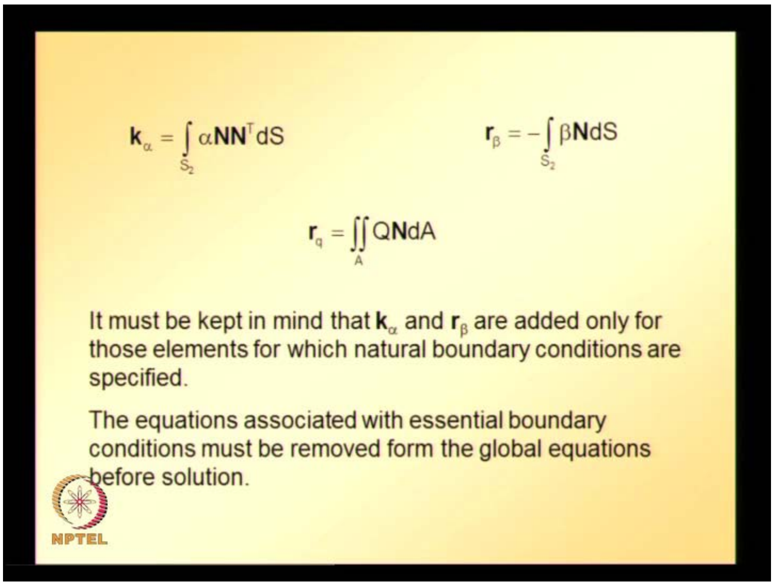

And it must be kept in mind that, k alpha, r beta are added only for those elements for which, natural boundary conditions are specified. We discussed all this. When we looked in detailed the derivation of finite element equations for general two dimensional boundary value problems. So, given a given the governing differential equation corresponding boundary conditions for a particular application, the first job is to identify what are these k x, k y, and p q coefficients. And then, we can easily assemble the finite element equations corresponding to that particular element, if you know the shape functions in derivative of shape functions. The equations associated with essential boundary conditions must be removed from the global equations before solution.

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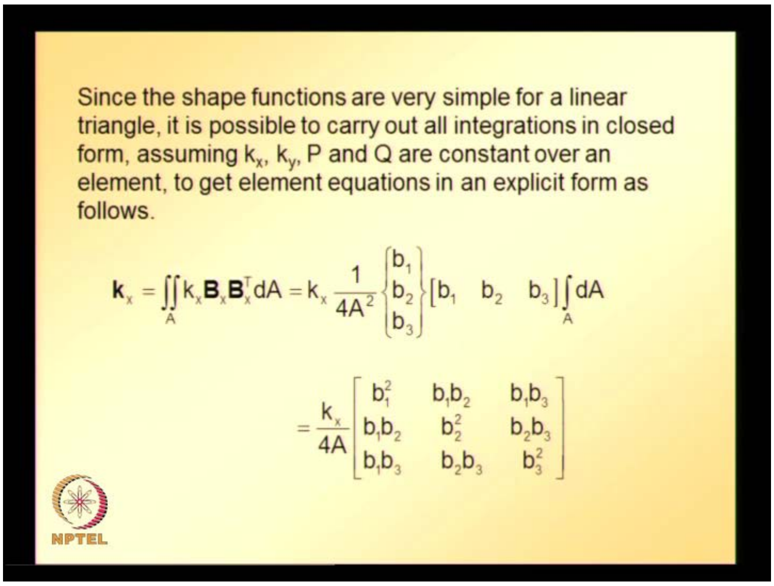

Since the shape functions are very simple for linear triangle, it is possible to carry out all integrations in closed form. Assuming k x, k y, P and Q are constant over element to get element equations and explicit form as follows. So, k x for linear triangle element is given by this, which can be further simplified.

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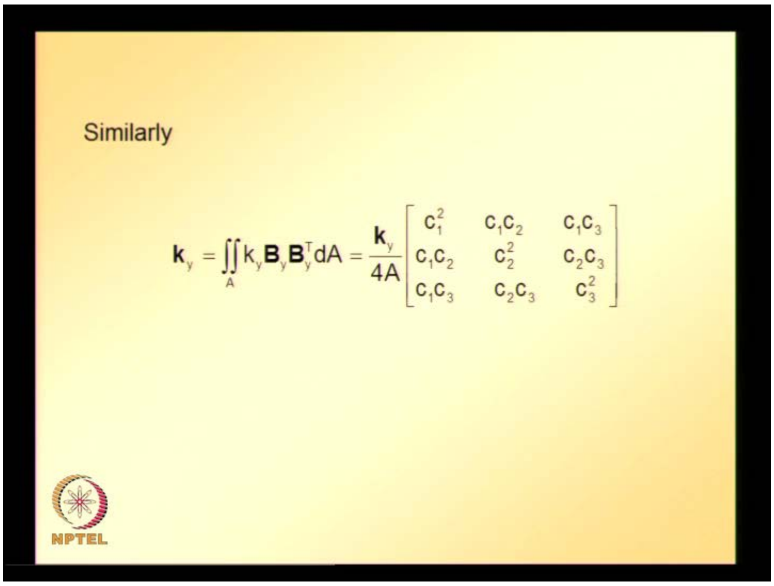

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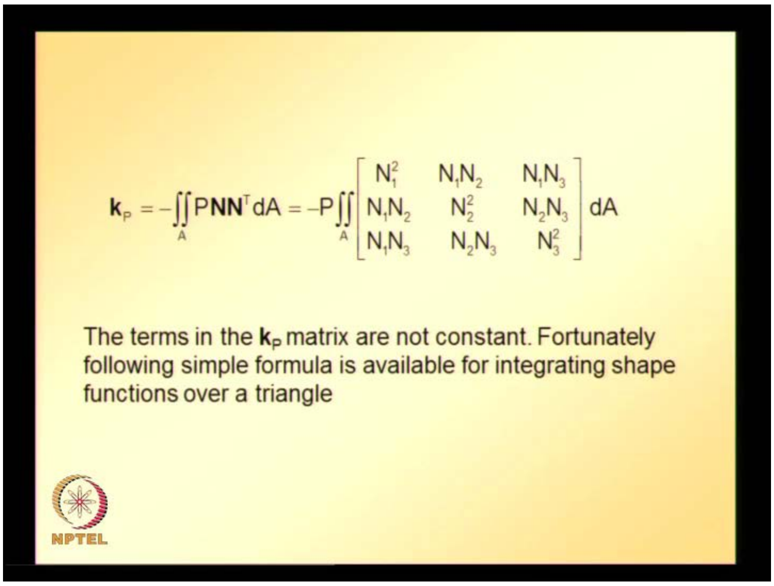

Similarly, k y and k p and to simplify the terms in k p, since shape functions are not constant; we required to use some formula. Fortunately following simple formula is available for integrating shape functions over an element.

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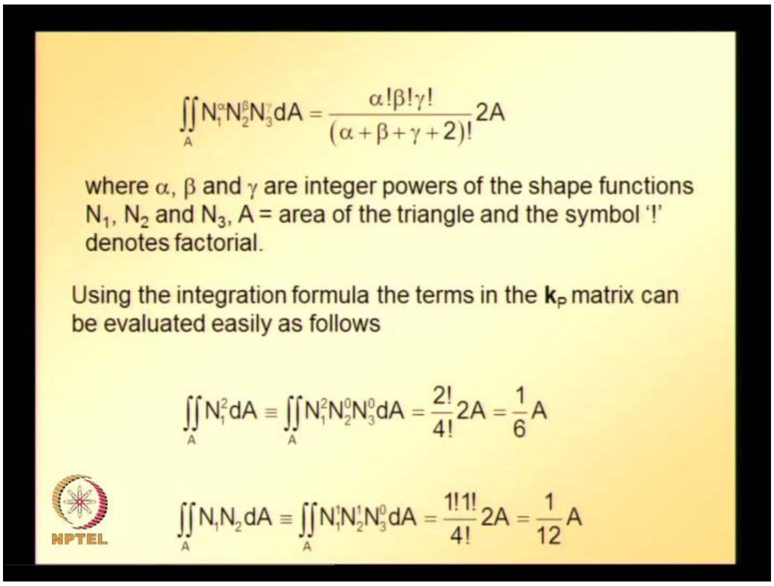

This is the formula, which you can use to simplify the terms in k p and where alpha, beta, gamma are the integer powers of shape functions N 1, N 2, and N 3. A is the area of triangle and the symbol exclamation denotes factorial and using integration formula the terms in k p can be evaluated as follows. One of the term is shown here.

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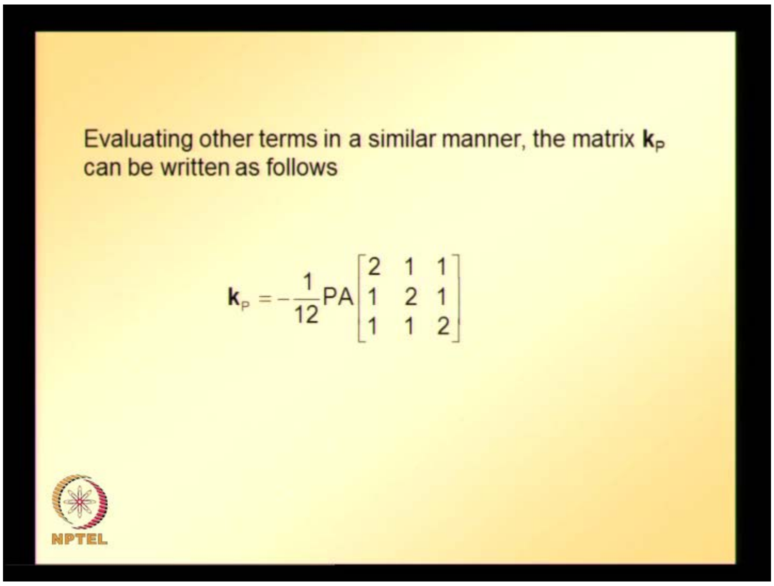

One more term and similarly, other terms can be evaluated. Evaluating other terms in a similar manner, the matrix k p can be written as follows.

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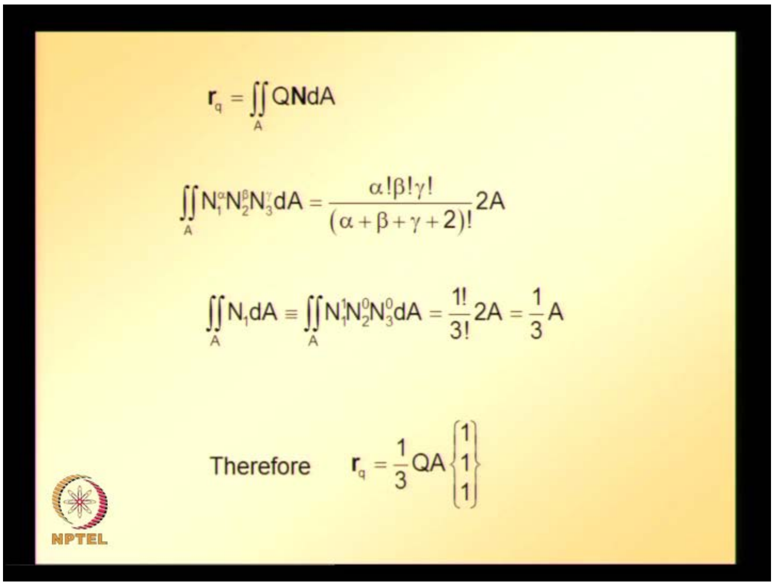

r q substituting the finite element shape functions, we can simplify this. And we can here also; we can use this formula, if required. The values of some of the terms are shown here. Therefore, r q is given by this.

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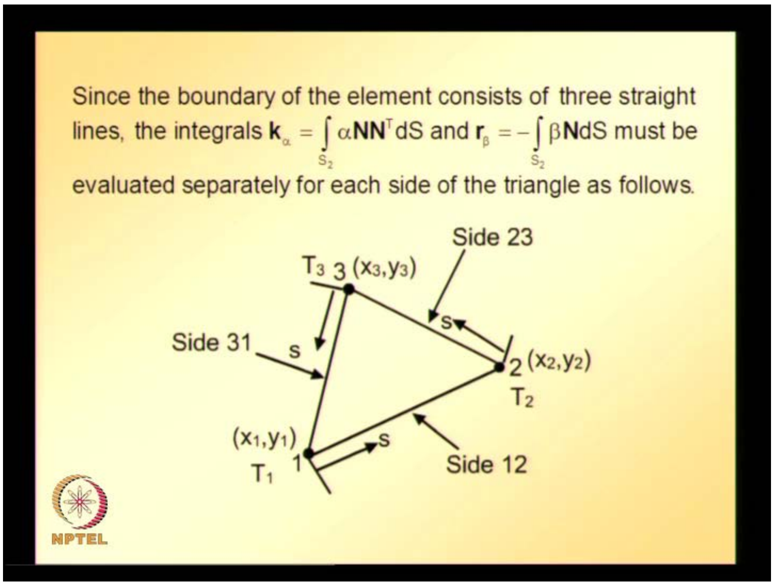

Since the boundary of the element consist of three straight lines, that is since we are dealing with linear triangle element, the integrals k alpha r beta must be evaluated separately for each side of triangle and the details are shown for one of the sides.

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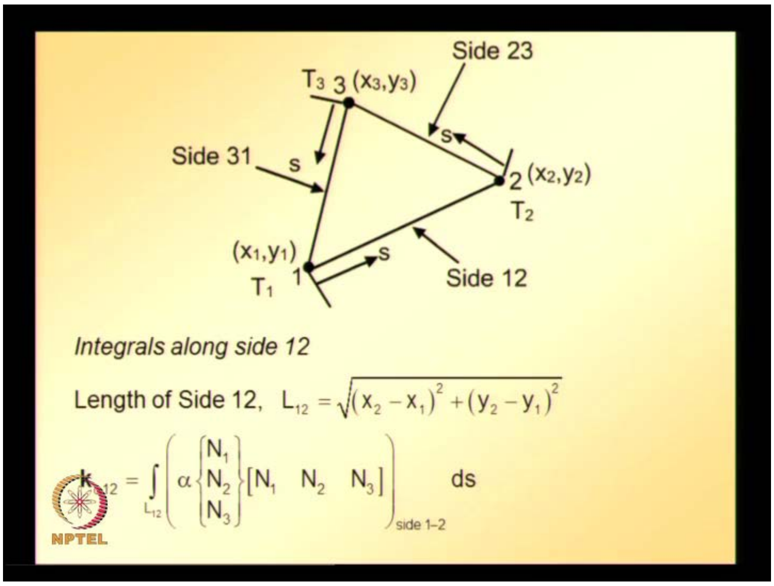

Integration along side 1 2: First, we required to find length of side and then, k alpha can be written like this.

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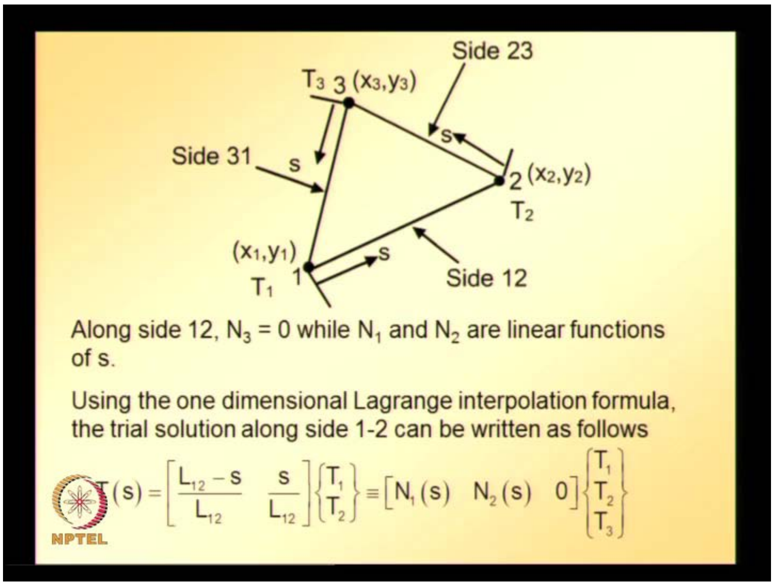

And we required to know, which shape functions are going to be non-zero alongside 12. N3 is going to be 0; N 1 and N 2 are linear functions of s local coordinate system. So, using one dimensional Lagrange interpolation formula using one dimensional Lagrange interpolation formula, the trial solution alongside 1 2 can be written as follows.

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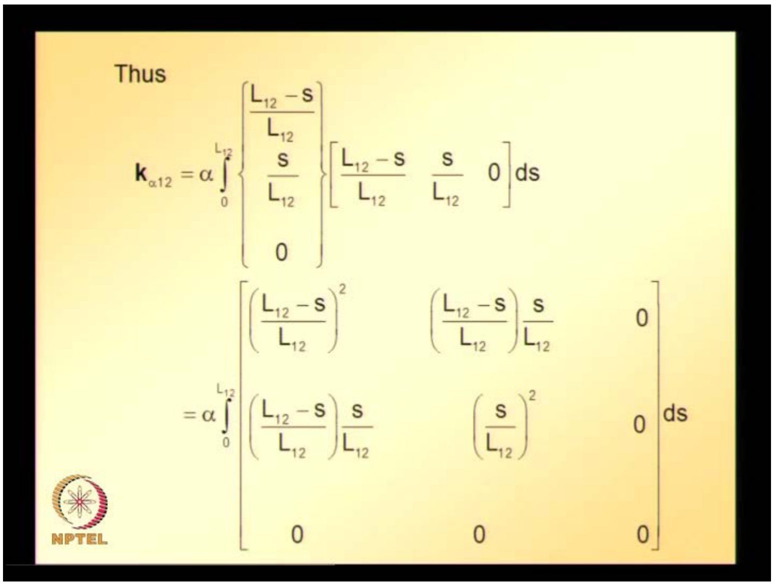

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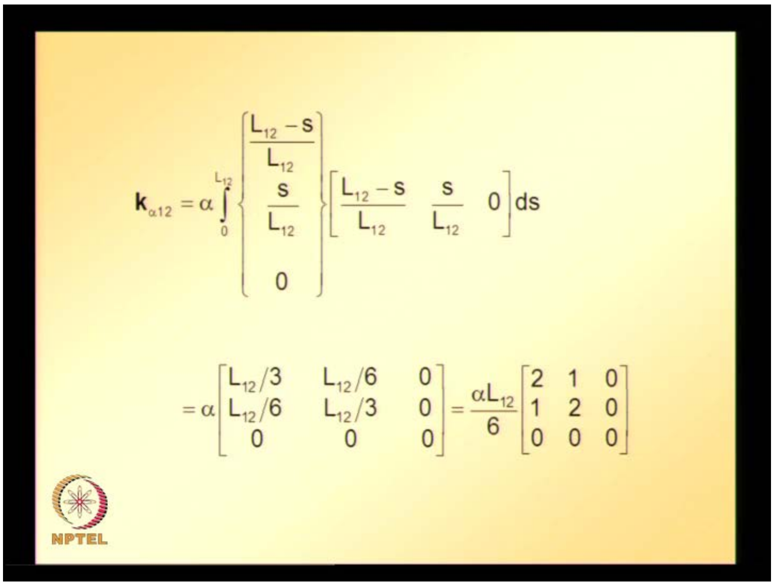

Substituting this into k alpha, we get or an approximate k alpha alongside 1 2 and simplified form of k alpha is given, here alpha is assumed to be constant. If alpha is not constant, then we need to include that inside the integral and then, you adopt numerical integration procedures.

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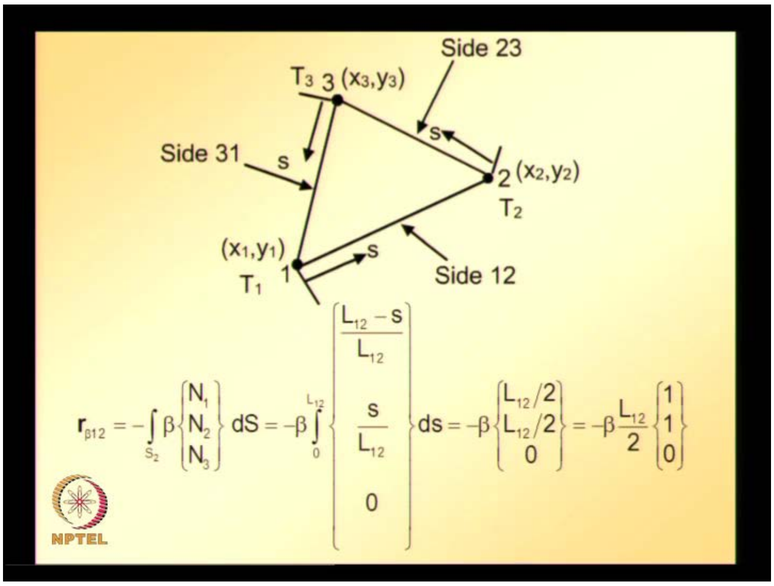

Similarly, r beta alongside 1 2, the details are shown here and what about other sides?

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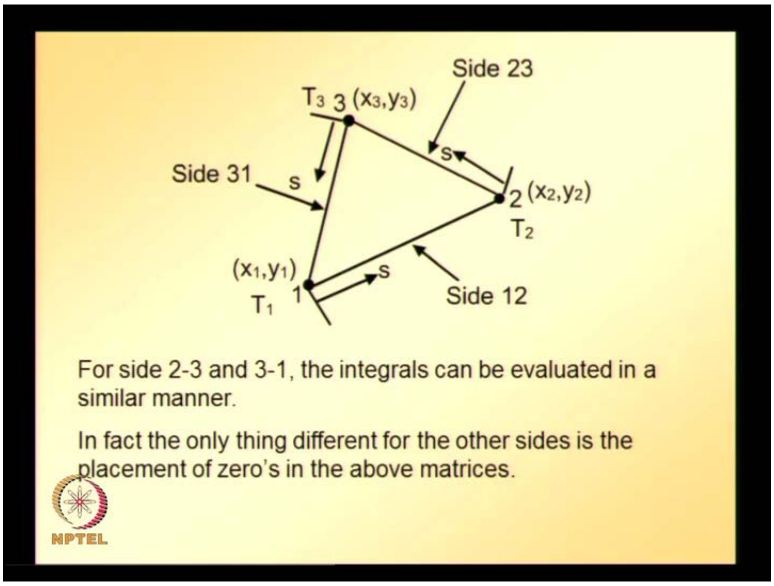

For other sides like side 2 3 and 3 1, integrals can be evaluated in a similar manner. And finally, we can get equations for all sides; infact, only one and while doing this, only thing that differs for each of the sides is the placement placement of zeros in the shape function matrix n.

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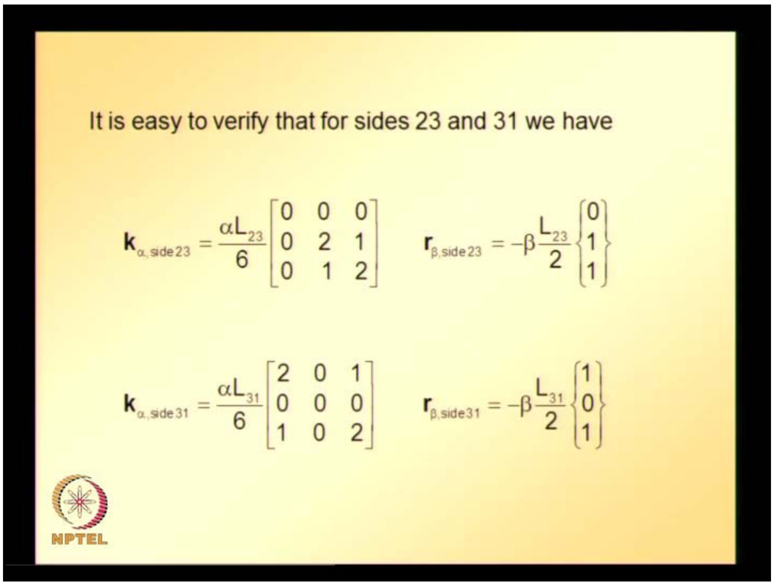

It is easy to verify that for sides 2 3 and 3 1, the corresponding k alpha r beta are like this. This is what, we have done when we are deriving finite element equations for general two dimensional boundary value problem. So, now we can use these finite element equations and identify, what are the corresponding k x, k y, p q in the torsion problem; torsion of prismatic bars and we can easily get the finite element equations.

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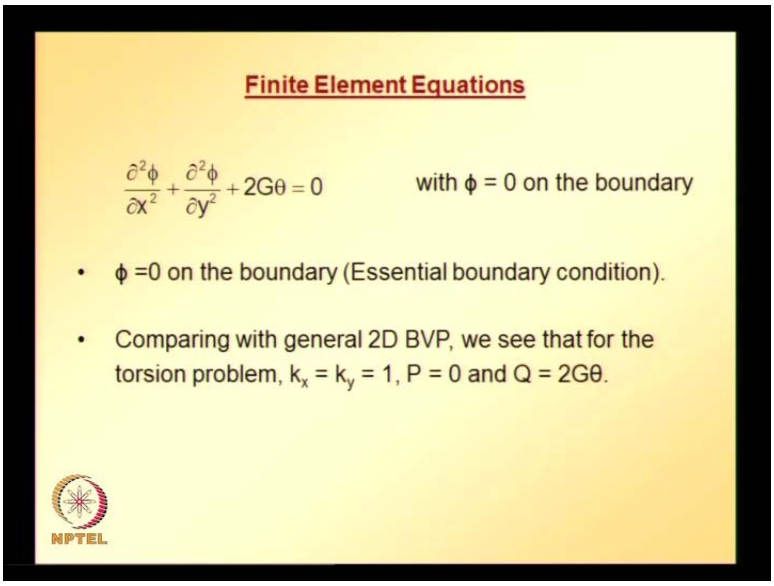

This is the governing differential equation subjected to the boundary condition that stress function is equal to 0. The essential boundary and comparing this with whatever, we just briefly review comparing this equation with the general two dimensional boundary value problem. We see that k x, k y is equal to 1; P is equal to 0; q is equal to 2 G theta. So, we can easily substitute this information into the equations, that we have just seen and we can get the equations corresponding to torsion of prismatic bars.

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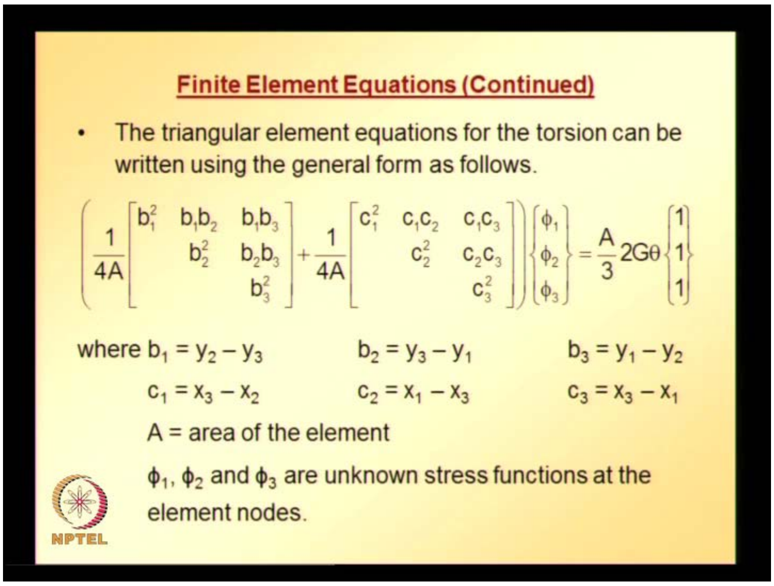

Triangular element equations for torsion can be written using the general formula as follows. We should also keep in a note about this coefficients b 1, b 2, b 3 what they mean? And if the if all the three coordinates if all the sorry if all the coordinates of three nodes of linear triangular element are known, we can easily calculate what is b 1, b 2, b 3 and also c 1, c 2, and c 3.

And A is area of triangle; area of the element; triangular element and phi 1, phi 2, and phi 3 are the unknown stress functions at the element nodes. So, in this problem of solving torsion of prismatic bars, we are basically solving for this stress functions. Once we know the nodal values of stress functions phi, we can calculate, what are the stress components, shear stress components?

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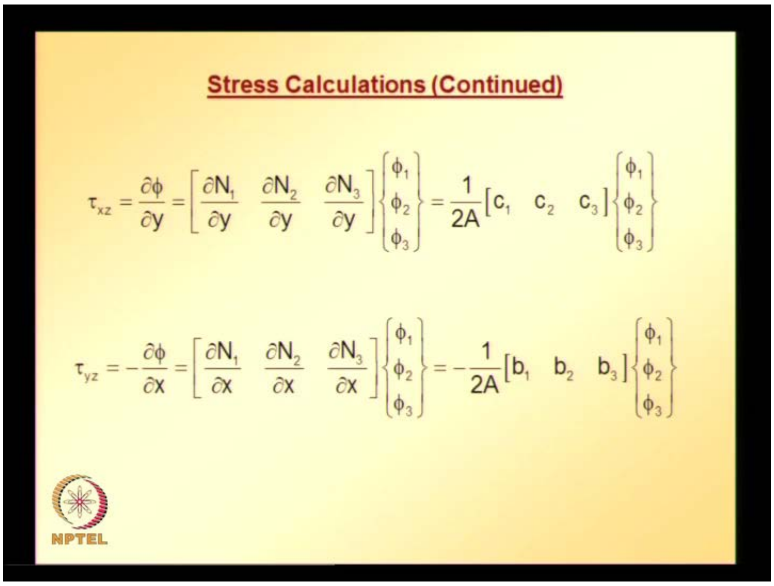

Once the nodal values are known, stresses can be calculated using element shape functions. So, the stress function stress potential sorry stress potential value at any point inside the element can be interpolated using this equation and taking derivative of that we get the stress components.

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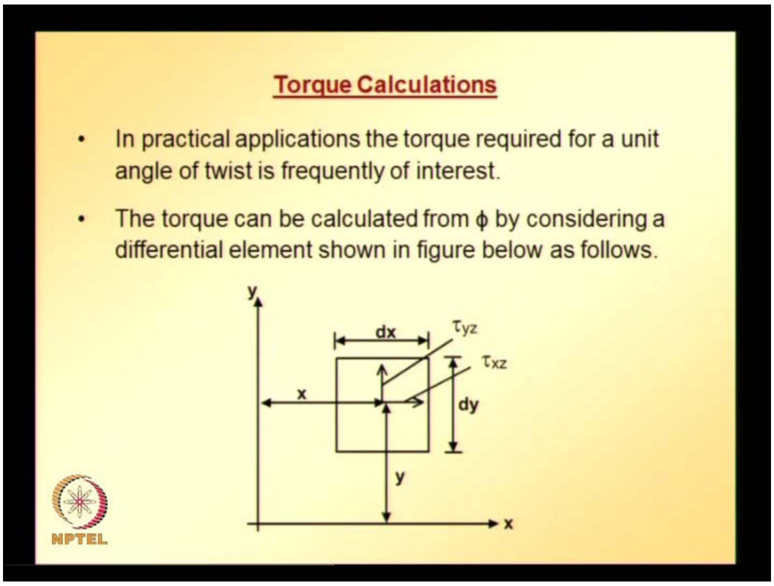

What about torque calculations? In practical applications, the torque required for unit angle of twist is frequently of interest. Torque can be calculated from the phi value by considering differential element as shown in the figure below. (No audio from 28:09 to28:23) And to develop the relation, we can take moments about origin.

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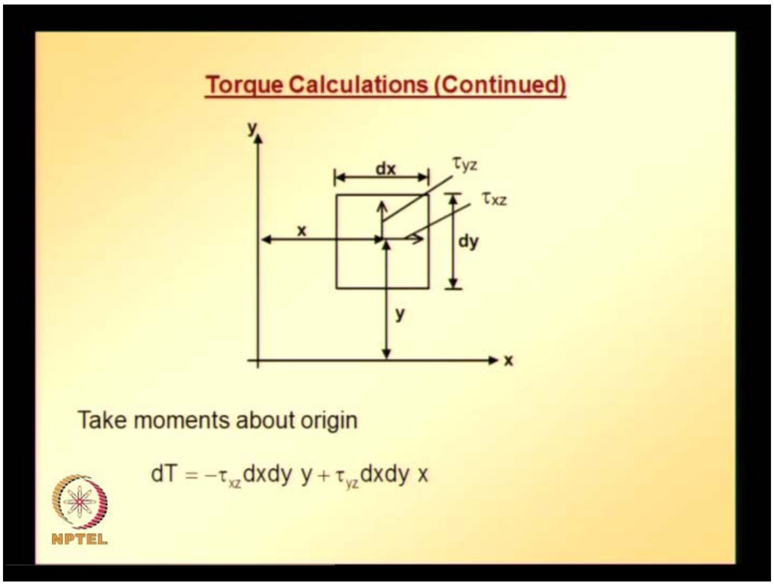

Take moments about origin, we get this equation and the total torque is given by integrating this over the entire area of cross section.

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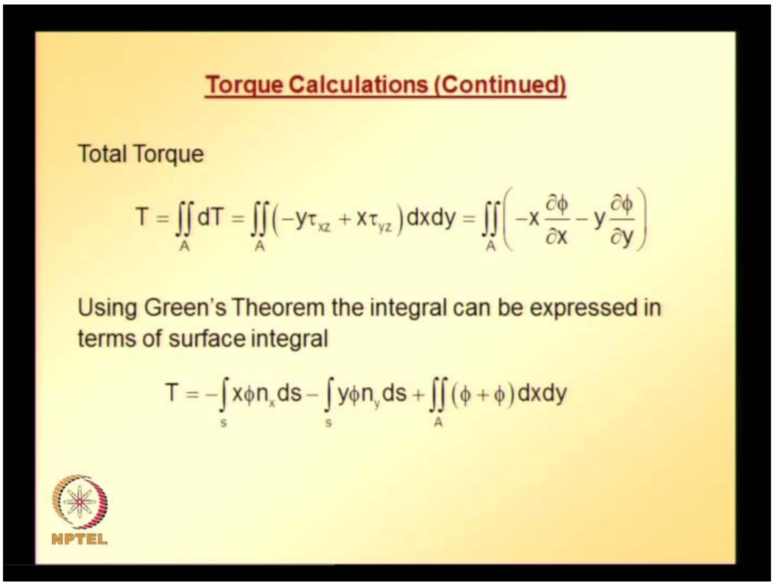

Total torque is given by this and this area integral can be expressed in terms of surface integral using Green's theorem. Using Green's theorem, the integral can be expressed in terms of surface integral. We have already looked at Green's theorem formulas. So, using those, we can write this tool sorry this equation and we also know that, the essential boundary conditions states that, stress potential phi is equal to 0 on the boundary. So, the first two terms, which are basically integrations over the surface, are going to disappear.

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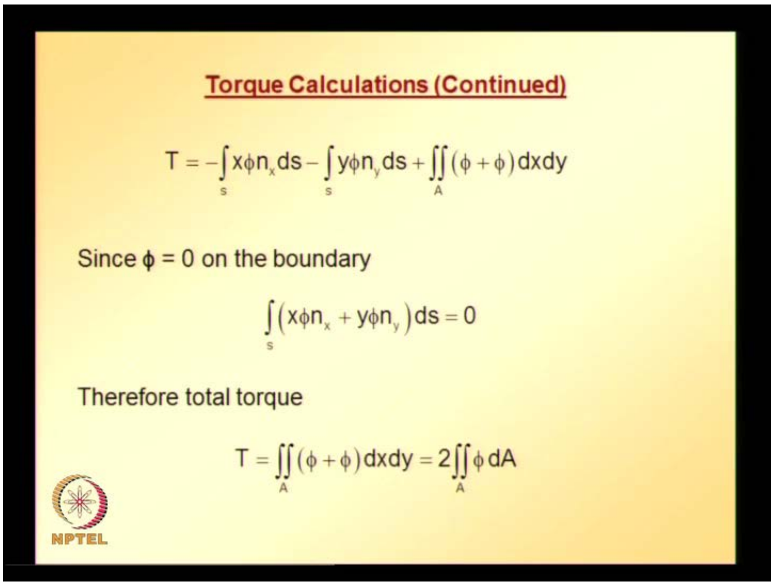

So, therefore total torque is given by this and how to calculate this using finite element the nodal values, that we obtained using finite element.

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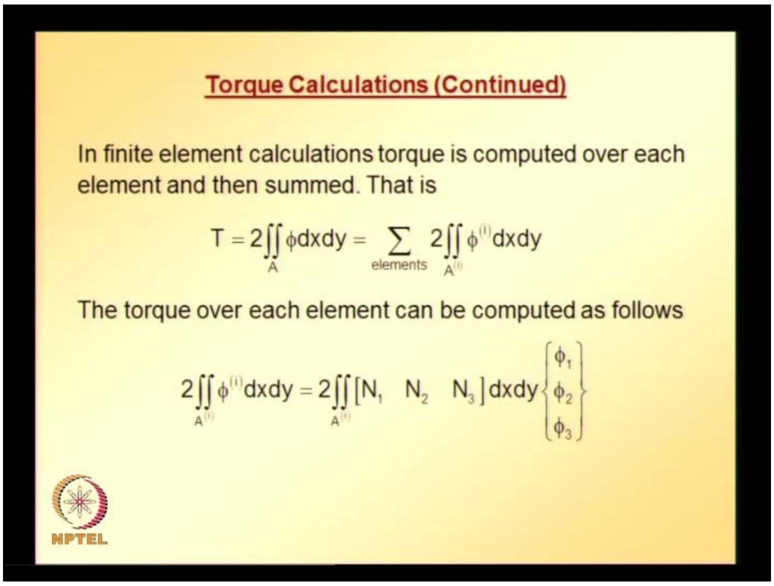

In finite element calculations, torque is computed over each element and then summed. That is, we will calculate torque for each of the elements and then, sum up over all the elements. Then, we get the total torque and for that, we required to find, what is the torque over each element? Torque over each element is given by this. Shape functions associated with that particular element and simplifying this equation for linear triangular element.

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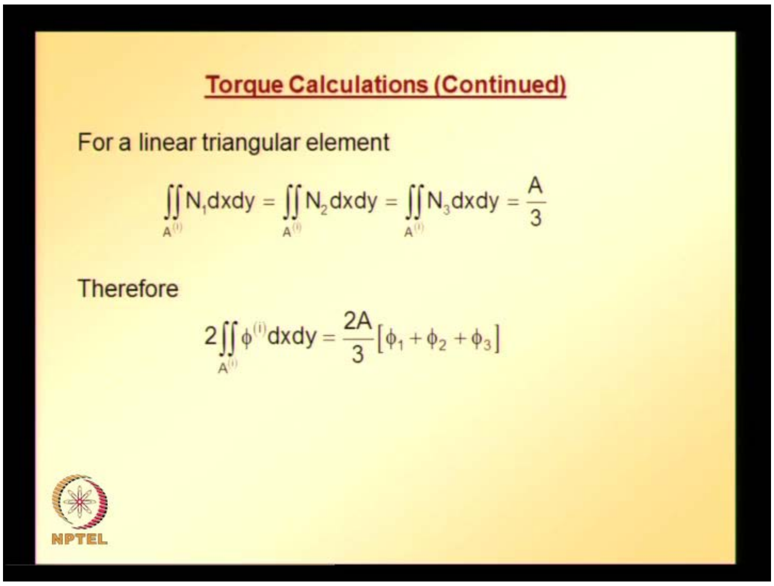

We know that, N 1 integrated over entire area of triangle; N 2 integrated over entire area of triangle; N 3 integrated over entire area of triangle is equal to A over 3. So, the previous equation for linear triangle element is simplifies to this one. So, total torque is basically, we need to add over all elements. For each element, it is two thirds area times or two times average stress function; stress potential value. Now, let us apply these equations to solve a problem.

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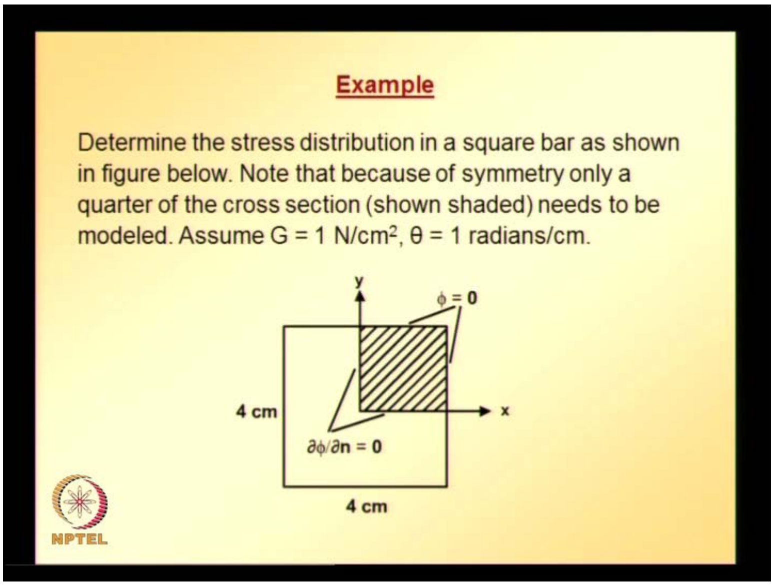

So, the problem statement is like this. Determine the stress distribution in a square bar shown in figure below. So, cross section of the bar is square; the dimension of the bar is so the cross section of the bar is square and the dimensions of cross sections are cross section is indicated 4 centimeter by 4 centimeter and we can notice that, there is symmetry. So, because of symmetry only, quarter of this cross section needs to be modeled.

The corresponding the values of shear modulus and also angle of twist are given here in the problem statement. So, we will model only the quarter portion that is shaded in the figure. The corresponding boundary conditions are also indicated. phi is equal to 0; stress potential is equal to 0 on the boundary and also derivative or stress potential is 0 on the sides; over which, symmetry boundary conditions are applied.

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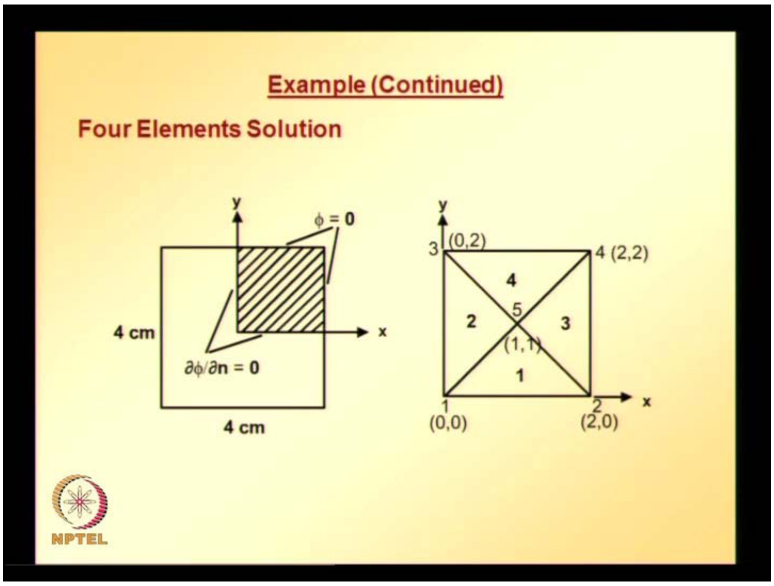

Now, let us take the quarter model and discretized that using four linear triangle elements four linear triangle elements. The discretization is shown with element numbers 1 2 3 4 and also nodes are numbered and corresponding coordinates; an x y coordinate system for each of the nodes is indicated. So, element 1 comprises of nodes 1 to 5; element 2 comprises of 1 5 3; element 3 comprises of 2 5 4; element 4 comprises of 3 5 4. Since each of the triangle elements are of same size, the element equations are going to be identical for each of these yes element stiffness equations are going to be identical for each of these elements.

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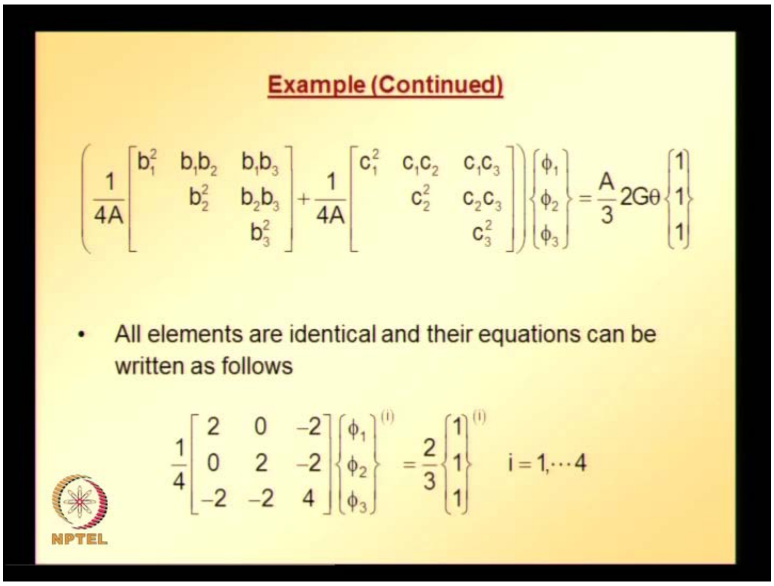

And substituting the values of the coordinates into this equation, we can actually get element equations for each of the elements. All elements are identical; their equations can be written as follows and here phi 1, phi 2, and phi 3 are in the local coordinate system. Locally, the node is 1 locally these nodes 1 2 3 can be different globally depending on the local to global node numbering mapping. So, all elements will have identical equations.

So, now let us look at, where the contribution from element 1 goes in, contribution from element 2 goes in and element 3 and element 4 goes into the global equation system. For that, we require to take a node of global node numbers and the corresponding local node numbers. And also please note that, in this problem that is torsion of prismatic or torsion of square bar at each node, we have only one unknown that is stress potential value. The contribution from element 1 goes into the locations depending on the numbering sequence.

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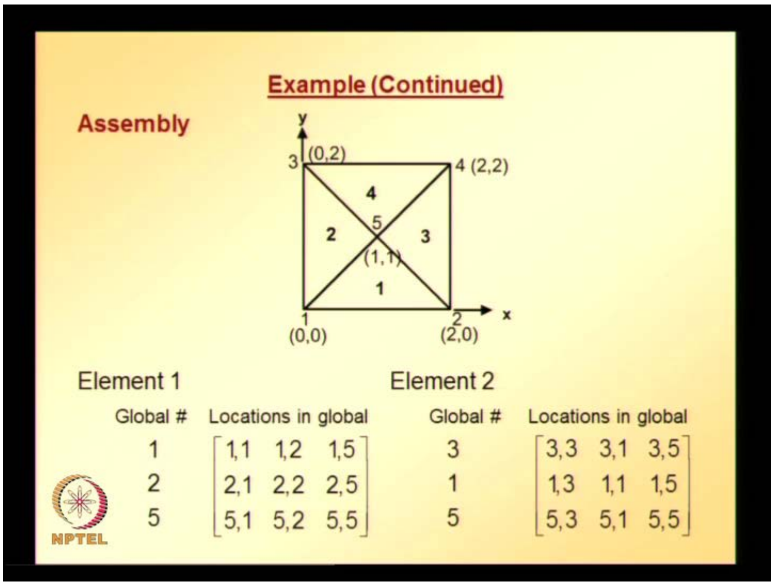

1 2 5 rows and columns of the global equation system and the locations in the global equation system is shown in the table or in a matrix form. Similarly, for element 2 the contribution goes in 3 5 1 sorry 3 1 5 rows and columns of the global equation system.

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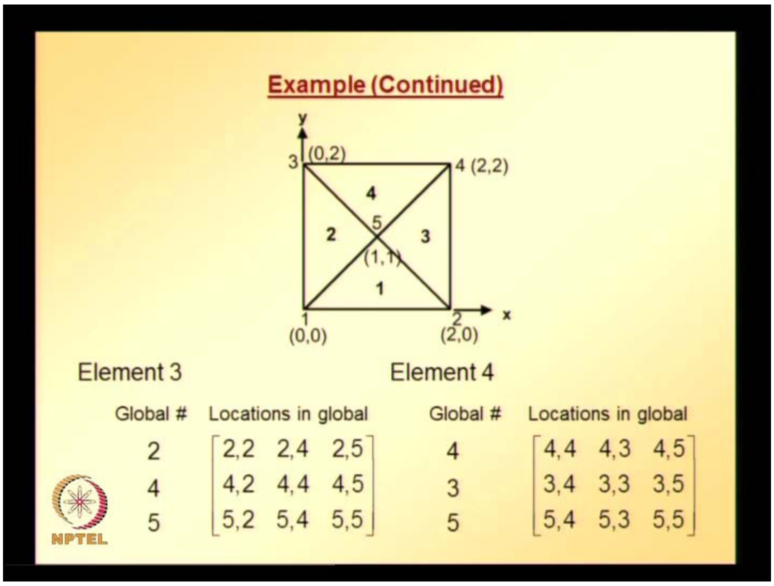

Element 3 contribution goes into 2 4 5 rows and columns of the global equation system. Element 4 contribution goes into 4 3 5 rows and columns of the global equation system.

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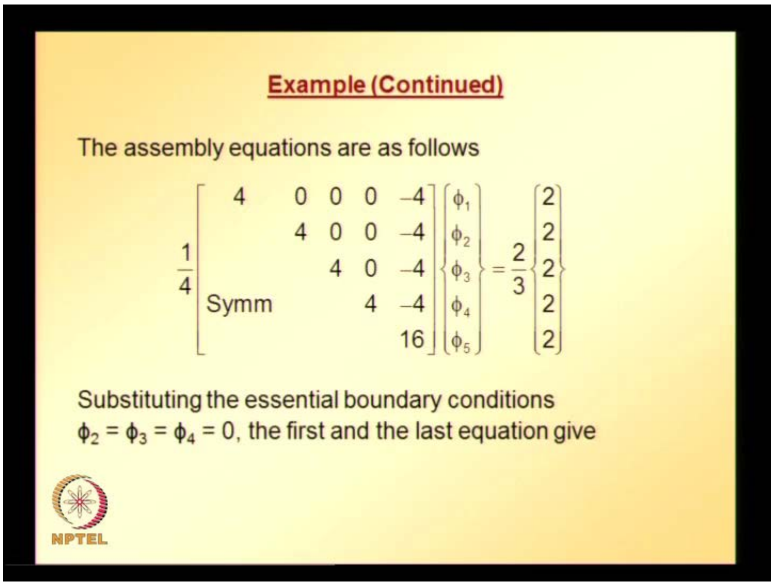

So, with that understanding, assembly of equations gives us this as a global equation system final global equation system. We know that, stress potential is equal to 0 on the boundary that is along nodes 2 4 and 3. So, substituting the essential boundary condition that is stress potential value at nodes 2 3 4 is equal to 0 will be left with only the first and last equations.

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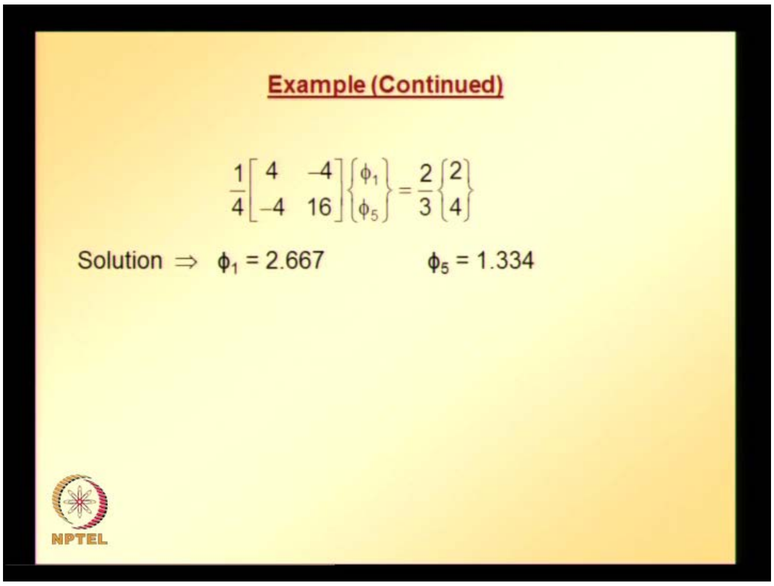

And solving those two equations, we get phi 1 and phi 5, as these values. Once we have the stress potential, we can easily calculate stresses. So, solution of phi 1 and phi 5 are these.

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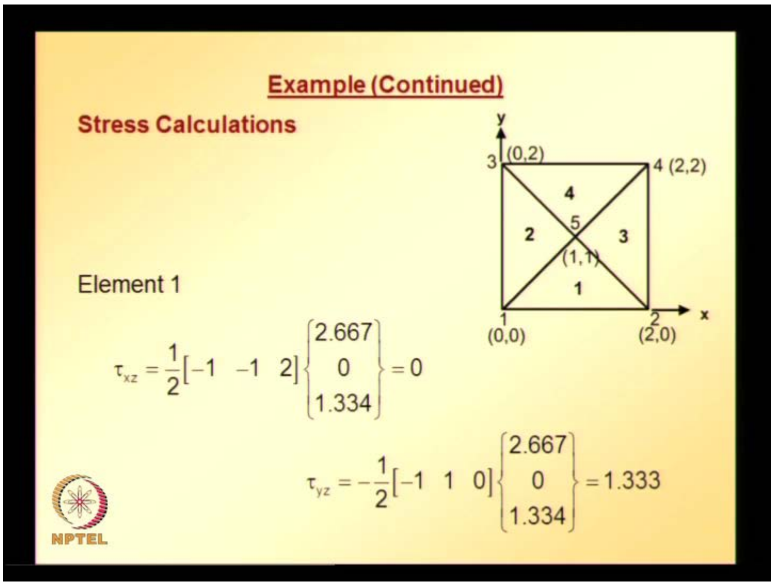

Stress calculations. Element 1 substituting the derivatives of shape functions and the nodal values corresponding to element 1, we get shear stress component tau x z as 0 and tau y z as 1.333.

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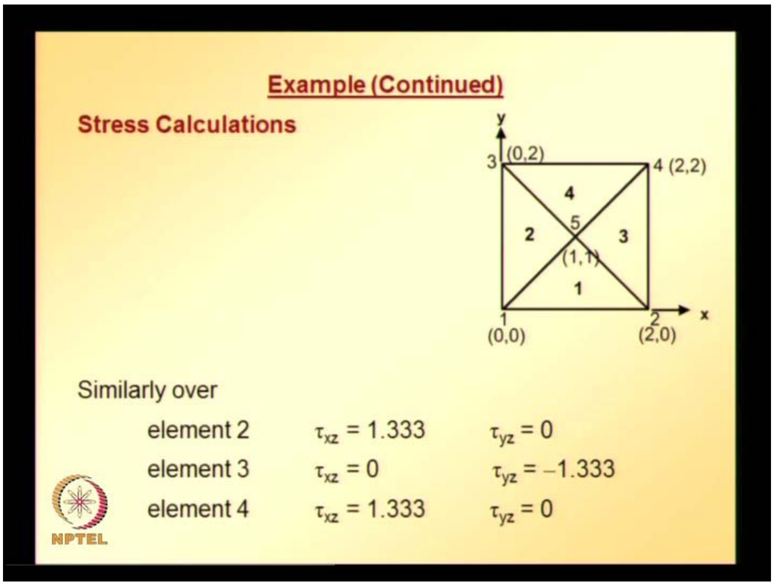

Similar calculations can be repeated for element 2, element 3, and element 4. Basically, we need to substitute this stress potential values at all the three nodes of element into the relation between shear stress components and stress potential in terms of derivatives of finite element shape functions and the nodal values of stress potential that we already looked at.

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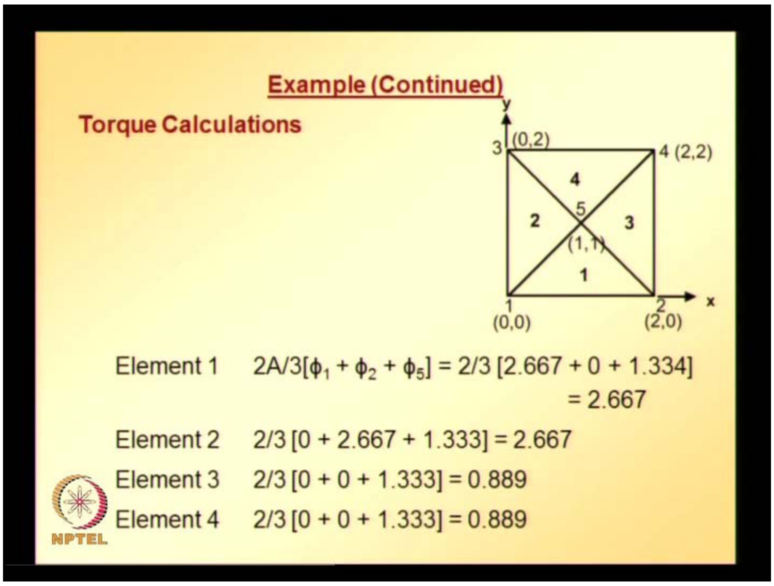

Once we have the stresses, we can also similarly calculate torque. Torque is basically two times the area of triangle times average value of stress potential in each of the element. So, for element 1, stress potential average is phi 1 plus phi 2 plus phi phi 5 divided by 3 that times 2 A that gives us 2.7 2.667; similarly, for element 2, element 3, and element 4. So, we have the torque values for each of the elements 1 2 3 4. So, total torque is going to be sum of all these torques for each of the elements.

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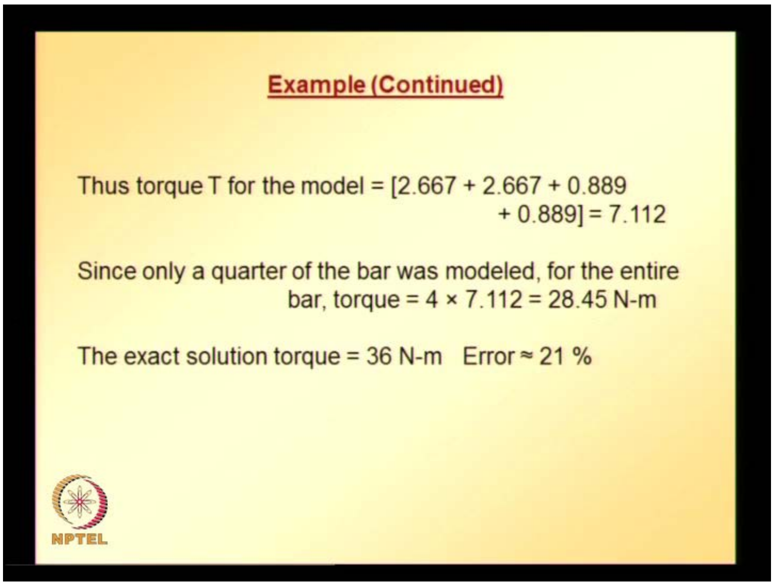

So, total torque for the quarter model is given by this one. Since only a quarter of bar was modeled for the entire bar, torque is 4 times the value; that is 4 times 7.112. It turns out that, it is 28.45 Newton meter; also this problem can be solved exactly; exact solution is 36 Newton meter. And if we compare with the value that we obtained 28.45 Newton meter with 36 Newton meter, it can be noticed that, there is 21 percent 21 percent error and that is because we used. First of all we use only linear elements and that two we use only 4 linear elements.

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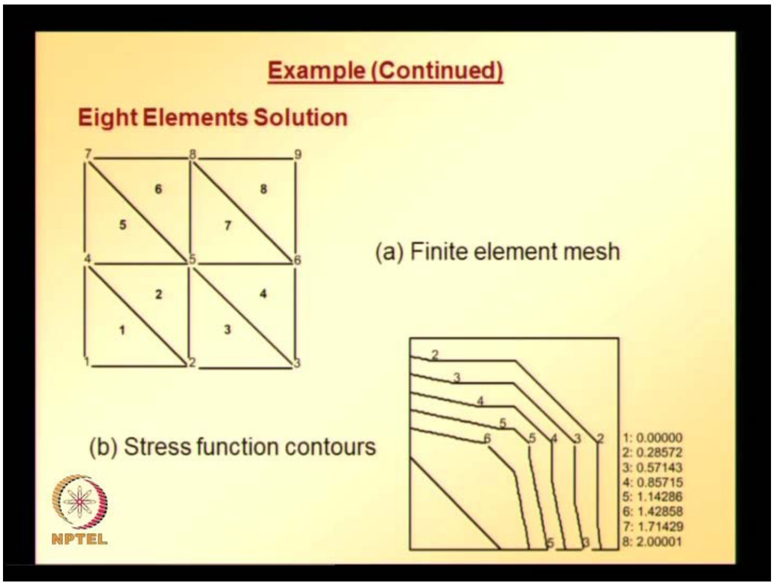

So, now let us repeat this calculation by dividing the quarter model into 8 triangle element. So, this is the finite element mesh and the procedure is same for each of the elements. All elements are identical for each of the elements. Once we write element equations for one of the elements, element equations are going to be identical for all the 8 elements. And applying the essential boundary condition, that nodes the stress potential value for the nodes on the boundary is equal to 0; that is 57,58,59,56, and 53 the stress potential values are 0.

So, applying those boundary conditions, we can solve for the stress potential value phi at rest of the nodes and then, we can interpolate. And then, we can find these stress stresses and torque as we did for 4 element solution. And once we do that and if you plot the stress function contours, they look like this for 8 element solution and component of stresses, stress contours, tau x z contours, tau y z contours.

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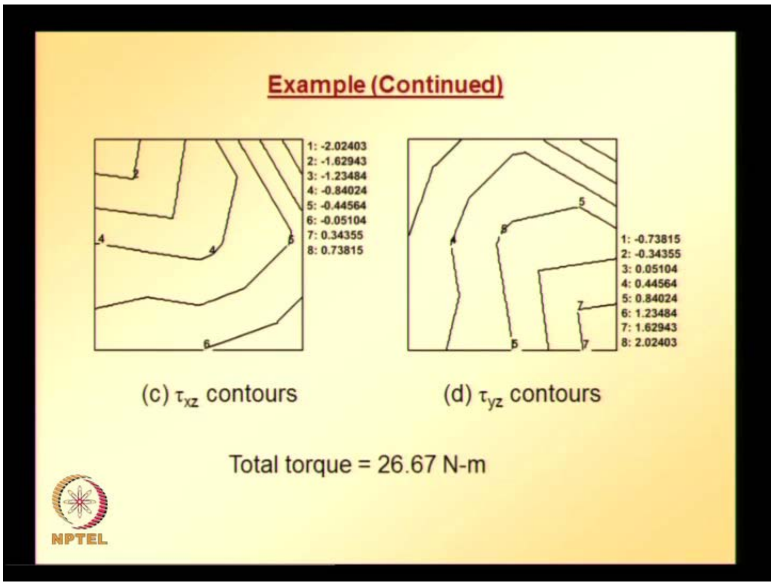

Similar to that we did for 4 element solution, we can calculate torque for each of the elements and then, we can find the total torque for the entire square bar and the total torque when 8 elements are adopted, turns out to be 26.67 Newton meter.

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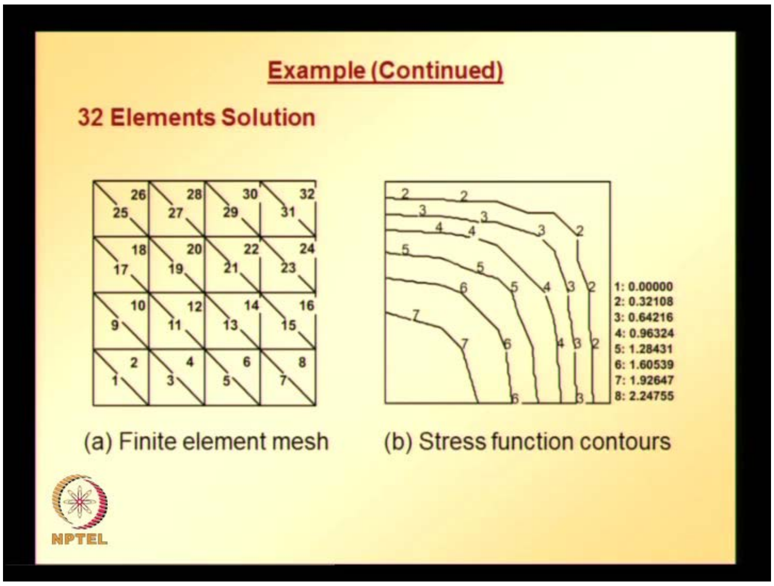

The calculation is also repeated by taking 32 elements and the finite element discretization is shown. Everything is similar. Since all elements are identical, finite element shape functions or finite element equations are going to be same for all elements. And again applying the condition that, stress potential value along the boundary is equal to 0 and we get the reduced equation system. Solving those equations, we get the stress potential at other nodes, and then we can do interpolation for calculations of stresses and torque and the stress function contours are given here.

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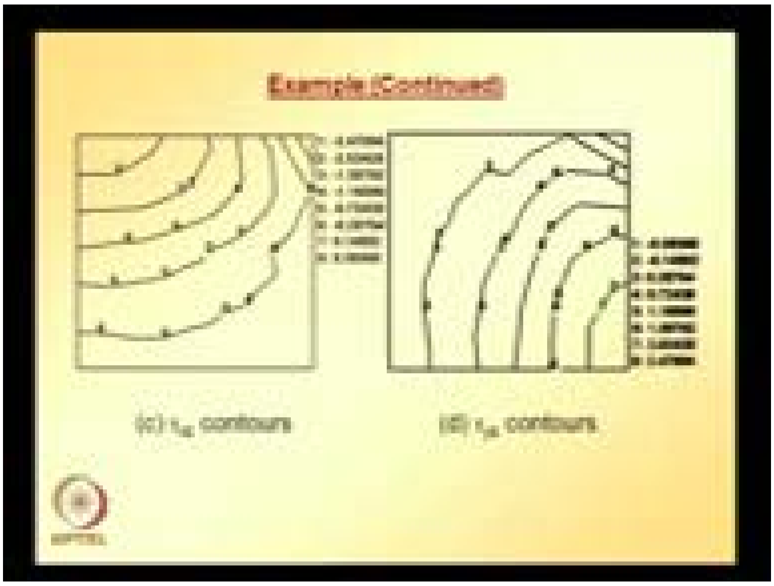

And shear stress components tau x z, tau y z contours are given here; and the total torque for 32 element solution turns out to be 34.26. So, we can see that solution is converging, as we increase the number of elements from 4 to 8 to 32.

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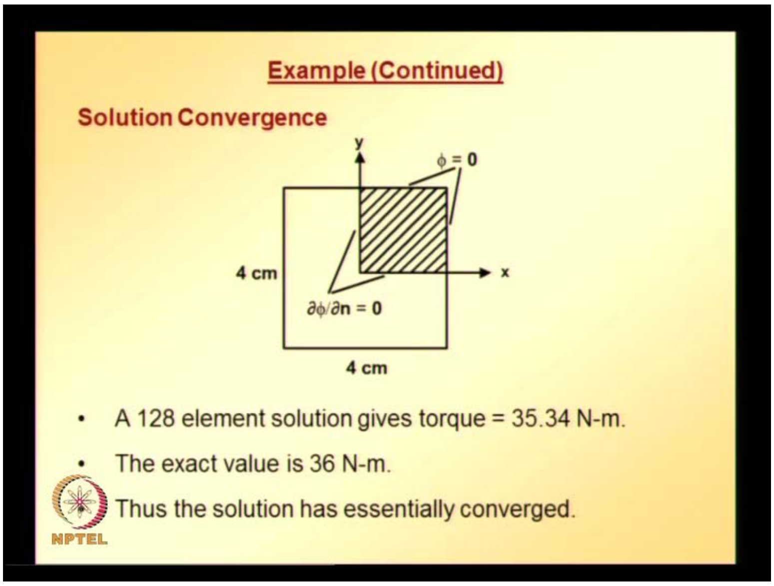

Similarly, but only thing is as the number of elements are increased, it is going to be difficult to do hand calculations. We need to automate it and we can write a computer code to do more refined finite element calculations using the automated code. So, here 28 128 element solution is shown; 128 element solutions gives torque 35.34 newton meter, whereas the exact value is 36 newton meter. So, basically the solution is converged or it is converging.

So, the finite element equations that we developed are correct. So, this is so this is the way, we can develop finite element equations for any two dimensional general two dimensional boundary value problem. Once we identify the corresponding coefficients k x, k y, p q, and identify the corresponding coefficient associated with essential boundary conditions.