Finite Element Analysis

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Lecture No. # 31

We will continue with, what we are doing in the last class. In the last class, we looked at some of the numerical integrations schemes like Newton-cotes rules with one point, two point, three point, four and five points. One point rule is called rectangle rule. Newton-cotes rule with one point is called rectangle rule. Newton-cotes rule with two point is nothing but trapezium rule, and Newton-cotes rule with three points is Simpson's rule. So, we looked at how to get the weights associated with these integration rules, and also we looked at dominant error term for each of these methods.

What is the order of dominant error term? And also, if the accuracy is not sufficient, we can go for repeated use of these rules like, we also looked at repeated trapezium rule, repeated Simpson's rule. And finally, what we have noted is Newton-cotes rules with n points integrates a function of order n minus 1 accurate or if we require to integrate a function of order n, we require to use Newton-cotes rule of order n plus 1. And then we looked at Gauss Legendre rules again one point, two points. For two point Gauss Legendre rule, we also looked in detail, how to get the locations of the points and also corresponding weights.

Basically, what we did is if we require to find four if we require to find n number of unknowns, we imposed condition such a way that we get n number of equation; so that we can solve these n number of equations for n number of unknowns uniquely. So, that is the approach, we followed to get the locations and weights of the integration points. Gauss Legendre rule is also called gauss quadrature, and why it is called gauss Legendre rule, because the location of integration points turns out to be the roots of Legendre polynomial.

And also we looked at In the last class, we looked at changing the limits of integration because some of these integration schemes require to have limits between minus 1 to 1.

Upper limit of integral should be 1 and lower limit of integral should be minus 1. In that case, whatever may be the limits of integration, we can change to minus 1 to 1 using some kind of transformation. We also looked at that one. And so far, whatever we have seen in yesterday's class; it is only one dimensional integration. In today's class, we will continue and look at even two-dimensional or how to evaluate a multi-dimensional integrals or multi integrals. So, before doing that, let us look at what is this Gauss-Leguerre rule?

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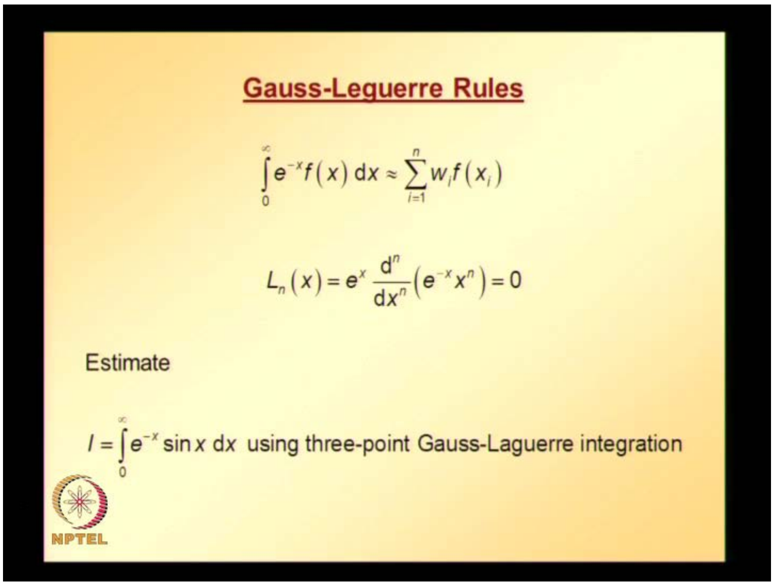

Before going for that, let us look at what is this Gauss-Leguerre rule? And this is this helps us to numerical evaluate the kind of integral that is shown in the equation. That is integral 0 to infinity e power minus x f(x) dx similar to Newton-cotes rules or gauss Legendre rules that we have seen. What basically we need to do is we need to evaluate function at some points over the domain between 0 to infinity and multiply with corresponding weights. And the number of points that we need to use depends on our choice and why this word Leguerre?

Because here it turns out that these locations of integration points or roots of this polynomial Leguerre polynomial; that is why, it is called Gauss-Leguerre rule. Let us see, how we can evaluate this integral 0 to infinity e power minus x sin x dx using threepoint Gauss-Leguerre integration. And similar to Gauss Legendre integration, we have well-documented the literature, the locations and corresponding weights of the points and they are given in this table. So, before we proceed in estimating this integral, let us look at that table.

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Gauss-Leguerre Integration Points & Weights

n	x_i	w_i
1	1.0	1.0
2	0.58578644	0.85355339
	3.41421356	0.14644661
3	0.41577456	0.71109301
	2.29428036	0.27851773
	6.28994508	0.01038926
4	0.32254769	0.60315410
	1.74576110	0.35741869
	4.53662030	0.03888791
	9.39507091	0.00053929

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So, depending on the number of points we require to use, we can select to the corresponding locations of integrations points and the corresponding weights from this table. So, the problem statement it is suggested to use three points. We select the location corresponding to n is equal to 3 and weight corresponding to n is equal to 3. So, we need to evaluate function at each of these points three points and multiply with the corresponding weights and sum them up. So, that is how we can estimate this estimate this integral; that is integral 0 to infinity e power x sin x dx.

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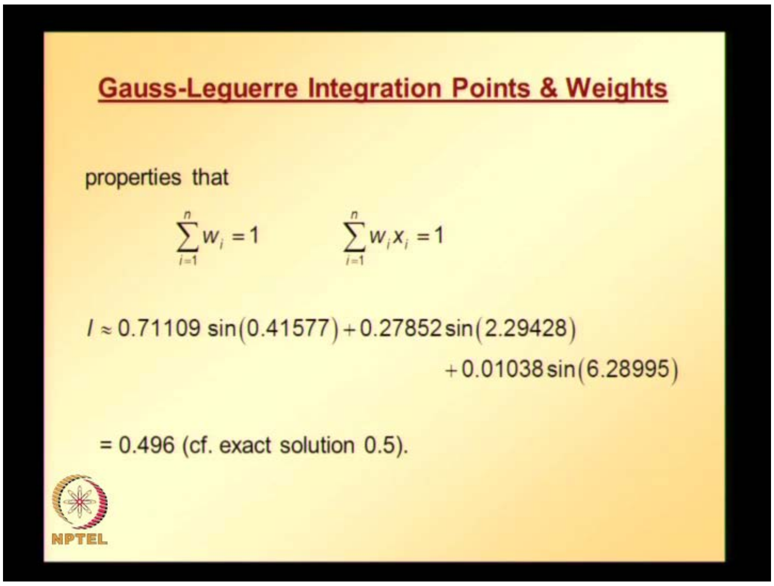

Before that, some of the important properties associated with the weights and locations in this Gauss-Leguerre rule are as follows. Sum of all weights is equal to 1 and weight times integration point location and if we sum up all the for all the points, it is going to be equal to 1. So, the locations in the corresponding weights satisfy these two conditions. And now coming to evaluation or estimation of the integral 0 to infinity e power minus x sin x dx, the details of calculations are given here. And it turns out that, simplification of these results in 0.496. We can also find exact solution for this particular problem, which is actually 0.5.

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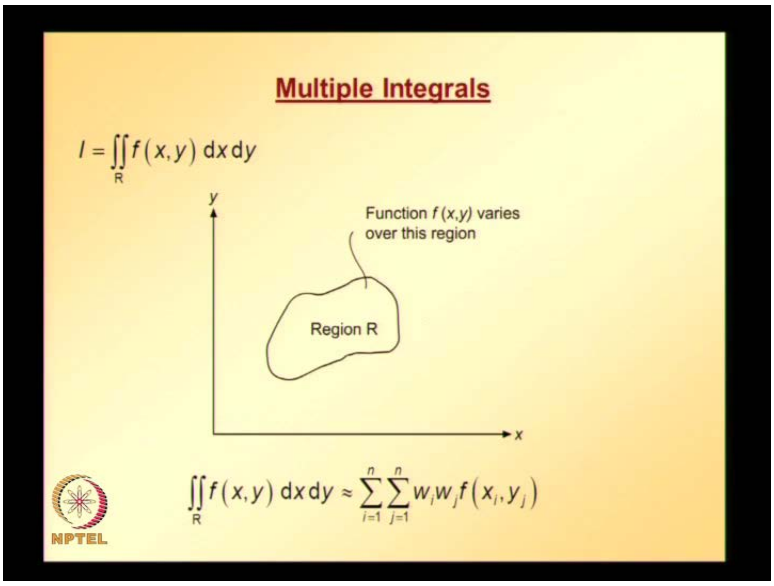

So, the estimate is in good agreement with the exact solution and if we required to get more accurate result, we can increase the order of integration; that is, we can go for four number of points. Let us look at, how to evaluate multiple integrals? That is an area integral like this; integral f(x) dx integrated f(x) dx dy integrated over the region R; here it is shown in this figure. So, we require to evaluate this function over this region and this function varies over this region that is why, f(x, y) is inside integral. If f(x, y) is constant, then we can pull out of the integral than integral d x d y over the region R is going to be the area of that region R.

So, now basically what we require to do? We need to find this integral. We required to decide, how many number of integration points you want to use along x direction and y direction. So, this integral f x f(x, y) and dx dy is approximated as function. This f(x, y) evaluated at certain number of points along x direction and y direction. Here, it is going to be a cartesian product of all the points and locations of the points and the corresponding weights. We can estimate this integral using this formula and which will be illustrated here.

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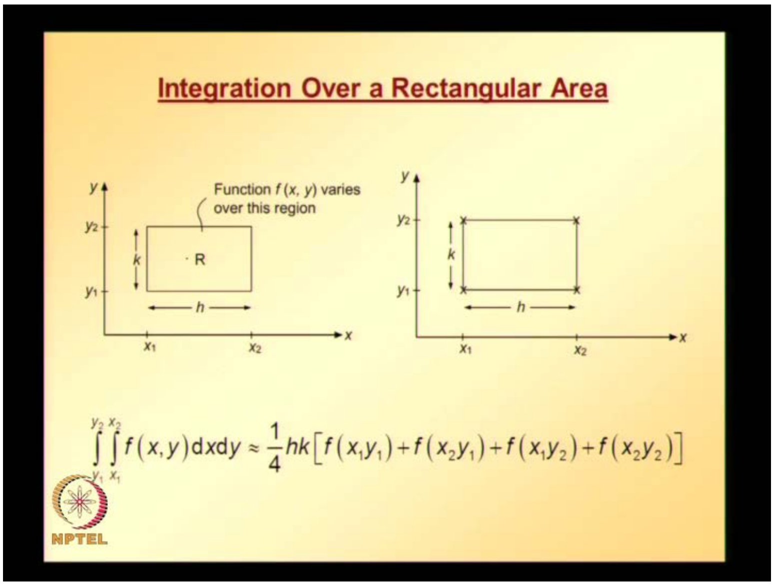

So, let us take a rectangular area like the in the figure, that is shown in the left hand side. And now, if you recall trapezium rule and we are actually basically using two points trapezium rule. So, the two points along x direction and two points along y directions are shown in the figure on the right hand side. The integration points are actually located at a at the corners of this rectangular area and the distance between these integration points along x direction is h; distance between the integration points along y direction is k.

So, we need to apply trapezium rule in both direction and this integral f(x, y) dx dy double integral between the limits x 1, x 2, y 1, y 2 is approximated as one fourth h, k. h is nothing but the distance between the points along x direction; k is nothing but, distance between the points along y direction and function evaluated at four points, that is function evaluated at x 1 y 1, x 2 y 2, x 1 y 2, x 2 y 1. If you recall trapezium rule, this is basically applying trapezium rule in both directions and taking cartesian product of that results in this equation.

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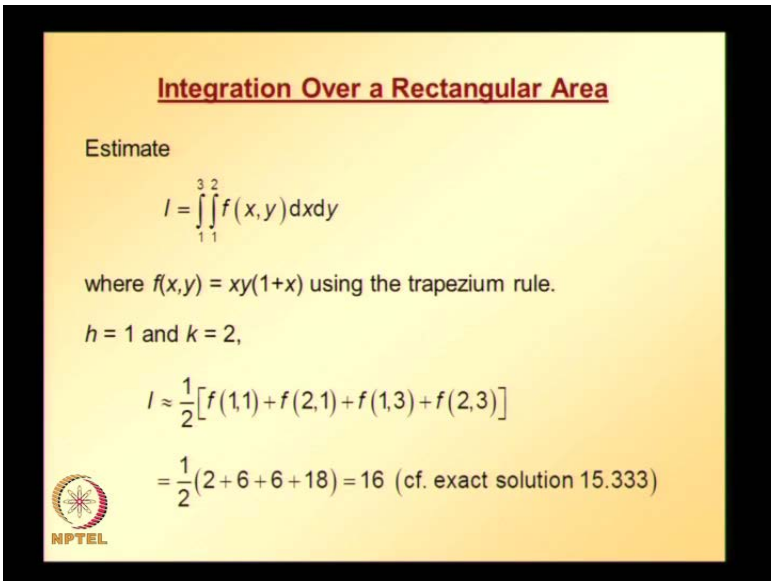

So, now let us apply this formula to estimate an integral like this. Integral 1 to 3, 1 to 2, f(x, y) dx dy, where this function f x y is x y times 1 plus x using trapezium rule. So, we need to apply the formula that we have just seen, that is we need to first identify what is h and what is k. h is nothing but the distance between the upper limit lower limit along x direction; k is nothing but distance between upper limit lower limit along y direction. So, h is equal to 1 in this case; 2 minus 1 is 1 and 3 minus 1 is 2. So, h is equal to 1 and k is equal to 2.

And then, apply the previous formula that we have just seen that results in this equation. So, we need to substitute what is the function value evaluated at each of these points f(1, 1), f(2, 1), f(1, 3), and f(2, 3). So, doing all that substitutions, we get this estimate 16 and the exact solution turns out to be 15.333. Since we used trapezium rule, the accuracy is little bit less. But if we adopt higher order rule like Simpson's rule, maybe we can reach the exact solution with more accuracy.

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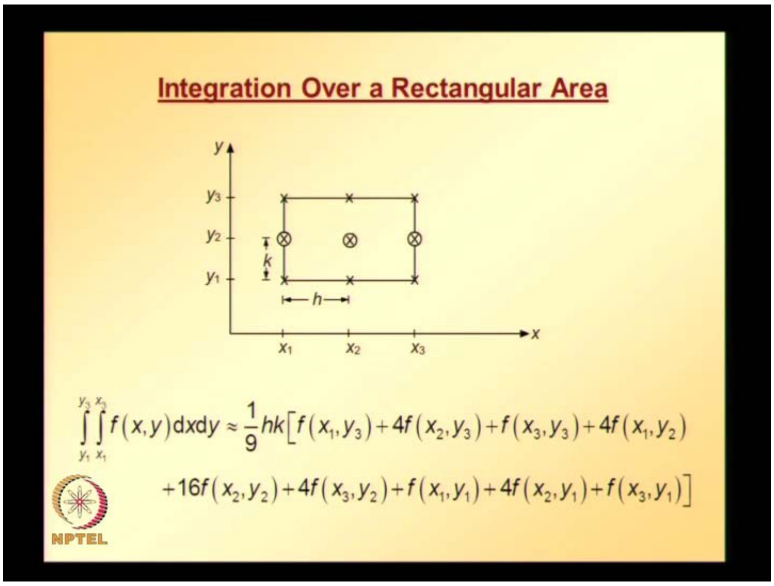

So, now let us see how to apply Simpson's rule over rectangular area? Basically, we need to evaluate function along three points at three points along x direction and at three points along y direction. Total, taking cartesian product of these three points along x direction and three points along y direction results in total nine points, which are shown there and the distance between the points are also indicated. Distances between points are x 2 minus x 1 is h and y 2 minus y 1 is k. So, with that understanding applying Simpson's rule along x direction and y direction separately taking cartesian product of that results in this.

So, area integral of f(x, y) between the limits x 1, x 2, y 1, y 2 is given by 1 over 9; h k times function evaluated at x1, y 3; four times function evaluated at x 2, y 3; function evaluated at x 3, y 3 and four times function evaluated at x 1, y 2 plus sixteen times function evaluated at x 2, y 2 plus four times function evaluated at x 3, y 2 plus function evaluated at x 2, y 1 plus four times function evaluated at x 2, y 1 plus function evaluated at x 3 y 1.

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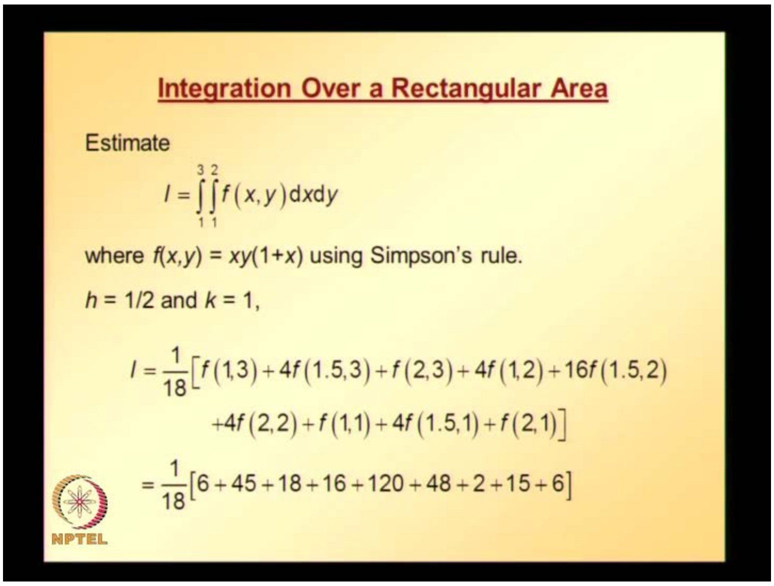

We can apply this to estimate that the integral, that we have already estimated using trapezium rule; this is what, we estimated. This is the integral that we estimated where f(x, y) is defined as x y times 1 plus x. So, now we need to apply Simpson's rule. So, first we required to identify what is h; what is k. h is nothing but half the difference between the limits upper limit and lower limit along x direction; k is half the difference between upper limit lower limit along y direction. So, those two results and h is equal to 1 over 2; k is equal to 1. And now, applying the formula we get this equation substituting the function value evaluated at each of the points that are in the equation.

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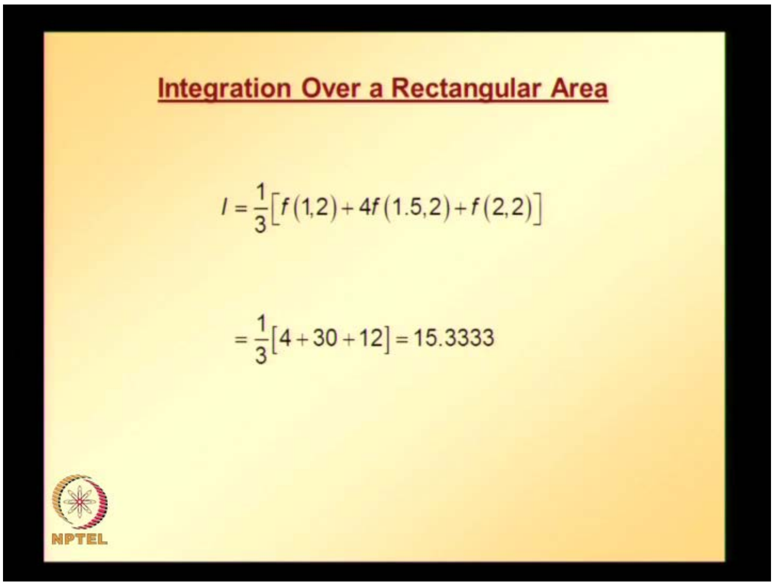

We get this one and finally and this becomes 15.333, which is actually the exact solution. So, when you evaluate this integral using Simpson's rule, we closely match or we almost get the same value as the exact solution. So, this is about integration over rectangular area using Newton-cotes rules. Again Newton-cotes rules require a function of order n requires n plus 1 number of points to get accurate or exact solution. So, we required to use n plus 1 number of points along x direction and n plus 1 number of points along y direction. Or if the function along x direction is of order n and the function is of order m along y direction, then we required to use n plus 1 along x direction; m plus 1 along y direction to get a exact solution using any of these rules; that is a trapezium rule or Simpson's rule for rectangular area.

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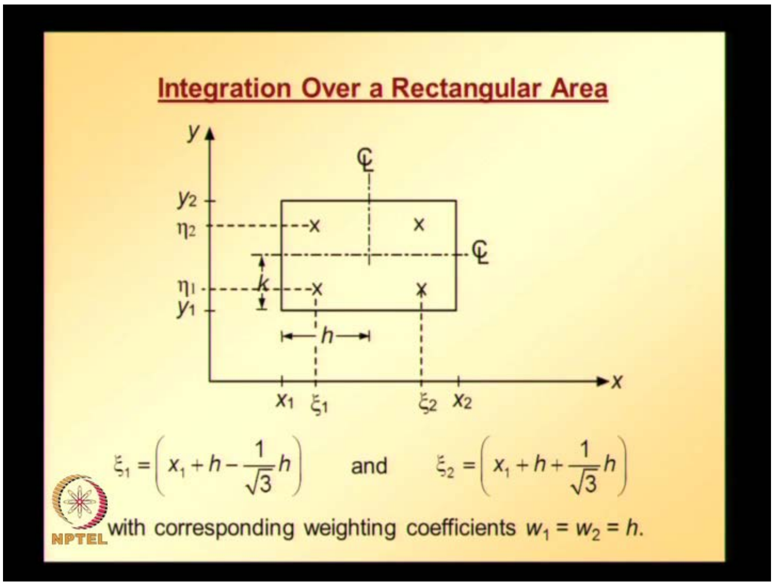

So, now let us look at, how to evaluate integration over rectangular area using a gauss quadrature rules or gauss Legendre rules. First we need to identify the locations of integration points. They are indicated in the figure rectangular area and then, since function is in terms of x y, we required to express these integration points in terms of x y. Or this is the relationship between the integration point location with respect to the local coordinate system, which is denoted there or which is indicated there in the figure. This is the relation; this equation gives relation between the psi coordinate system and x coordinate system. So, psi 1 is given by x 1 plus h minus 1 over root 3 times h and psi 2 is x 1 plus h minus 1 over root 3 times h, where h is half the dimension of the rectangular area along x axis. The corresponding weights at these two points is equal to h. This is the coordinates of psi in terms of x.

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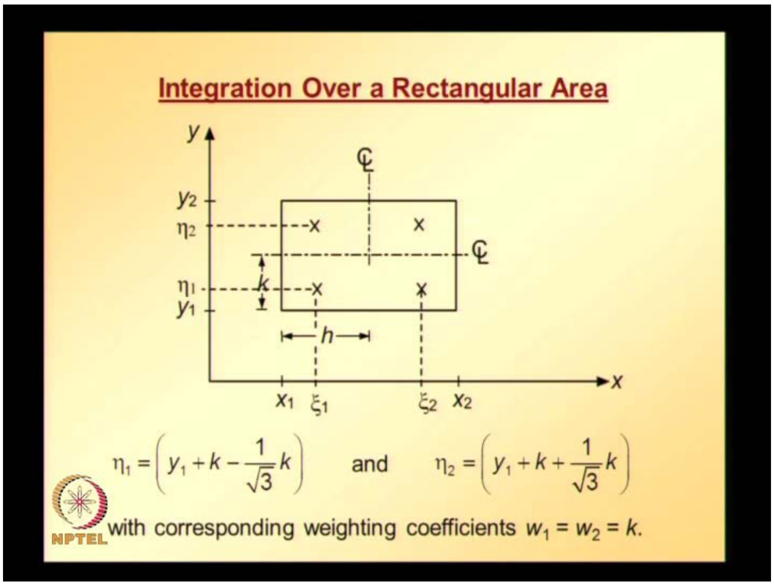

Similarly, we can express eta in terms of y; eta 1 is equal to y 1 plus k minus 1 over root 3 times k and eta 2 is y 1 plus k plus 1 over root 3 times k and the corresponding coefficients or weight functions or weights are w 1, w 2 is equal to k; that is half the dimension of the rectangular area along y direction. Once we have these locations and the weights, what we required to do is, we need to take cartesian product of this and multiply with corresponding weights and sum them up.

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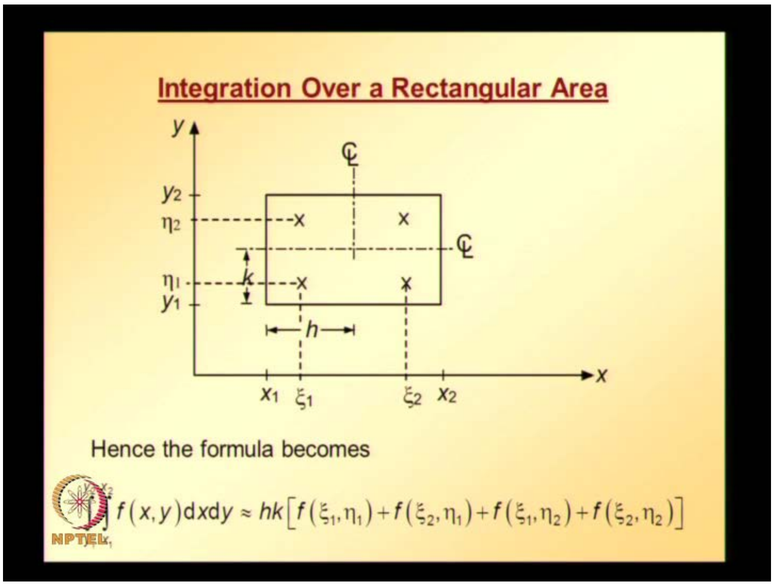

Then, we are going to get the approximate value of integral. So, hence the formula becomes integral f(x, y) over the rectangular region x 1 to x 2, y 1 to y 2 is approximated as weight along x direction times weight along y direction that is h times k multiplied by function value evaluated at all the four points. And sum of the functions values evaluated along at all the four points.

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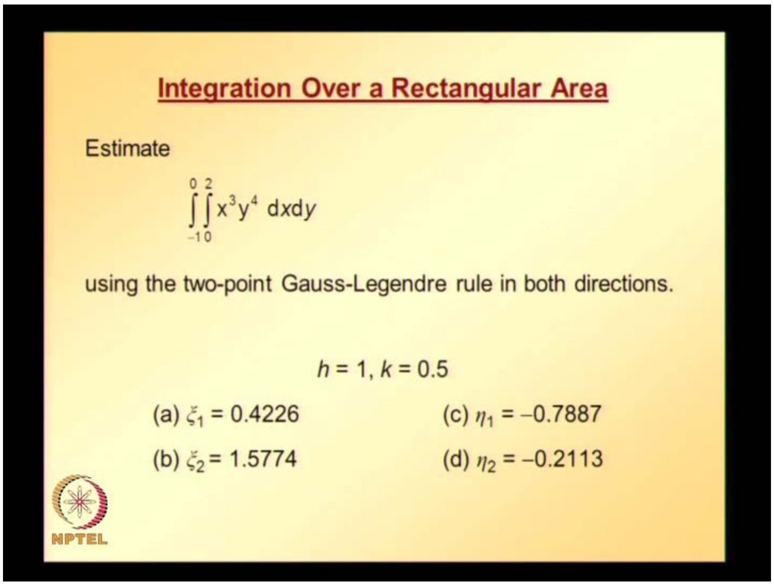

So, now let us apply this formula to evaluate this integral. Integral x cube y 4 y power 4, we need to integrate over this area 0 to 2, minus 1 to 0. First let us check, what is the order of this integrant along x direction? It is 3. So, 2 n minus 1 is equal to 3. So, it turns out n is equal to 2 and along y direction, 2 n minus 1 is equal to 4. So, n is equal to 2.5. We need to round it up to 3. So, to evaluate this integral accurately or exactly, we required to use two points along x direction and three points along y direction. But here in the problem statement, it is suggested to use two points along both directions.

So, we can expect some kind of errors, when we evaluate this integral using this two point Gauss Legendre rule in both directions. First, we required to identify what is h; what is k. h is nothing but half the dimension along x direction; that is 2 minus 0 divided by 2 which is 1. Similarly, y is half the dimension along y direction; that is 0 minus half minus 1 divided by 2 is equal to 0.5. That is h is equal to 1; k is equal to 0.5 and the integration points are given by cartesian product of this psi 1, psi 2, eta 1, and eta 2. So, now we have weights and the corresponding locations of the integration points.

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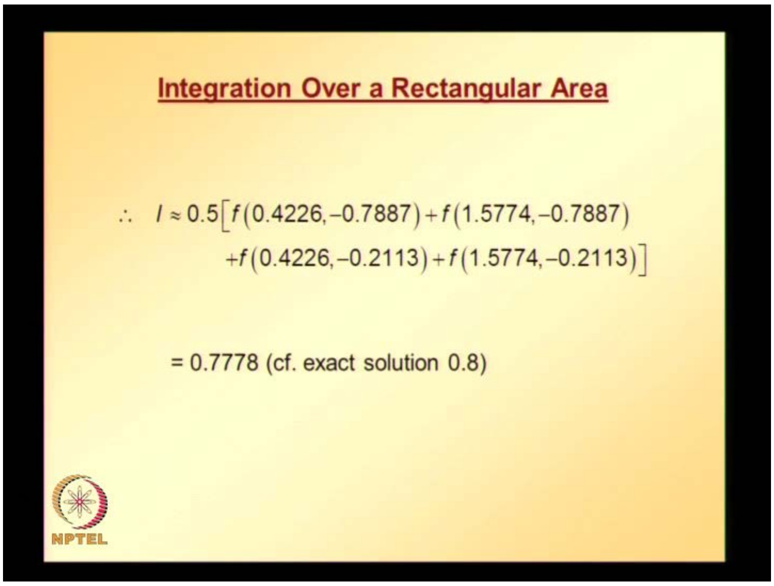

So, we can evaluate the integrant and those locations and multiply with corresponding weights and sum them up. We are going to get the estimate of the integral, which turns out to be 0.7778, whereas the exact solution is 0.8. So, this is what I mentioned. Since we under integrated that is we use less number of integration points than required, the solution may not be accurate. There may be some error in the solution and that is what is reflecting here. So, now let us see, how to use. So, we use two points along x direction; two points along y direction.

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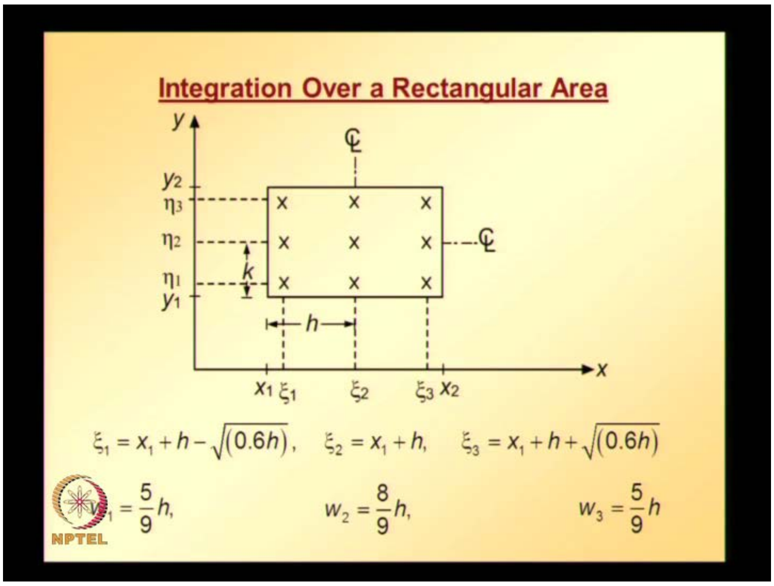

So, now let us see how to evaluate integration over rectangular area, if we adopt three integration points along x direction, y direction. So, we are having three points along x direction that is psi 1, psi 2, and psi 3; along y direction eta 1, eta 2, and eta 3. Again similar to the previous case, half the dimension along x direction is denoted with h; half the dimension along y direction is denoted with k. So, once we have this we have the values of psi 1, psi 2, psi 3, eta 1, eta 2, eta 3, we can take Cartesian product of this and get the locations of all the integration points.

And it each integration point weight along psi direction times weight along eta direction, we need to multiply and then, we get the total weight. So, we first identify what are these psi 1, psi 2, and psi 3 in terms of x coordinate or x x 1. So, psi 1 is x 1 plus h minus root of 0.6 h, psi 2 is x 1 plus h, psi 3 is x 1 plus h plus root of 0.6 h. These can value or these expressions crazily we verified, once we knew the locations of these integration points. Similarly, what are the weights at these locations? Weights at psi 1 weight at psi 1 is 5 over 9 times h; h is half the width or the dimension of the rectangular area along x direction; w 2 is 8 over 9 times h; w 3 is 5 over 9 times h.

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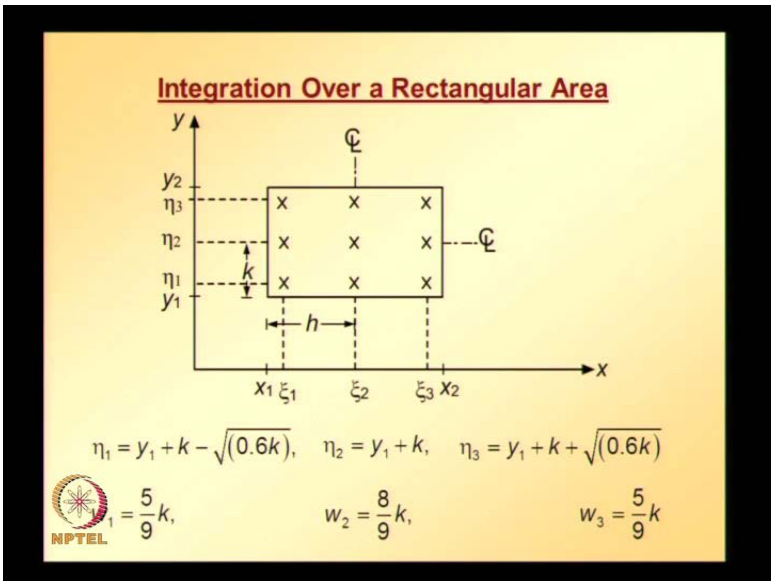

Similarly, eta 1 eta 1 is y 1 plus h minus root of 0.6 times k; eta 2 is y 1 plus h; eta 3 is y 1 plus sorry y 1 plus k plus 0.6 times k, where k is half the dimension of the rectangular area along y direction. w 1 one is 5 over 9 times k; w 2 is 8 over 9 times k; w 3 is 5 over 9 times k. So, we have the locations psi 1, psi 2, psi 3, eta 1, eta 2, eta 3.

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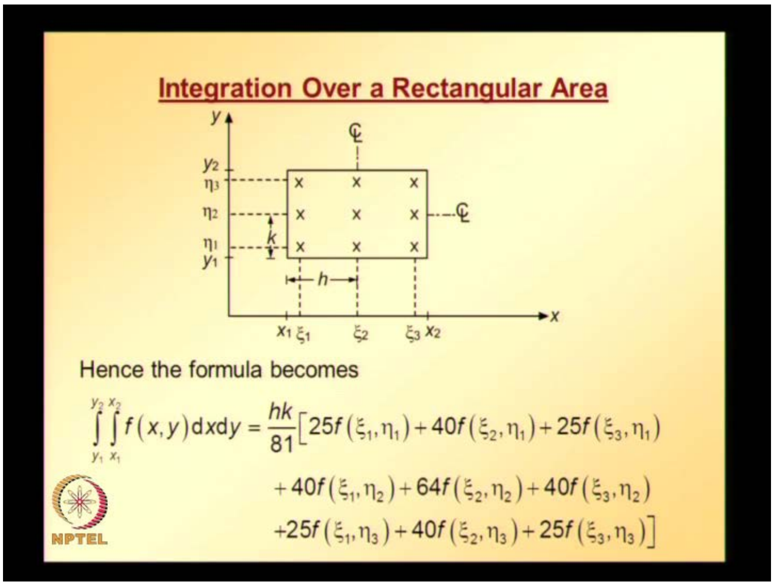

So, we can evaluate integral f(x, y) between the limits x 1 x 2, y 1 y 2 using this formula. So, we need to evaluate the function at each of the points psi 1, eta 1, psi 2, eta 1, psi 3, eta 3 and rest of the points and multiply with the corresponding coefficients and weights. We get the approximate value of integral.

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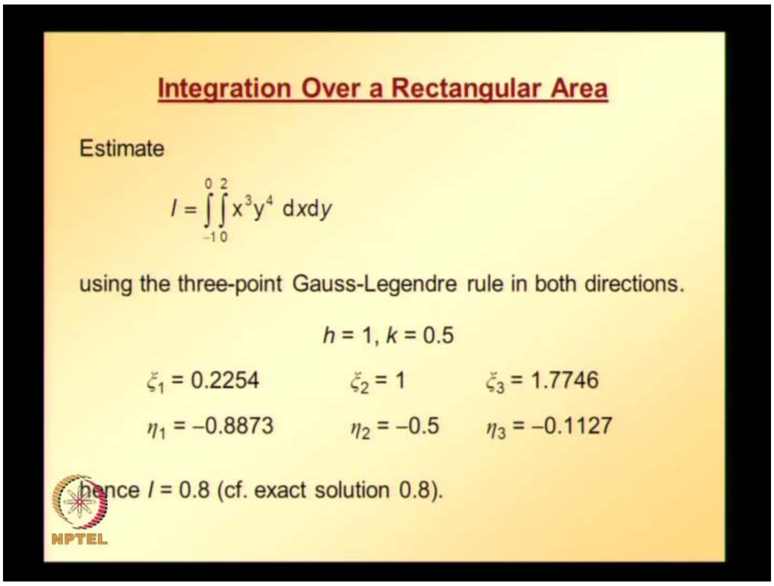

So, we can apply this formula to estimate this integral again, which we already did with two point Gauss Legendre rule. So, the polynomial or the integrant is of order 3 along x direction; is of order 4 along y direction. So, to estimate this integral accurately, we already discussed that, we required to use along atleast two points along x direction; atleast three points along y direction. So, if we use more number of points along x direction. We are not going to get any more increase in accuracy but accuracy remains same. But along y direction (( )), we use two points. Now, we use three points and see whether we match the exact solution or not.

So, using three points Gauss Legendre rule in both directions and identifying h and k. h turns out to be 1 and k turns out to be 0.5 and these are the integration point locations. And when we substitute these integration point locations and evaluate the integrant at these locations and multiply substitute the corresponding values with previous formula, we get this integral value. The estimate of this integral value to be 0.8, which matches very well with exact solution; since we actually use the required number of points along y direction.

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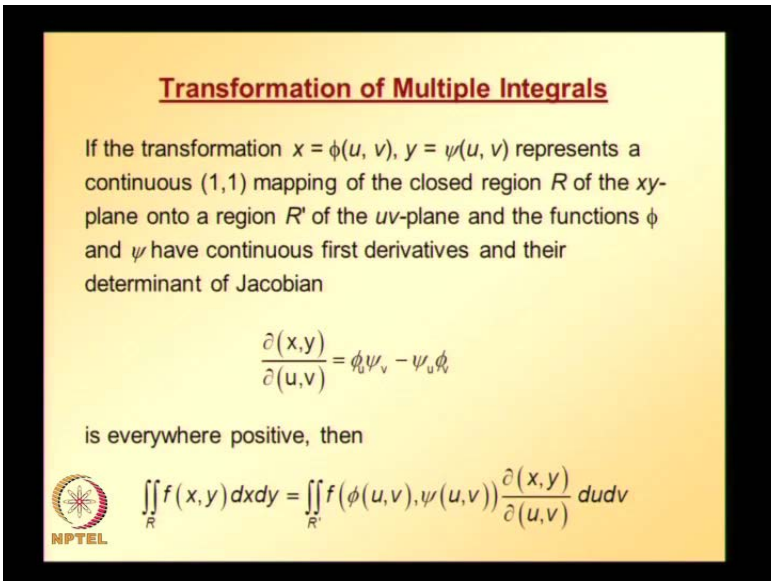

One more important thing that may be helpful for evaluation of multiple integrals is given here. If the transformation from x and y to use x and y space to u and v space is like this. x is equal to phi times, phi as a function of u and v and psi as a function of u and v represents a continuous 1 to 1 mapping of closed region R of x y plane into the region R prime of u v plane. Then, the function phi and psi have continuous first derivatives. So, these are the requirements and their determinant of Jacobian is given by this; then determinant is positive everywhere.

Then, the integral f(x, y) dx dy evaluated over region R can actually be transformed into the way; that is shown on the right hand side of the equation. That is double integral R prime, f as a function of phi and psi, which in turn are functions of u and v multiplied by determinant of Jacobian d u d v. So, this kind of transformation we can use to transform the limits of integration from one set of values to another set of values. So, this formula may be of help when we are evaluating multiple integrals. So, that is why it is given here.

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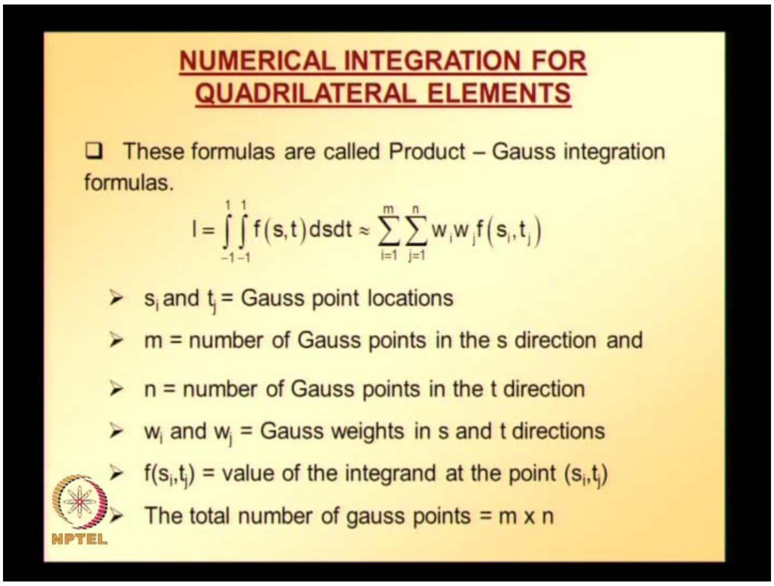

So, now let us look at how to apply these concepts for numerical integration over quadrilateral elements. These formulas are called product integration formulas. Basically, these are same as what we already seen; integral minus 1 to 1minus 1 to 1 double integral minus 1 to 1 minus 1 to 1 f(s, t) ds t is, we need to select certain number of points along s direction; certain amount of points along t direction. And evaluate the function at these points; multiply with the corresponding weights; weight along s direction times; weight along t direction.

So, s and t are the locations of integration points; m is the number of integration points along s direction and n is number of integration points along t direction. And w i, w j are the weights along s and t directions and f s t, f(s i, t j) is function value at the integration point. And the total numbers of integration points are number of integration points along s direction times number of integration points along t direction; that is m times n.

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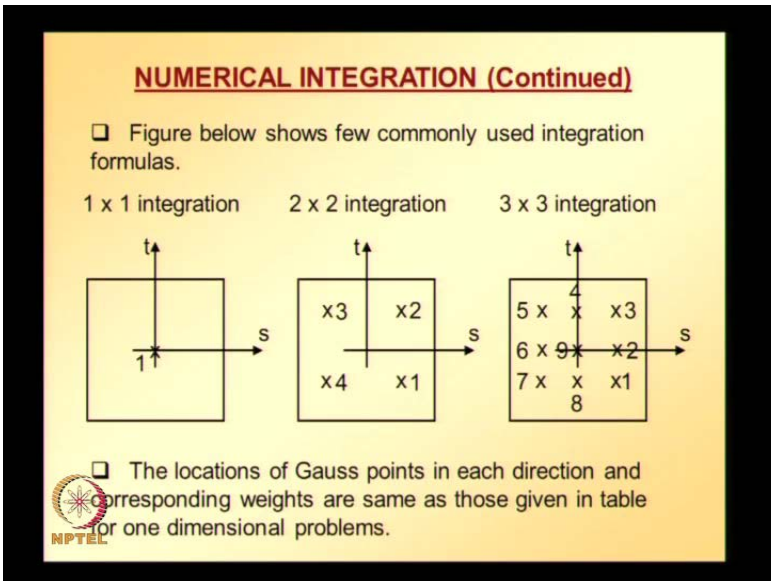

So, now let us see what is by meant by 1 by 1 integration, 2 by 2 integration, 3 by 3 integration? The figure below shows few commonly used integration formulas; 1 by 1 integration, 2 by 2 integration, 3 by 3 integration. There is If it is 1 by 1 integration, one point along s direction; one point along t direction. If it is 2 by 2 integration, two points along s direction; two points along t direction. If it is 3 by 3 integration three points along s direction; three points along t direction. The locations of Gauss points in each direction and the corresponding weights are same as those given in the table for one dimensional problems, which we already looked at. So, they are repeated here for more number of integration points.

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Gauss Quadrature

Gauss Points (± x_i)	Weights (w_i)
n = 2	
0.57735 02691 89626	1.00000 00000 00000
n = 3	
0.00000 00000 00000	0.88888 88888 88888
0.77459 66692 41483	0.55555 55555 55555
n = 4	
0.33998 10435 84856	0.65214 51548 62546
0.86113 63115 94053	0.34785 48451 37454
n = 5	
0.00000 00000 00000	0.56888 88888 88889
0.53846 93101 05683	0.47862 86704 99366
0.90617 98459 38664	0.23692 68850 56189

So, now let us look at the table. Here the locations and weights for integration points 1 2 sorry 1 is not given; 2 3 4 5 are given in this table and one is the location is 0 and integration the weight at that point is 2. So, n is equal to 1; the integration point location is 0 and weight is equal to 2. So, it is not given in the table; but n is equal to 2, n is equal to 3 and n is equal to 4 and 5. The locations and weights are given here.

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Gauss Points (± x_i)	Weights (w_i)
n = 6	
0.23861 91860 83197	0.46791 39345 72691
0.66120 93864 66265	0.36076 15730 48139
0.93246 95142 03152	0.17132 44923 79170
n = 7	
0.00000 00000 00000	0.41795 91836 73469
0.40584 51513 77397	0.38183 00505 05119
0.74153 11855 99394	0.27970 53914 89277
0.94910 79123 42759	0.12948 49661 68870
n = 8	
0.18343 46424 95650	0.36268 37833 78362
0.52553 24099 16329	0.31370 66458 77887
0.79666 64774 13627	0.22238 10344 53374
0.96028 98564 97536	0.10122 85362 90376

And similarly, a continuation of this table is here with rest of the points; n is equal to 6 7 8 and 9 and 10, number of integration points.

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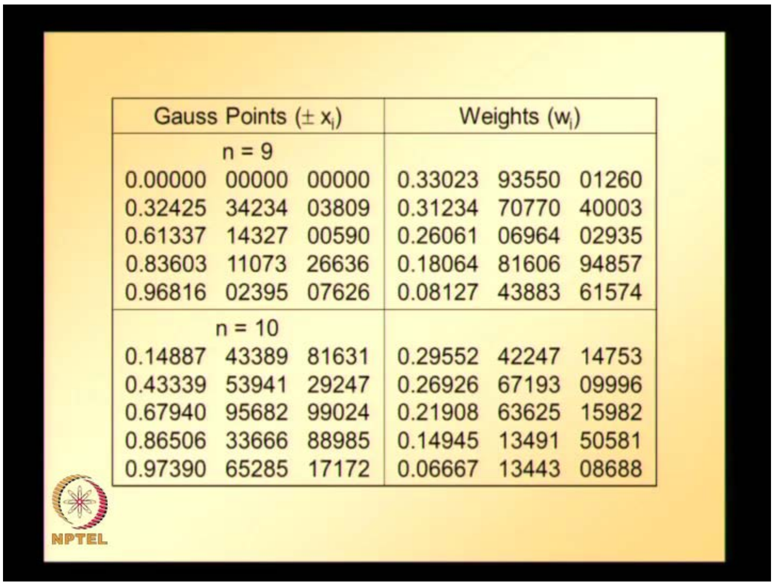

Gauss Points (± x_i)	Weights (w_i)
n = 9	
0.00000 00000 00000	0.33023 93550 01260
0.32425 34234 03809	0.31234 70770 40003
0.61337 14327 00590	0.26061 06964 02935
0.83603 11073 26636	0.18064 81606 94857
0.96816 02395 07626	0.08127 43883 61574
n = 10	
0.14887 43389 81631	0.29552 42247 14753
0.43339 53941 29247	0.26926 67193 09996
0.67940 95682 99024	0.21908 63625 15982
0.86506 33666 88985	0.14945 13491 50581
0.97390 65285 17172	0.06667 13443 08688

So, looking at this table, we can actually find the locations of integration points for two dimensional integrations or three-dimensional integration. And here if you see these tables, there is a gap of the certain number of significant digits. So, that space is nothing but depending on the accuracy that is required; we can go for more number of significant digits. Otherwise, we can chop off at that location, where the space is there depending on the accuracy that is required.

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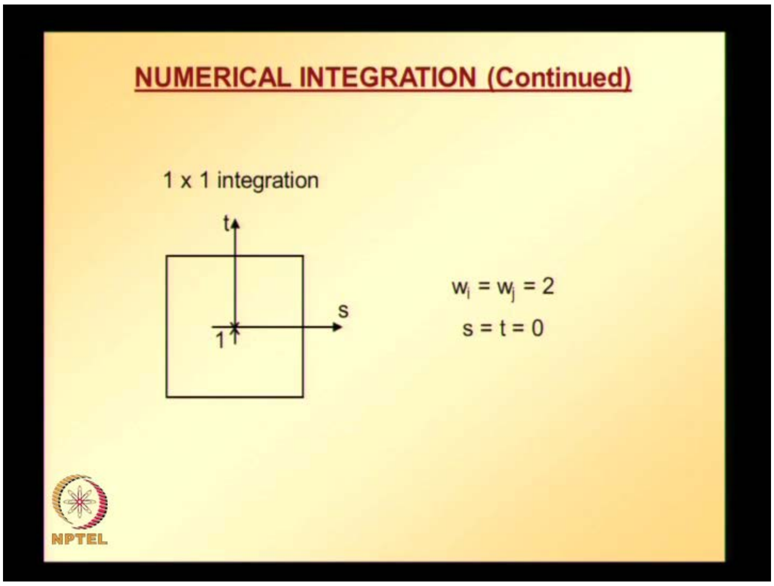

So, 1 by 1 integration what is the location and the corresponding weight? They are given here. Weight along s direction is same as weight along t direction is equal to 2 and location is s is equal to 2 and t is equal to s is equal to 0; t is equal to 0. So, we need to evaluate function at 0 0 and multiply with weight 4.

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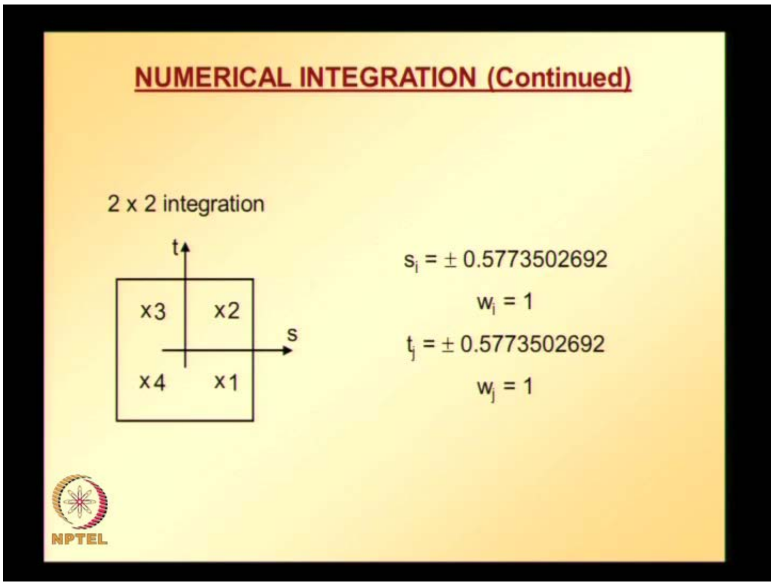

2 by 2 integration: locations along s direction and locations along t direction and weight along s direction; weight along t direction are given here. So, by taking cartesian product of this s values s 1 2 and t 1 2, we get the locations of all the four integration points and the weight at each integration point is along s direction times weight along t direction.

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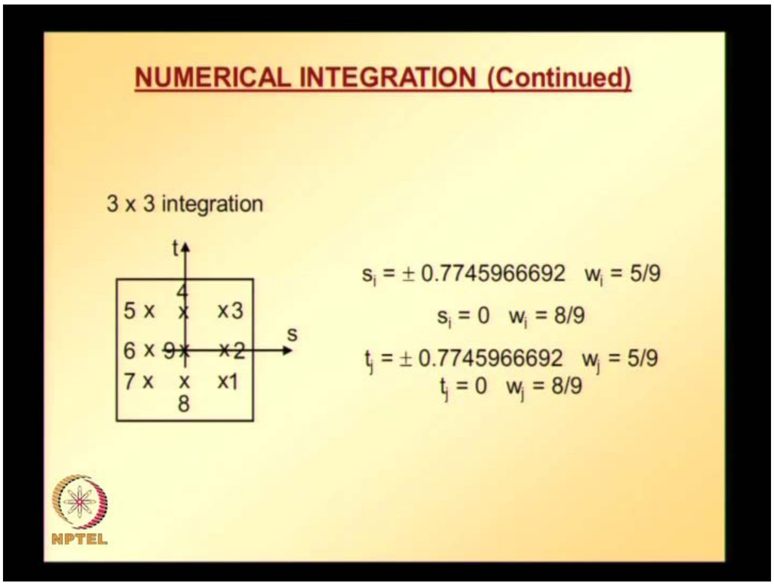

Similarly, 3 by 3 integration: the locations of integration points along s direction and the locations of integration point along t direction are given; the corresponding weights are given. So, by taking cartesian product, we can get the coordinates of all integration points and this.

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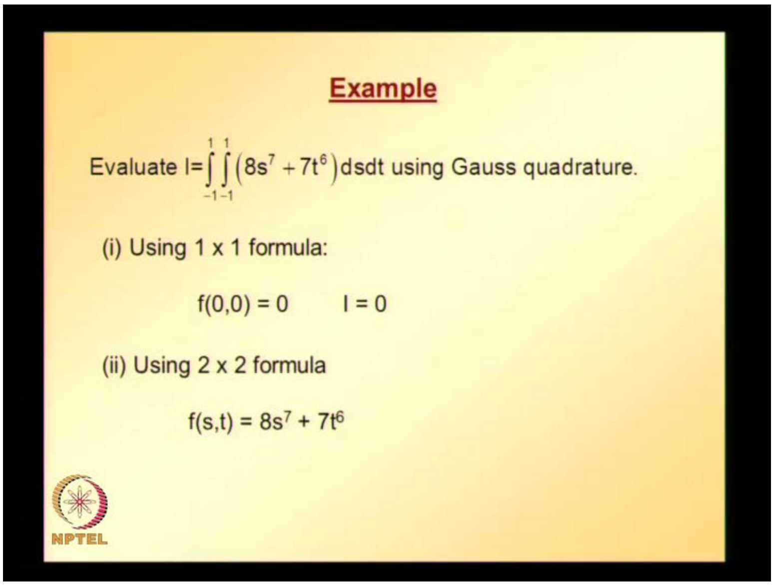

We can apply to evaluate this integral using Gauss quadrature and before we proceed, we see if we see the integral the highest order of highest order along s direction is 7 and highest order of this integrant along t direction is 6. 2 n minus 1 is equal to 7 and n is equal to 4 and 2 n minus 1 is equal to 6, results in n is equal to 3.5 or 4. To evaluate this integral accurately, we required to use four points along s directions; four points along t direction. If you use less number of points, we get error. So, let us verify whether that is true or not. So, using 1 by 1 integration formula, so we need to for 1 by 1 integration, we need to evaluate function at s is equal to 0; t is equal to 0 and multiply with weight 4. Function evaluated at s is equal to 0; t is equal to 0 turns out to be 0 and the weight is 4; so, four times 0 is 0. The estimate of this integral using one point one formula is 0. (No audio from 38:40 to 38:52)

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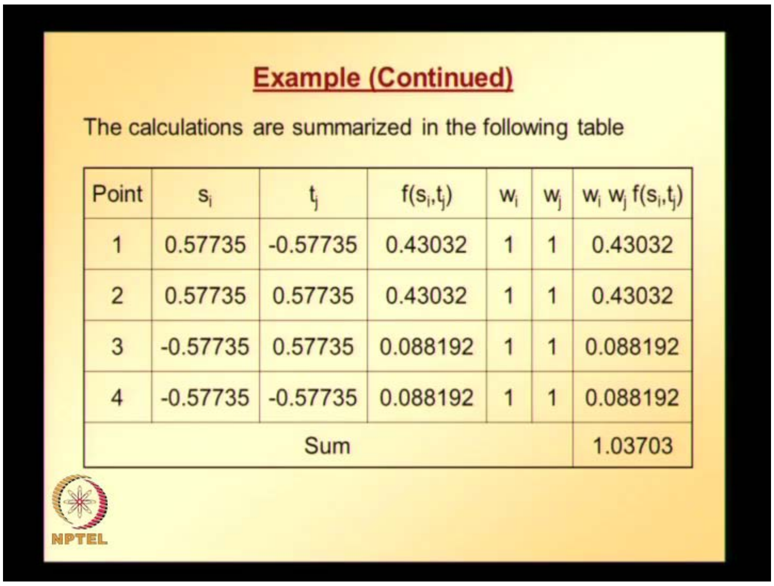

Point	s_i	t_j	$f(s_i,t_j)$	w_i	w_j	$w_i\, w_j\, f(s_i,t_j)$
1	0.57735	-0.57735	0.43032	1	1	0.43032
2	0.57735	0.57735	0.43032	1	1	0.43032
3	-0.57735	0.57735	0.088192	1	1	0.088192
4	-0.57735	-0.57735	0.088192	1	1	0.088192
Sum						1.03703

Using 2 by 2 integration formula, calculations are summarized in this table. 0.1 s and t locations and the corresponding function value at this point first point, the corresponding s and t values, weight along s direction and weight along t direction weight along s direction times and weight along t direction times function value at 0.1. Similarly, for 0.2, 0.3, 0.4 all the details are provided in the table. And the estimate of the integral is sum of all the values corresponding to 1 2 3 4 points in the last column. If we sum up, we get the integral estimate which turns out to be 1.03703.

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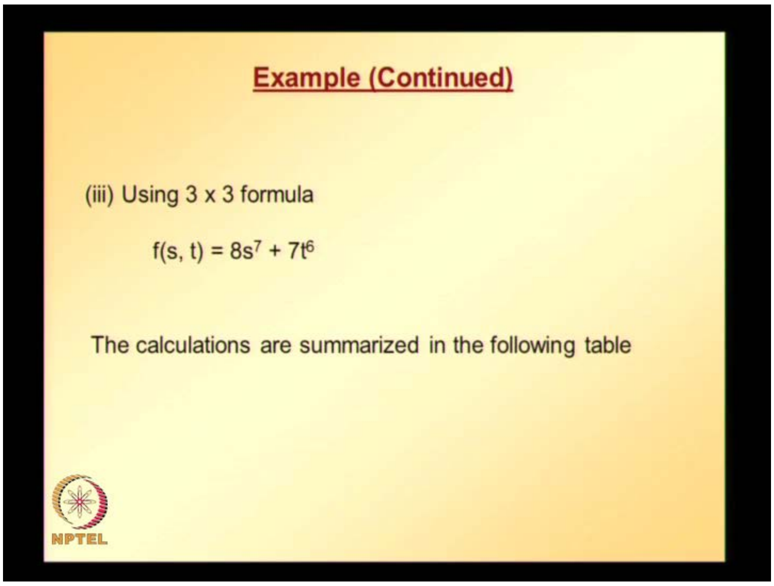

Now, let us try evaluating this integral using 3 by 3 formula. So, the function is shown here. So, we are trying to use three points along s direction; three points along t direction. So, similar to the previous case, the calculations are summarized in a tabular format.

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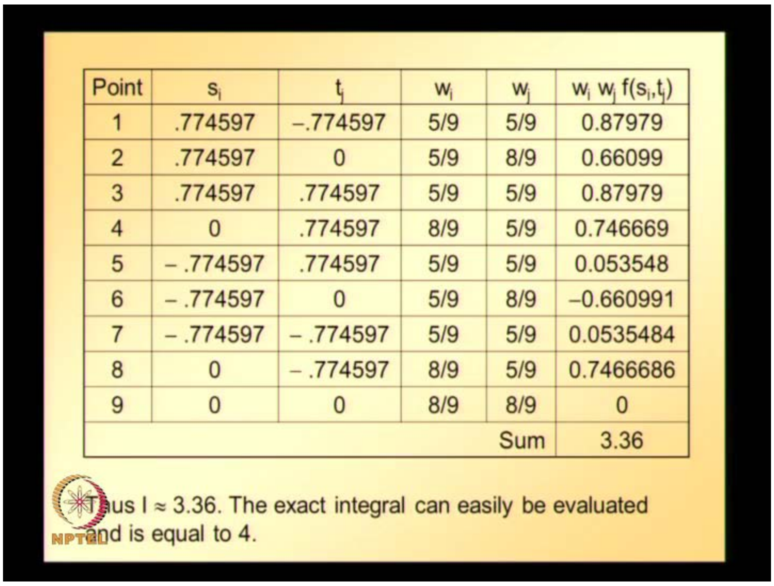

Point	s_i	t_j	w_i	w_j	$w_i\, w_j\, f(s_i,t_j)$
1	.774597	–.774597	5/9	5/9	0.87979
2	.774597	0	5/9	8/9	0.66099
3	.774597	.774597	5/9	5/9	0.87979
4	0	.774597	8/9	5/9	0.746669
5	– .774597	.774597	5/9	5/9	0.053548
6	– .774597	0	5/9	8/9	–0.660991
7	– .774597	– .774597	5/9	5/9	0.0535484
8	0	– .774597	8/9	5/9	0.7466686
9	0	0	8/9	8/9	0
				Sum	3.36

Thus $I \approx 3.36$. The exact integral can easily be evaluated and is equal to 4.

There are total nine integration points, corresponding s values t values are given; corresponding weights are given. The estimate of integral is nothing but sum of all the values corresponding to 0.1 to 9, which appear in the last column and if you sum up, it turns out to be 3.36. The estimate of this integral using 3 by 3 integration is 3.36, whereas the exact integral value is going to be 4. (No audio from 41:17 to 41:27) So, to increase the accuracy here, we required to go for 4 by 4 integration.

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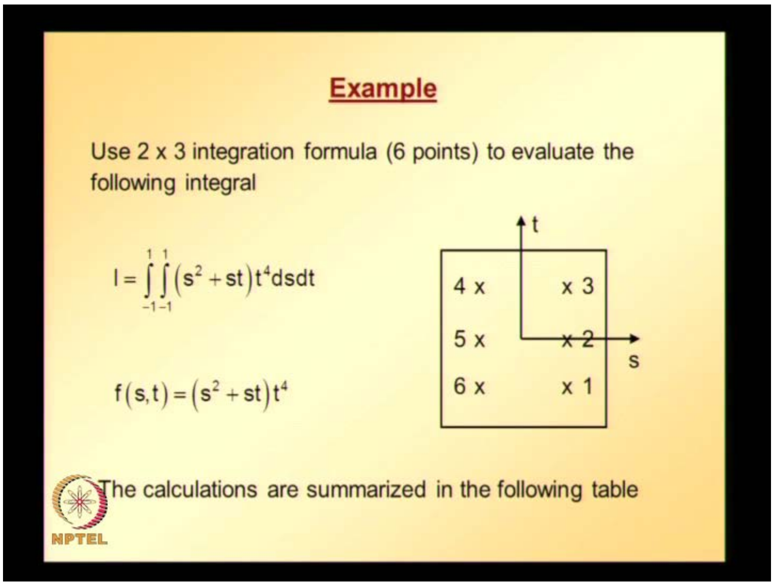

So, now let us take another integral and it is suggested to use to evaluate this integral. It is suggested to use two points along s direction; two three points along t direction. So, let us check whether this leads to the exact solution or not. The highest order along s direction the highest order of integrant along s direction is 2. So, 2 n minus 1 is equal to 2. So, n is equal to 1.5 or we required to round it off to side two. Similarly, the order of integrant along t direction is 5. So, 2 minus 1 is equal to 5. So, n is equal to 3.

So, we required to use two integration points along s direction; three integration points along t direction to evaluate this integral accurately. So, that is what is suggested. So, the locations of these integration six integration points along s and t directions are as shown in the figure and this is the integrant and the calculations are summarized. So, we required to evaluate this function at each of these points and multiply with the corresponding weights. The corresponding weights are weight along s direction times weight along t direction.

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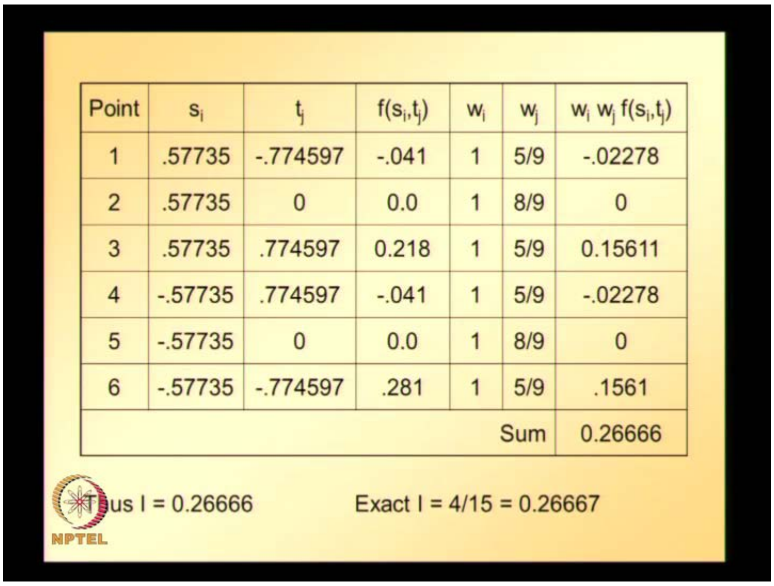

Point	s_i	t_j	$f(s_i,t_j)$	w_i	w_j	$w_i\ w_j\ f(s_i,t_j)$
1	.57735	-.774597	-.041	1	5/9	-.02278
2	.57735	0	0.0	1	8/9	0
3	.57735	.774597	0.218	1	5/9	0.15611
4	-.57735	.774597	-.041	1	5/9	-.02278
5	-.57735	0	0.0	1	8/9	0
6	-.57735	-.774597	.281	1	5/9	.1561
					Sum	0.26666

Thus I = 0.26666 Exact I = 4/15 = 0.26667

Again as in the previous cases, some of the value corresponding to all the six points 1 to 6 appearing in the last column. If you sum up, we get the estimate of the integral. And so, 2 by 3 integration results in 0.26666 and the exact solution also is going to be the same. This is about applying Gauss Legendre rules for evaluating integrals over rectangular area.

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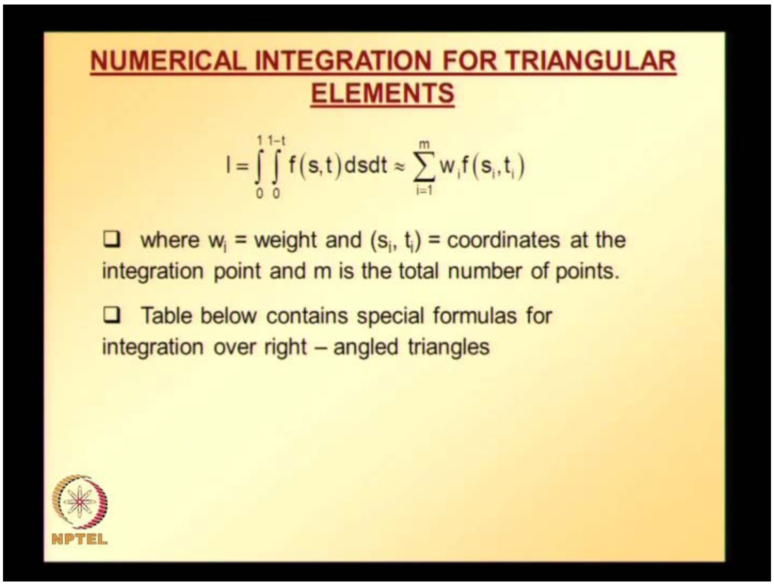

So, now let us look at integration for triangular elements. and this integration for rectangular elements And integration for triangular elements, these details are very important for finite element element equation assembly or finite element equation evaluation. So, integration over a triangular area is nothing but this kind of integral between the limits 0 to 1 minus t along s direction and between the limits 0 to 1 along t direction. We need to evaluate this integral f(s, t), double integral f(s, t) between the limits 0 to minus 1 minus 1 t sorry 0 to 1 minus t and 0 to 1 and that can be estimated similar to what we did for rectangular elements.

So, we need to find the integration points first and once we have the integration point location, evaluate the function at those integration point location, multiply with weight and sum up the contributions from all the integration points. So, where w i is the weight; s i t j are the coordinates of the integration points; m is the total number of points. And the table below shows the locations and the corresponding weights for the integration points and weights for integrating over a right angled right angled triangle.

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NUMERICAL INTEGRATION (Continued)

Integration Points and Weights for Triangles

Number of points	Degree of accuracy	Integration Points s	Integration Points t	Weights
1	1	1/3	1/3	1/2
3	2	1/6	1/6	1/6
		2/3	1/6	1/6
		1/6	2/3	1/6
4	3	1/3	1/3	-9/32
		1/5	1/5	25/96
		3/5	1/5	25/96
		1/5	3/5	25/96

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So, whatever the whatever may be the triangle for integration purpose, it is going to be mapped onto a right angle triangle, where s goes from 0 to 1 and t goes from 0 to 1. So, number of integration points and the degree of accuracy associated with adopting this number of integration points and the locations of integration points and corresponding weights are given in this table.

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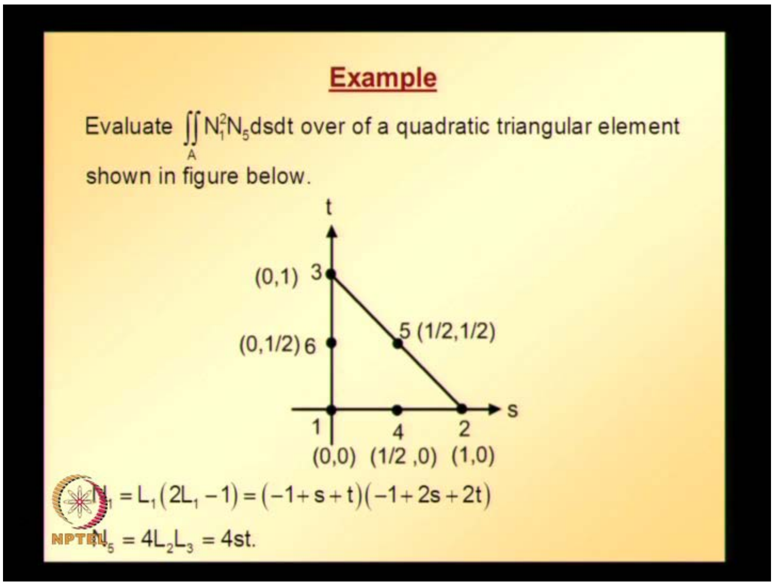

So, this helps us to evaluate this kind of integrals over a right angled triangle that is shown there, where the coordinates of all the key points are shown. In the figure, where this N 1 and N 5 are nothing but a quadratic or six node triangular element shape functions and they are given by this one. N 1 is equal to minus 1 plus s plus t times minus 1 plus 2 s plus 2 t and N 5 is given by four times s times t. So, we need to substitute this N 1 and N 5 values into the area integral N 1 square N 5, ds dt, and then we can decide the number of integration points. And then, we can evaluate the function at those those points and multiply with the corresponding weights to estimate this integral.

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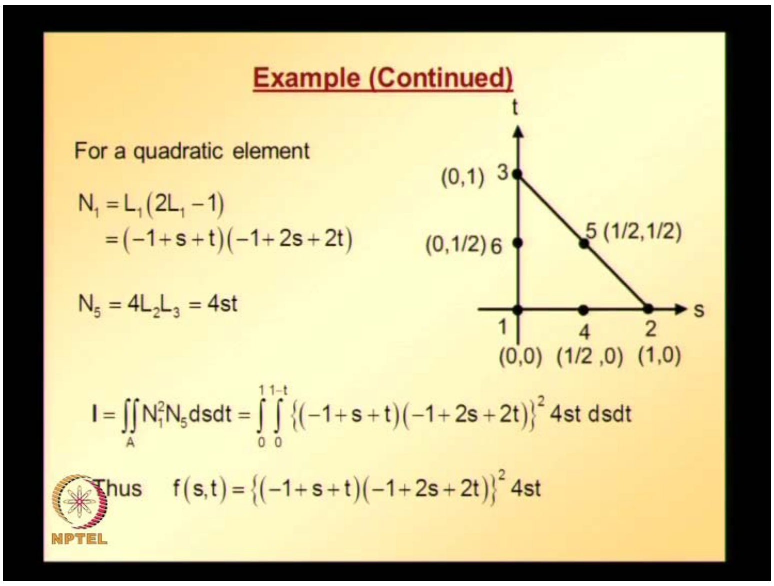

So, for a quadratic element, N 1 is this; N 5 is this. So, the integral becomes this one. So, integrant is nothing but minus 1 times sorry minus 1 plus s plus t times minus 1 plus 2 s plus 2 t whole square times 4 s t. So, that is the integrant.

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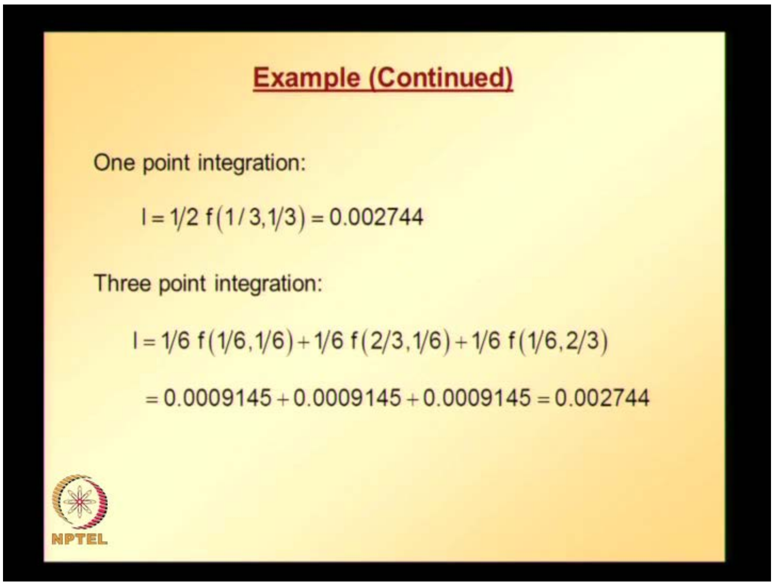

So, one point integration; the location of integration point is one third one third and the weight is half. So, one point integration results in 0.002744, three point integration results in this equation and which simplifies to 0.002744.

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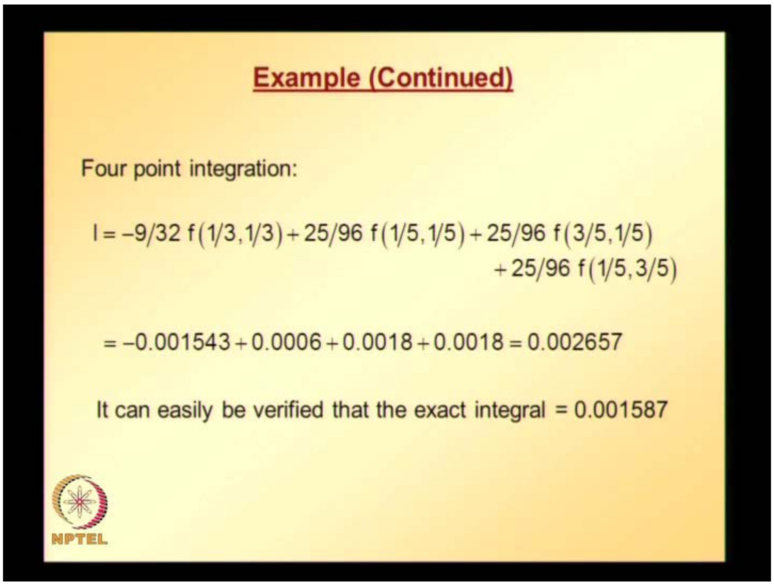

And four point integration results in 0.002657 and the exact solution or exact value of this integral is 0.001587. So, we required touse more number of points, and the corresponding locations and weights can easily be found in any of the standard text books on numerical methods or finite element method. So, this is basically the end of numerical integration; details of numerical integrations. This is the reason, why we adopt Gaussian quadrature for evaluating integrals in finite elements. For three-dimensional case, the extension to one more dimension is straightforward. So, the locations and weights can easily be extended for three dimensions following similar steps as we have done from one-dimensional to two-dimensional case.