Finite Element Analysis
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Lecture No. # 30

In the next few lectures, let us look at various types of integration schemes. And this helps us to get an idea why Gaussian quadrature is preferred for finite element calculations.

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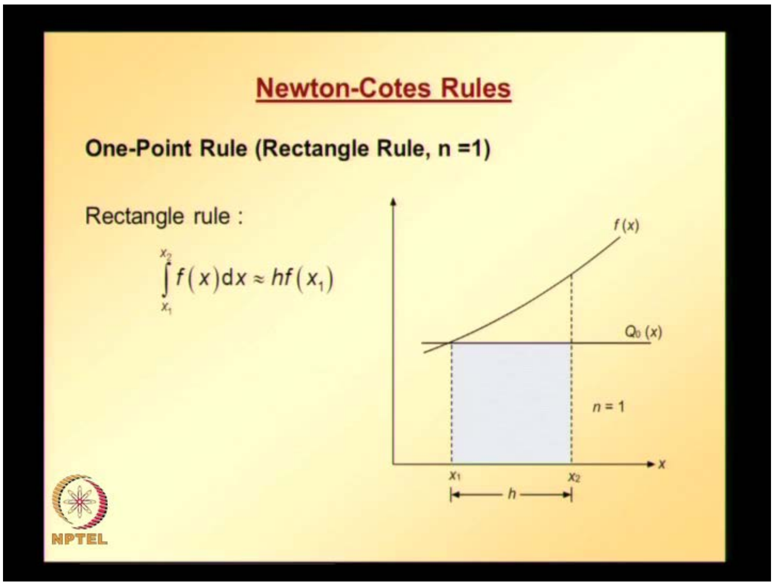

To start with it, let us look at what are called Newton-cotes rules. And under that we have the first one, one-point rule or rectangle rule. And rectangle rule basically for numerical integration purpose we are going to evaluate function at one point, so n is equal to 1. Basically, in rectangle rule if we have a function f (x) to evaluate integral f (x) within the limits x1 to x 2, it is approximated as the function value at one of the points that is lower bound that is x1 function value at x1 multiplied by a weight. Here, for rectangle rule weight is going to be the distance from x 2 minus x1 is going to be the weight, and function evaluated at a single point that is x1. By doing this basically, what we are doing is whatever with the type of function whether it is constant, linear, quadratic or any order of function where basically approximating that function as a constant function Q naught as shown in the schematic there.

Basically rectangle rule is good for evaluating integrals where f (x) is a constant. Since we are evaluating function at only one point as you can see in the figure, the difference between the area enclosed by f(x) between points x1, x 2. And the area enclosed by the curve q not between the points x1, x2. This difference in area is going to be the error that is associated with this rectangle rule, that is the accuracy that we are losing when we adopt rectangle rule if the function is not a constant function, but for a constant function rectangle rule or one-point rule or Newton-cotes rules with n is equal to 1 results in accurate solution.

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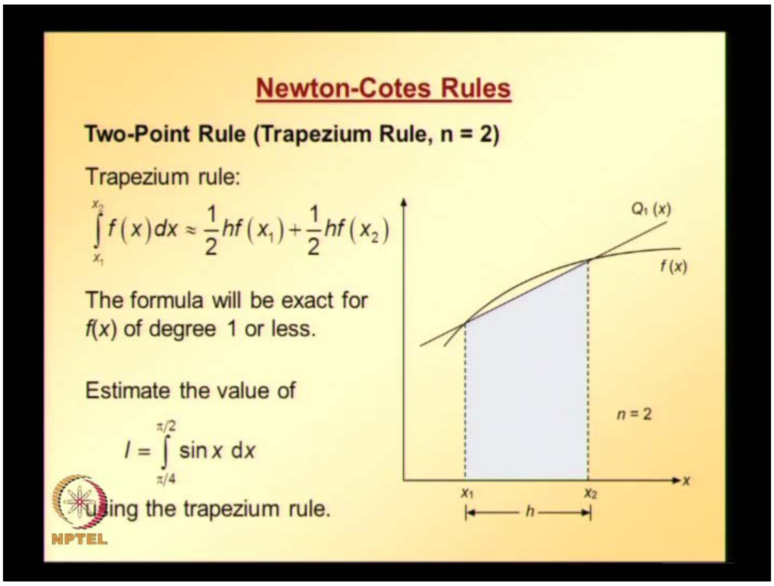

Now, let us to look at two-point rule or trapezium rule. The function is going to be evaluated at two points n is equal to 2 as in the figure function f (x) is shown. Basically, in trapezium rule to integrate f (x) between x1 and x2 what we are going to do is? We are going to evaluate this function f (x) at two points. The two points are pre-decided and two points are the limits between, which we want to find this integral this f(x) is going to be evaluated at lower limit x1 and f x is going to be evaluated at upper limit x2.

Once we have this function values, the function value at x1 is multiplied with a weight which is half the distance between x1 and x2 or x2 and x1. Similarly, the function value at x2 is multiplied with a weight which is equal to half the distance between x2 and x1 is h over 2.

Basically in trapezium rule, we are replacing whatever function that function we are replacing with a linear function q1 as shown in this schematic. Whatever a error associated with this trapezium rule is the difference in area between the area enclosed between the area enclosed by the curve between the points x1, x2. And the area enclosed by q1 between points x1 and x2. The difference between these two areas is going to be the error associated with trapezium rule, and trapezium rule is going to be accurate, if function f (x) is a linear function or trapezium rule is going to be accurate, if the function order of the function is 0 or 1. Even for constant function trapezium rule is going to result in accurate solution, so this is about trapezium rule. The formula will be exact for function of degree one or less.

To illustrate the procedure let us try to evaluate this integral sine x between the limits phi over 4 and phi over 2, so lower limit is phi over 4, upper limit is phi over 2 using trapezium rule. Basically what we require is, we need to find what is the distance between upper limit and lower limit phi over 2 minus phi over 4 that is going to be h. Once we have h, we need to evaluate function that is sine x at phi over 4, and sine x at phi over 2 multiplied with corresponding weights, which is going to be h divided by 2. In this case it is going to be phi over 4 times half which is going to be phi over 8 all those details are given here.

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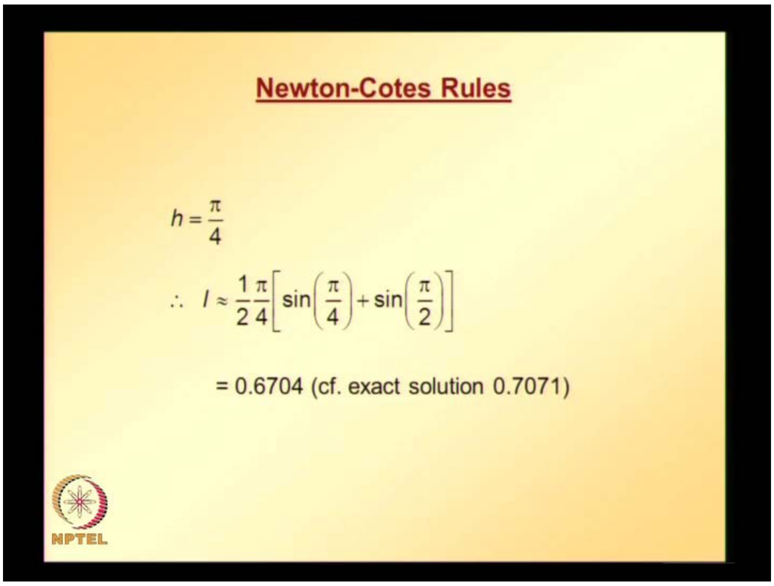

When h is equal to phi over 4 for this particular case, therefore the integral is approximated in this manner and the approximate solution turns out to be 0.6704. And this function can also be integrated exactly. So, the exact solution if we compare with exact solution, the approximate solution that we just obtained using trapezium rule it turns out that exact solution is 0.7071 that is the accuracy that we get using trapezium rule.

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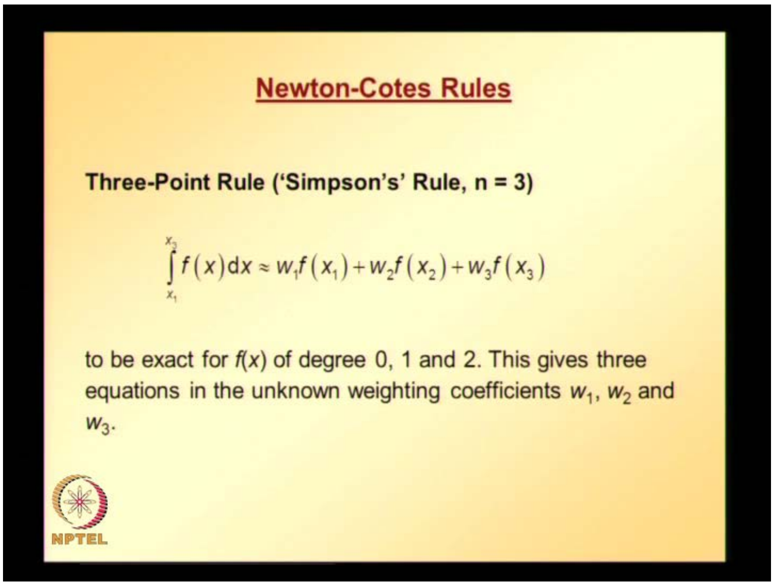

Now, let us go and see what is called Simpson's rule or three point-rule where we are going to evaluate function at three points. So, n is equal to 3 integral f (x) between the limits x1, x 3 intentionally the upper limit is named as x 3. If one wants to evaluate this function integral of this function between limits x1 and x3 that is approximated using Simpson's rule in this manner weight function at x1 multiplied by function value at x1 plus, weight function at x2 multiplied by function value at x2, weight function at x3 multiplied by function value at x3. So, x1 let us see where this x2 is this x2 should be between x1 and x3. Once we know upper limit and lower limit we can easily figure out what is x1 value, what is x2 value, and what is x3 value. We can easily evaluate function at these points and only thing unknowns are W1, W2, and W3.

Let us see how we can evaluate this W1, W2, W3 .And the procedure that we are going to follow for evaluating these weights is going to be a general procedure which also helps us to understand in gauss quadrature how locations of integration points and weights are obtained. The Simpson's rule is going to be exact for a function of degree 012. And this gives three equations in three unknown waiting functions W1, W2, and W3. Basically, if you see three-point rule, it is going to be exact for a function of order two. If you take trapezium rule, it is going to be exact for a function of order one. And if you take rectangle rule, it is going to be exact for function of order zero. So, what we can conclude from here is Newton-cotes rules is that if you adopt n points, we can integrate a function of order up to n minus 1 accurately using Newton-cotes rules.

Newton-cotes rules can either be rectangle rule, trapezium rule or Simpson's rule or we can go for n is equal to 4 or n is equal to 5, whatever it may be Newton-cotes rules are going to be exact for a function with n points. Newton-cotes rules with n points are going to be exact for a function of order n minus 1.

Now, let us see the procedure how to find these weights W1, W2, W3 to get to solve for 3 unknowns, we know that we require 3 equations to get unique solution for 3 unknowns. To evaluate this W1, W2, and W3 we are going to come up with 3 equations in terms of W1, W2, W3. And solve these 3 unknowns using these 3 equations that is the basically the procedure to get these weights W1, W2, and W3.

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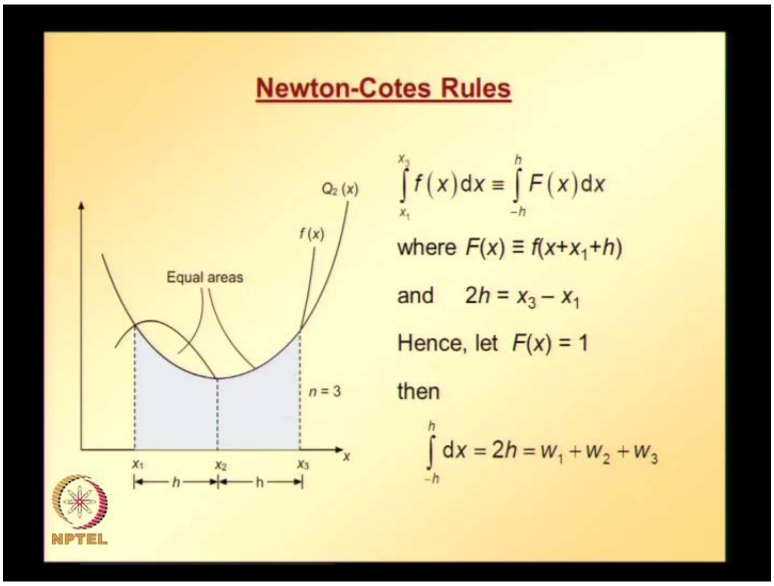

Now, let us look at the procedure. Basically in Simpson's rule a function f (x) is replaced or it is approximated as a quadratic function, q2 as shown in the figure. And this function is evaluated at 3 points x1, x2, and x3. And x2 is midway between x1 and x3 and the distance between x1 and x2 is h, x2 x3 is h. So, our job is to evaluate this function integral f(x) between the limits x1 and x3. Since x1 corresponds to minus h and x2 corresponds x3 corresponds to h. If you take x2 as origin, x1 is at a distance minus h and x2 is x3 is at a distance h. We can replace these limits x1 x3 with minus h and h. Similarly, since the function is initially is with respect to the origin that is shown with respect to the x axis. Since, we are defining a new axis this function also needs to be expressed in terms of new origin that is x2. So, this f(x) is replaced with F(X) it can be easily verified that this F(X) and f(x) values at any point along the curve are going to be the same. So, the distance between x3 and x1 is 2h.

The procedure goes like this we need to determine 3 unknowns W1, W2, and W3. We will start with a F (X) is equal to 1. When F(X) is equal to 1 integral minus h to h .And 1dx becomes 2h and the function value taken at any point that is x1, x2, x3 is going to be the same which is going to be equal to 1. The previous approximation W1 times function evaluated at x1 plus W2 times function evaluated at x2 plus W3 times function evaluated at x3 it becomes 2h is equal to W1 plus W2 plus W3 we got one equation now.

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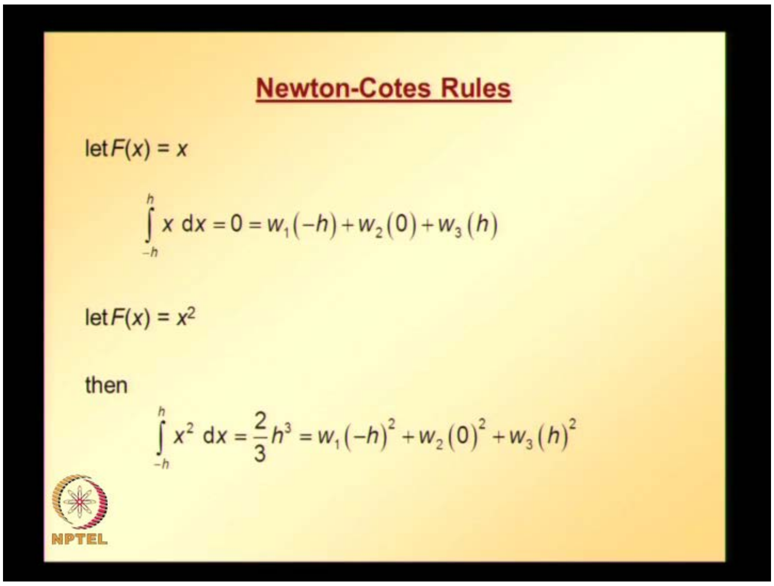

Let us say F(x) is equal to x, again apply that equation which we have already seen. So, replace f (x) with x integrate between the limits minus h and h, x d x and since x is a odd function. And we are integrating odd function over a symmetrical domain minus h to h. The integral turns out to be 0. And this should be equal to or this is approximated as W1 times function evaluated at x1here in this case x1 is nothing but minus h. So, W1 times function itself is x f(x) is equal to x function evaluated at x1 is equal to minus h, function evaluated at x2 is equal to 0. Function evaluated at x3 is equal to h. Using this information, we get this equation W1 times minus h plus W2 to times 0 plus W3 times h. If we simplify it, we get equation in terms of W1 and W3 and since the W2 or h function evaluated at x2 is equal to 0.

Now, let us get another equation by letting f is equal to x square .And that results in the second equation that is shown that is minus h to h x square d x, x square even function. And we are integrating over a symmetrical domain. So, we are going to get some non-zero value which turns out to be 2 over 3 h q is equal to W1 times function evaluated at x1. And since x1 is same as minus h that is going to be minus h square.W2 times function evaluated at x2, x2 is same as 0. It is going to be 0 square. W3 times function evaluated at x3 function evaluated at x3, function evaluated x3 square is h .It becomes W3 times h square that is how we obtained this equation. We got 3 equations since we need to determine 3 unknowns. We use this condition that f (x) is equal to 1, f(x) is equal to x, f(x) is equal to x square. And using these conditions we got 3 equations.

Now, we can solve these three equations for these three unknowns W1, W2, and W3. We imposed while deriving these weights W1, W2, W3 itself we impose these conditions that the function has to be exact up to quadratic, that is x square Simpson's rule is going to be exact for a function of order 2 that is quadratic function.

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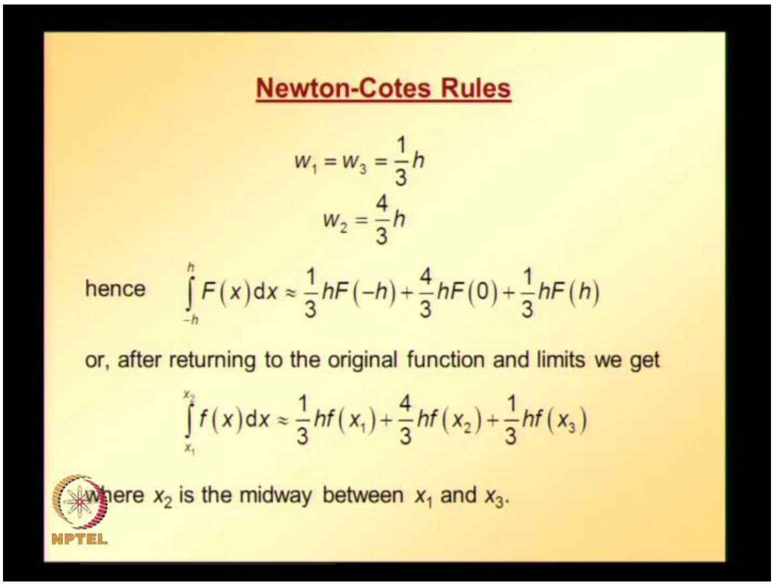

With solving these three equations, we get W1 is equal to W3 which turns out to be h over 3 W2 that is turns out to be four h over 3. And the function or the integral f(x) between the limits minus h and h is approximated as one third h times function evaluated at minus h, four third times function evaluated at 0 plus one third times function evaluated at h. The first one should be at minus h or after returning to the original function and its limits we get this equation. And where we need to keep a note at x2 is midway between x1 and x3. Basically, this is the procedure that we can generalize for any number of points any number of points.

Newton-cotes rules of any number of points to obtain the weights this is the procedure that we can generalize. If one wants to derive Newton-cotes rules, weights for a Quartic function that is a fourth order function then one needs to start getting four sorry five equations. We need to start with f(x) is equal to 1, f (x) is equal to x, f(x) is equal to x square, f (x) is equal to x cube, f(x) is equal to x power 4. If it is a cubic function, we need to impose the condition f(x) is equal to 1, f(x) is equal to x, f (x) is equal to x square, f (x) is equal to x cube. If it is Quartic function we already discussed .And if it is fifth order function, we need to impose conditions f(x) is equal to 1 2. And f (x) is equal to x power 5. We get six equations there we need to use six weights or we need to use Newton-cotes rules n is equal to 6.

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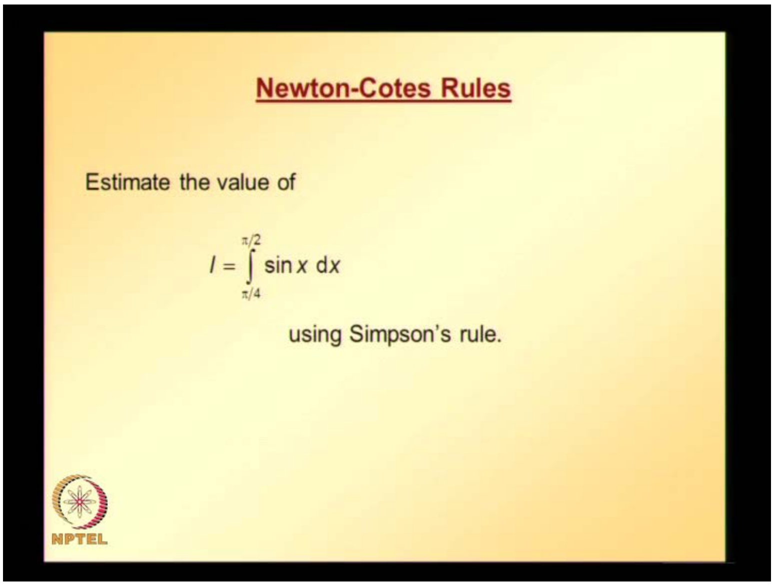

Now, let us take this example which we already solved using trapezium rule integral sine x between the limits phi over 4, and phi over 2 using Simpson's rule. x1 is going to be phi over 4, x 3 is going to be phi over 2, x2 is going to be between that is midway between phi over 4 and phi over 2.

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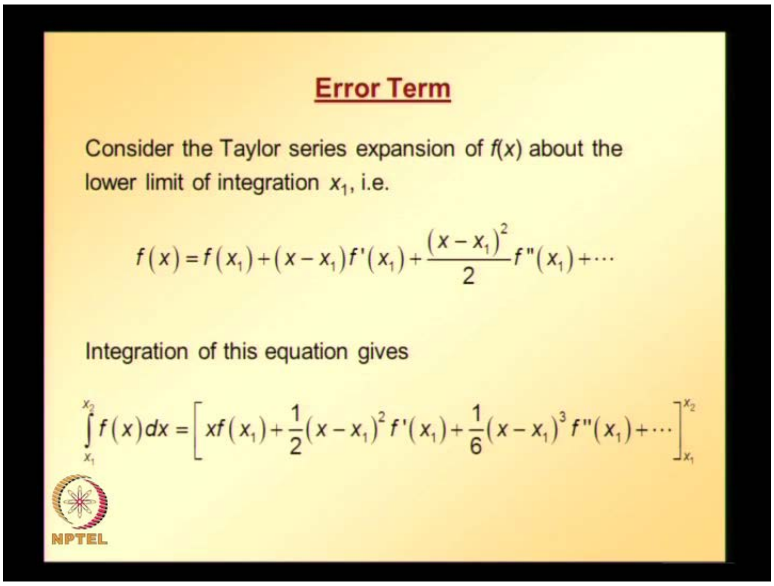

Solution h is equal to phi over 8 and function value evaluated at these 3 points, and multiplied by the corresponding weights results which can be simplified to this. And we and we compare this solution with exact solution which is 0.7071, we can observe that it is very closely matching with exact solution, because Simpson's rule we actually evaluated function at more number of points.

Now, let us look at what is the associated error with each of these 3 rules, 3 Newton-cotes rules that is rectangle rule, trapezium rule and Simpson's rule to do that. To do that lets consider a Taylor series expansion f of x about the lower limit of integration x1. If we do Taylor series expansion of f(x) about the about x1 we get this equation. And integrating this equation on both sides between the limits x1 and x2 results in this which can be simplified to truncated by rectangle rule.

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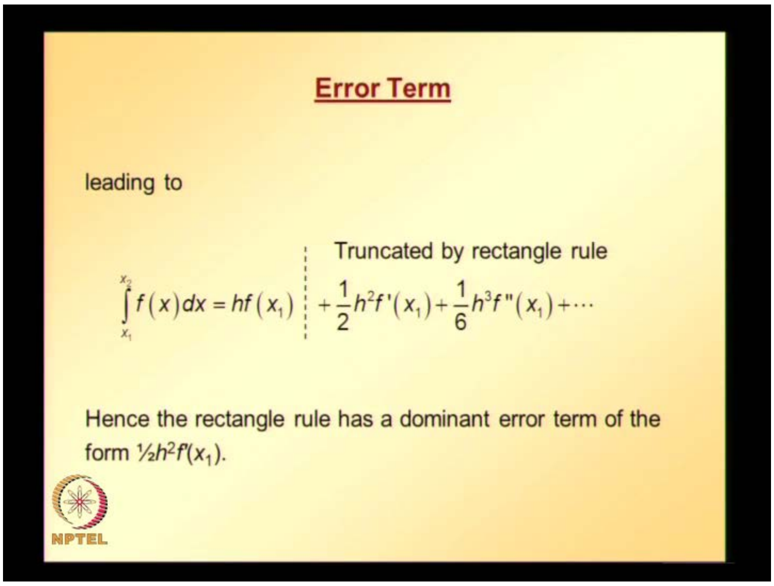

Basically, if you recall for rectangle rule to integrate function between the limits x1, x2 what we are doing is we are evaluating function at x1 lower limit .And multiplying with a weight which is equal to x2 minus x1. And we are not considering the higher order terms that are shown in the series there. So, all the higher order terms are truncated by the rectangle rule. The terms that are truncated by the rectangle rule are shown in the equation. Rectangle rule has a dominant error term of order or of the form h o, x square over two first derivate of f evaluated at x1. So, this is the error associated with rectangle rule. Similarly, we can derive or we can find what is the error associated with trapezium rule.

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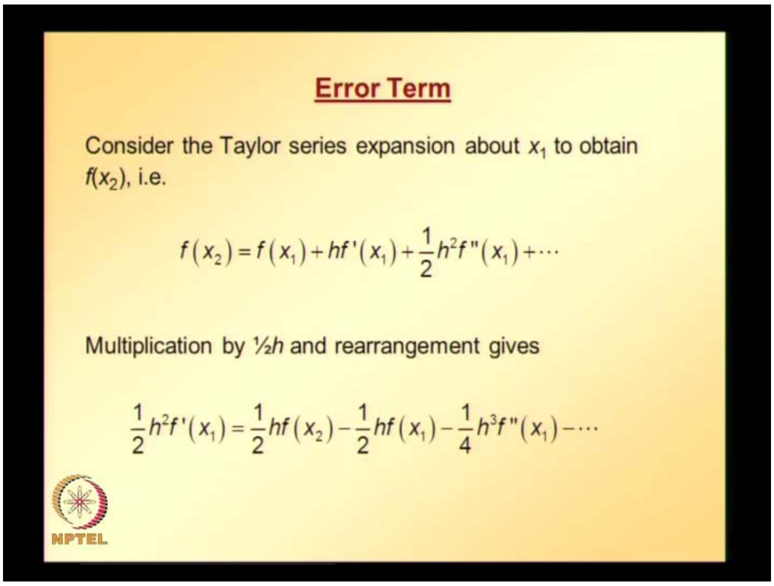

Consider Taylor series expansion about x1 to obtain x2, we get this and multiplying by h over 2 and rearranging, we get this equation. Now, what we are going to do is? We are going to substitute this; that is half h square times first derivative of f evaluated x1 into the previous equation that we already looked for when we are deriving error term for rectangle rule, which is reproduced here.

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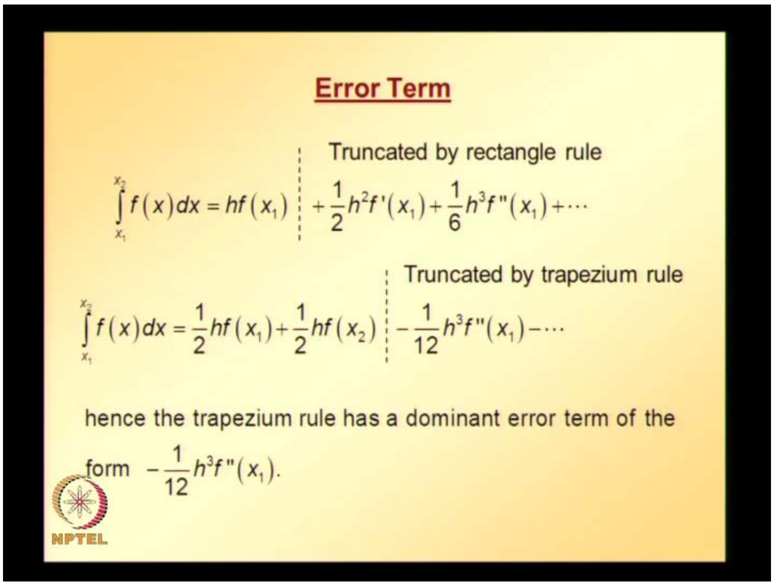

We are going to substitute the previous equation into this equation. And that results in this equation. And if you see the second equation, basically it is same as the formula that we looked for trapezium rule. Except we have some higher order terms which are truncated by trapezium rule. Trapezium rule has a dominant error term of whatever term highest term that we did not consider in the trapezium rule; that is minus one over twelve h cube second derivative of function evaluated at x1. This is the dominant error term associated with trapezium rule.

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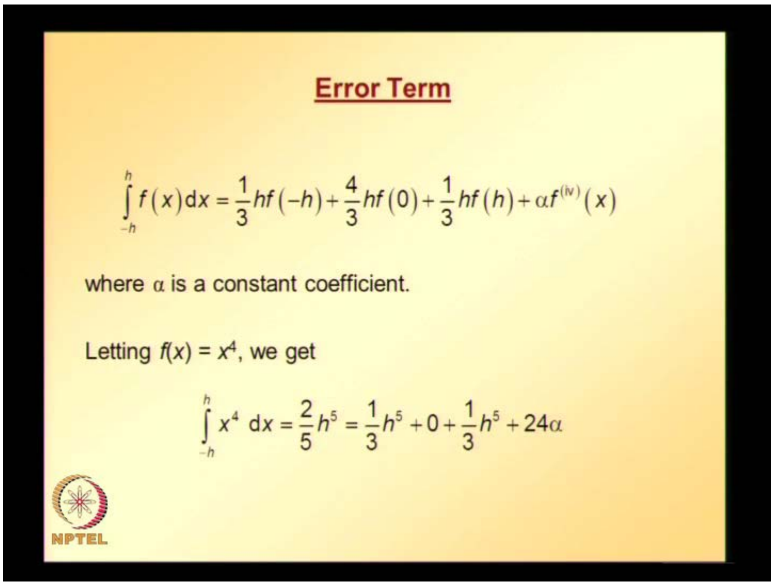

Similarly, we can derive error term dominant error term associated with Simpson's rule. Basically in Simpson's rule we approximated function in this manner. And where higher order terms are grouped into the form or the dominant error term is shown as alpha times fourth derivative of function.

Now, let us take where α is a constant coefficient. And then let us take a function let f(x) be x power 4 when we substitute this fourth derivative of x power 4 is four times, three times two that is going to be fourth derivative of f(x) is going to be 24. And alpha is there, so 24 plus 24 alpha that is the error dominant error term associated with Simpson's rule. Now, we can also evaluate this function exactly that is minus h to h x power 4 d x x power 4 is an even function. And we are actually integrating it over a symmetrical domain, so the integral value turns out to be a non-zero value which is going to be 2 over 5, h power 5, so we can easily back calculate, what is alpha?

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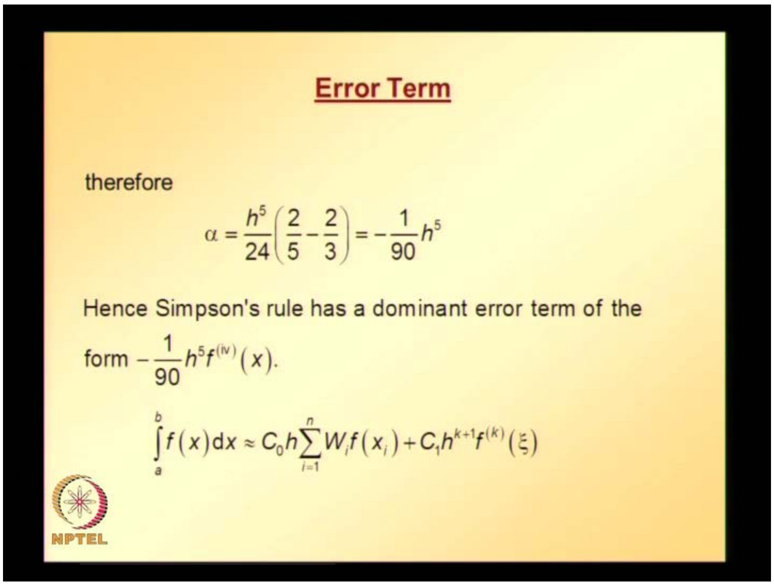

The dominant error term associated with this Simpson's rule is minus 1 over 90, h power 4, fourth derivative of function. So, this is how we can actually find what is the dominant term associated with any of these Newton-cotes rules. To summarize a function between the limits or integral between the limits a to b f x dx can be approximated using this Newton-cotes rules using this formula. The second one second formula sorry second term of this formula is associated with the dominant error term for that particular rule. And where the constant C's c naught W's and c 1 all those details are given in this table.

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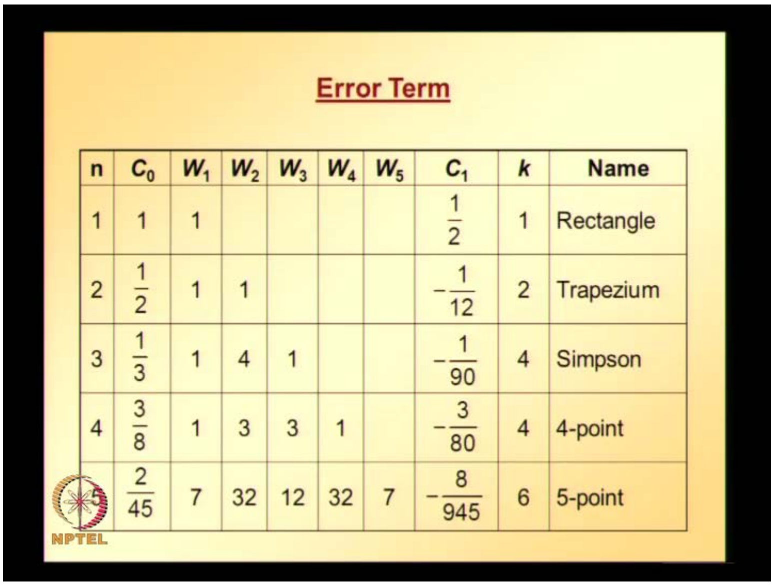

If one wants to select n is equal to 5 they can easily go through this table. And find what is c not and corresponding W's, and also C1 gives you idea about the a dominant error term this is about Newton-cotes rules that is rectangle rule, trapezium rule, Simpson's rule. Sometimes, we may not evaluate any integral at a single stretch using any of these rules instead of that we can use them repeatedly.

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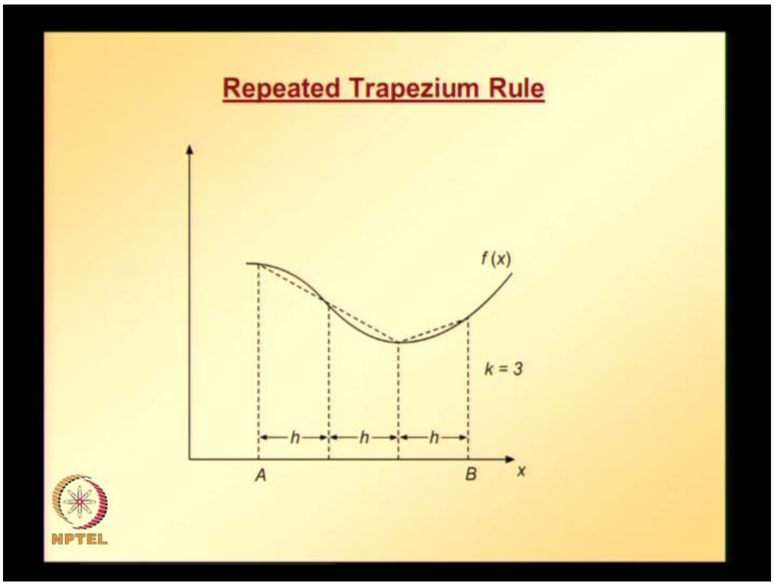

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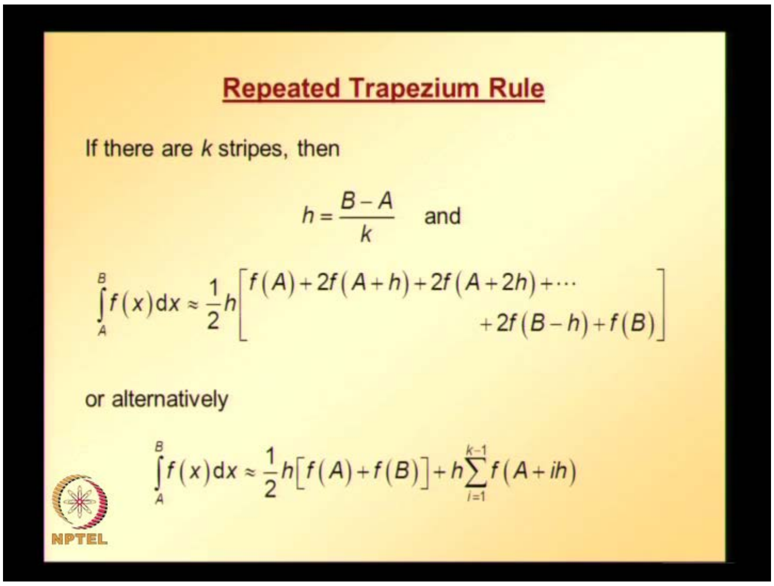

Now, let us look at what is called repeated trapezium rule. Instead of integrating a function between the limits A to B at a single stretch, we can actually divide this into number of strips here. For example, function f (x) needs to be evaluated between the limits A to B. And the region A to B is divided into three strips. And width of each strip is B minus A divided by k, where k is the number of strips. Here in this case, it is 3 in the previous figure whatever we have seen k is equal to 3, but it can be something else for a different problem. For this each strip, we need to apply trapezium rule. And sum it up then we are going to get the repeated trapezium rule for this problem or for this integral. And that is going to be as shown in the equation. And this equation can be rearranged in this manner, so this is about repeated trapezium rule.

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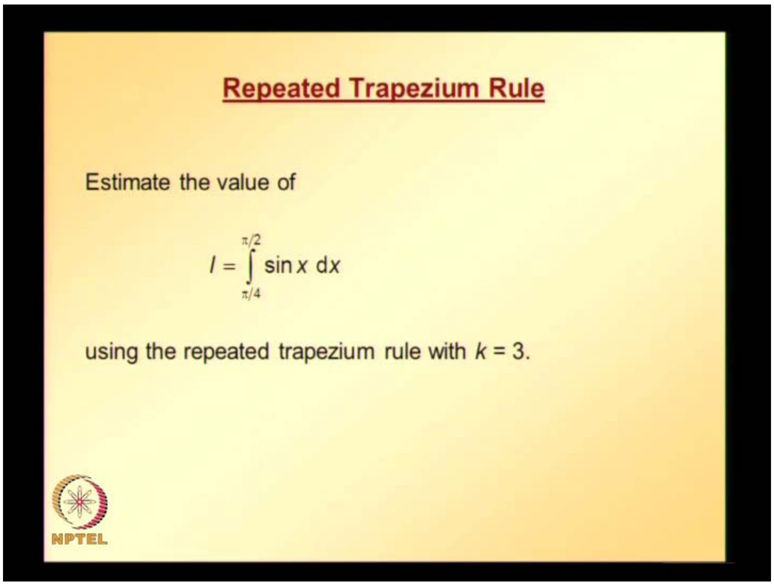

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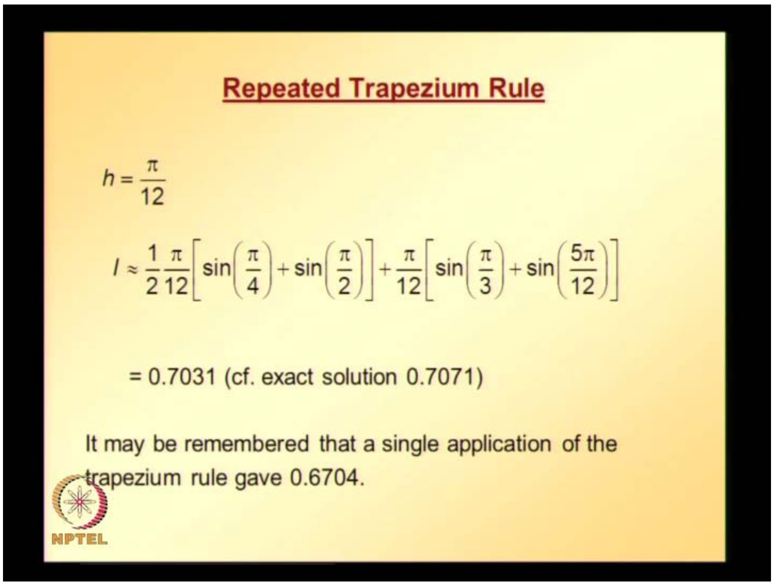

The previous examples example, which we already considered can be solved using repeated trapezium rule with k is equal to 3 applying this formula only thing is we need to know what is h here, h is going to be upper limit minus lower limit divided by k. Doing calculation it turns out that k is equal to phi over 12 for this problem. And applying this formula we get approximate value of the integral as given in this equation. And simplification of this results in this value 0.7031. And a comparison of this with exact solution which is 0.7071 shows that the accuracy is improved. It may be remembered that a single application of trapezium rule gave 0.6704.

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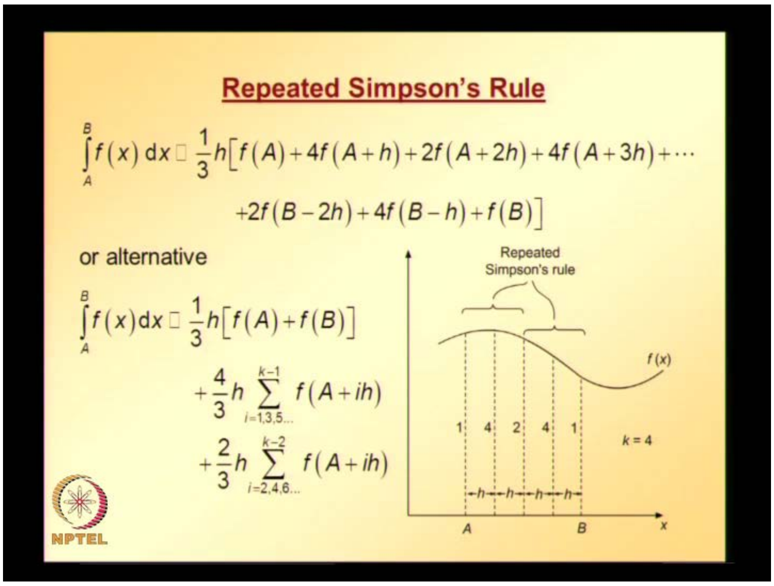

Now, let us look at repeated Simpson's rule. We need to apply same l logic instead of integrating at a single stretch a function or integral between the limits A to B, we divide this width A to B into number of strips of width h. So, h value depends on the number of strips, and the distance between A to B when we apply this repeated Simpson's rule integral f(x) between the limits a to b turns out to be this one or which can be further simplified. Here I think, because of fund problem approximation symbol is not coming properly.

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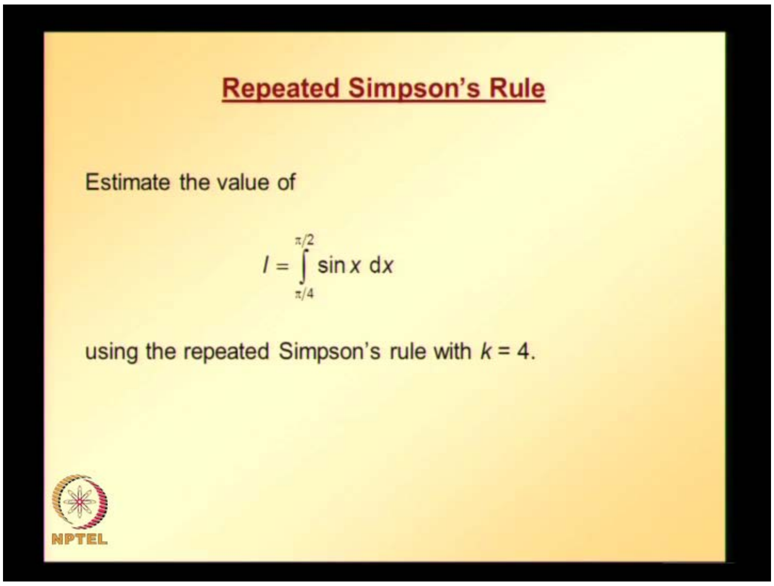

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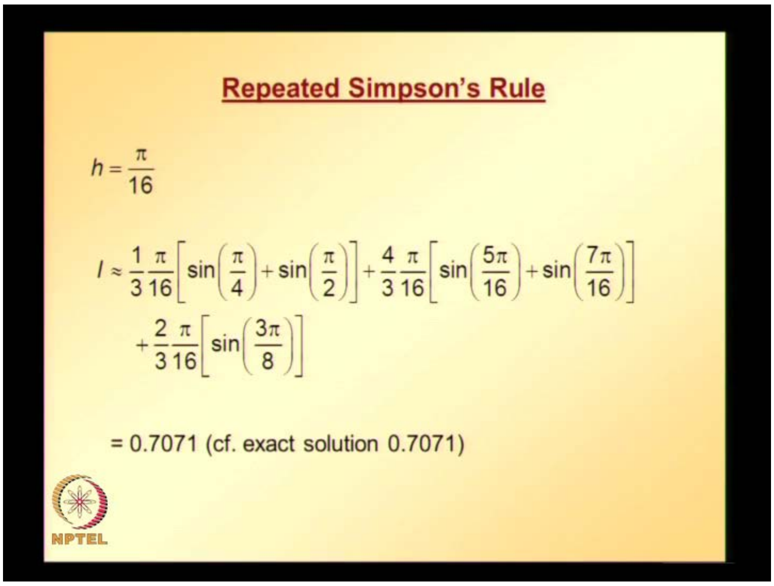

Now let us apply Simpson's rule to estimate this integral taking k is equal to 4 width of each strip is going to be upper limit minus lower limit divided by 4. And it turns out that h is equal to phi over 16. And applying repeated Simpson's rule results in this approximation of the given integral which can be further simplified as 0.7071. And exact solution matches very accurately with exact solution the four significant digits that are shown there so, this is about Newton-cotes rules.

In summary, Newton-cotes rule of n number of point results in exact solution of integral of order n minus 1 or if we want to integrate a function of order n exactly, we need to use a Newton-cotes rules having n plus 1 number of points. If we want to integrate a function of order 0, that is a constant function we need to use 1 point. And if we want to integrate a linear function, we need to use 2 point. If we want to integrate a quadratic function, we need to use 3 points.

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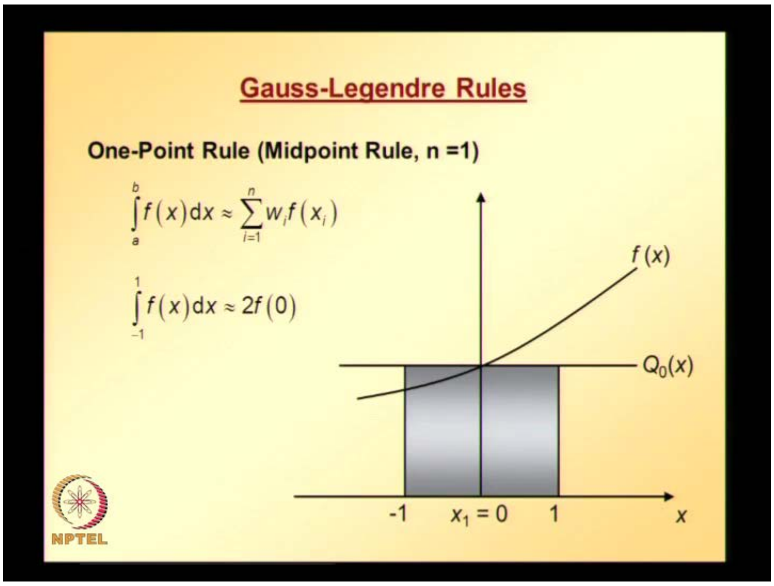

Now, let us look at Gaussian quadrature also called as gauss Legendre rules. We will understand why Legendre word is there. Now in gauss Legendre rule basically for one-point rule or its also called midpoint rule that is n is equal to 1.We are going to evaluate function at a single point which is midpoint or this is the equation that is shown here is a general equation for gauss Legendre rules or gauss quadrature that is integral f (x) between the limits A to B is approximated. As function evaluated at certain points multiplied by the corresponding weights for one-point rule n is going to be equal to 1.Function sorry integral minus 1 to 1 f x dx is approximated as some weight times function evaluated at midpoint of minus 1 to 1. What is midpoint of minus 1 to 1? It is 0. So, function is evaluated at 0.

And weight here is 2, so two times function evaluated at 0. Basically, whatever may be the order of function whether it is constant, linear, quadratic, cubic, quadratic or fifth order function. Basically it is replaced with a constant function as shown in the figure. f (x) is replaced with q not and in one-point rule this integral minus 1 to 1 f(x) dx is evaluated by using the function evaluated at the midpoint of the range over, which integration needs to be performed multiplied by the weight which is turns out to be 2 here for this one-point rule.

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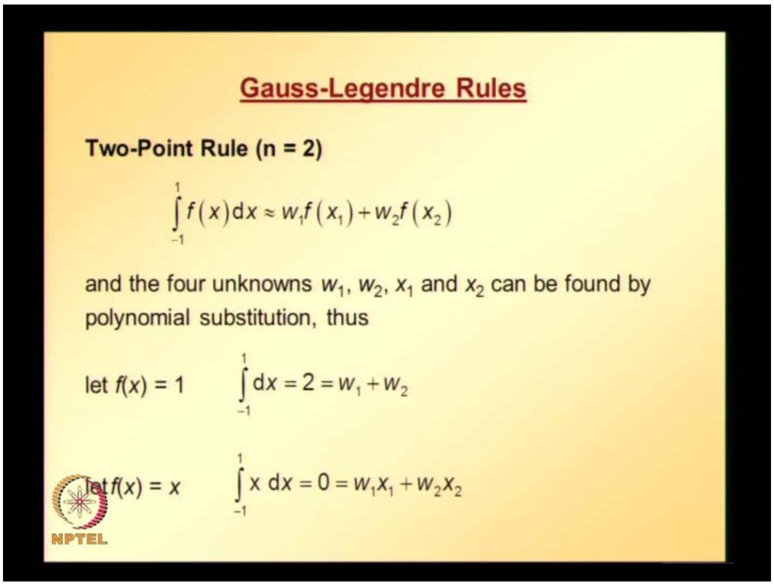

Now let us look at how this two is obtained, how zero is obtained for location, how is obtained for the weight? We will understand it better when we look at three point rule. Now, let us look at two-point rule that is n is equal to 2 again function f (x) or the integral f (x) minus 1 to 1 dx is approximated as W1 times f (x1) function evaluated at x1 plus W2 two times function evaluated at x2. Now, let us understand how to get these weights, and weights W1, W2 and x1, and x2. Just before I made a statement that the procedure will be cleared when we take n is equal to 3, but let us look here itself for two point rule how to get this weights and locations x1 and x2.

There are four unknowns to be determine x1, x2, W1, W2. So, one can easily guess we require four equations to uniquely determine these W1 W2 and x1 x2, So, what we will do is we will play the same or we will use the same logic as what we did for getting the weights for Simpson's rule. Let us start with f(x) is equal to 1. We need to get 4 equations, so let us start with f (x) is equal to 1, f (x) is equal to x, f(x) is equal to x square, f(x) is equal to x cube and we get 4 equations. When we substitute f(x) is equal to 1.We get the equation integral 1 dx between the limits minus 1to 1 is going to be 2. And substituting f (x) function evaluated x1 is equal to 1, function evaluated at x2 is equal to 1, we get this equation. Similarly, at f (x) is equal to x again x is a odd function we are integrating over a symmetrical integral domain, integral turns out to be 0 .And f(x) is equal to x means function evaluated at x1 is going to be x1. And function evaluated x2 is going to be x2. We will get this equation 0 is equal to W1, x1 plus W2 x2.

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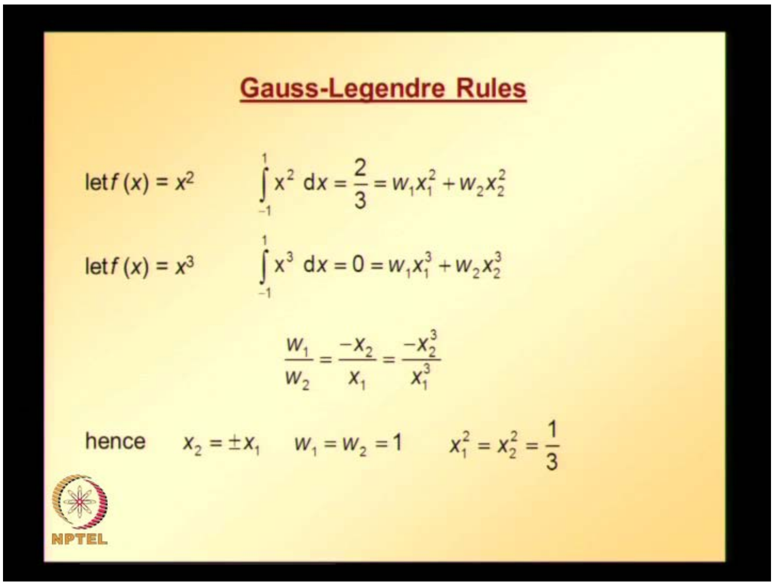

Now, let f(x) is equal to x square, again x square is an even function integrating this over a symmetrical domain results in a non-zero value, which is going to be 2 over 3. And function evaluated at x1 is going to be x1 square. And function evaluated at x2 is going to be x2 square which gives us this equation 2 over 3 is equal to W1 x square plus W2 x square, W1 x1 square plus W2 x2 square sorry, f (x) is equal to x cube, again x cube is odd function where integrating over symmetrical domain integral turns out to be 0, f(x1) function evaluated x1 is going to be x1 cube, function evaluated at x2 is going to be x2 cube that results in this equation 0 is equal to W1 x1 cube plus W2 x2 cube. So, we got the four equations that we actually looking for. We can solve these four equations for the four unknowns x1, x2, W1, and W2. And the solution results in these equations x1 or x2 is plus or minus x1, W1 is equal to W2 is equal to 1. And x1 square is equal to x2 square is equal to one third. And simplification of previous equation results in this and W1 is equal to 1 and W2 is equal to 1.

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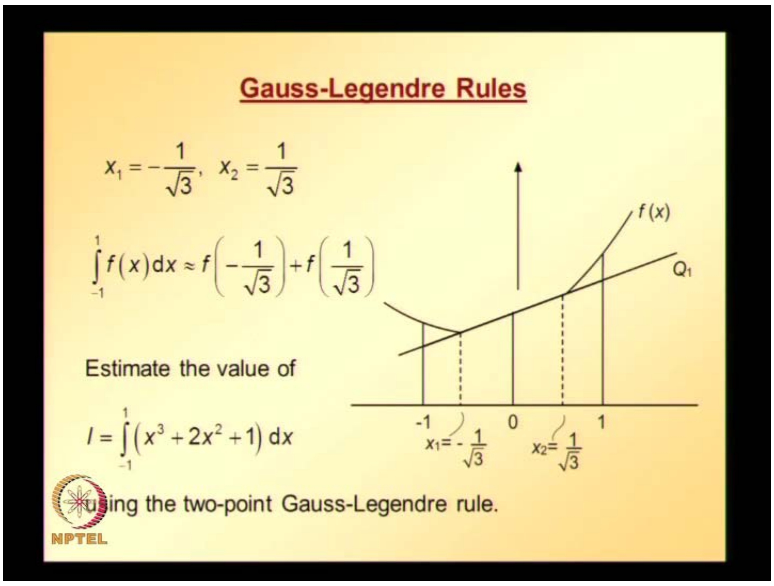

According to two-point gauss-Legendre rule, we are going to evaluate function at minus 1 over root 3 and multiply with corresponding weights. And add them together that is corresponding weights are 1. It is basically function evaluated at minus 1 over root 3 plus function evaluated at 1 over root 3. And this is what is schematically shown in the figure. The original function f x is replaced with a linear function or function q1 original function is replaced with a function q1. And the function is evaluated at two points x1 and x2. Estimate the value of this integral and the highest order of integrand here is 3.

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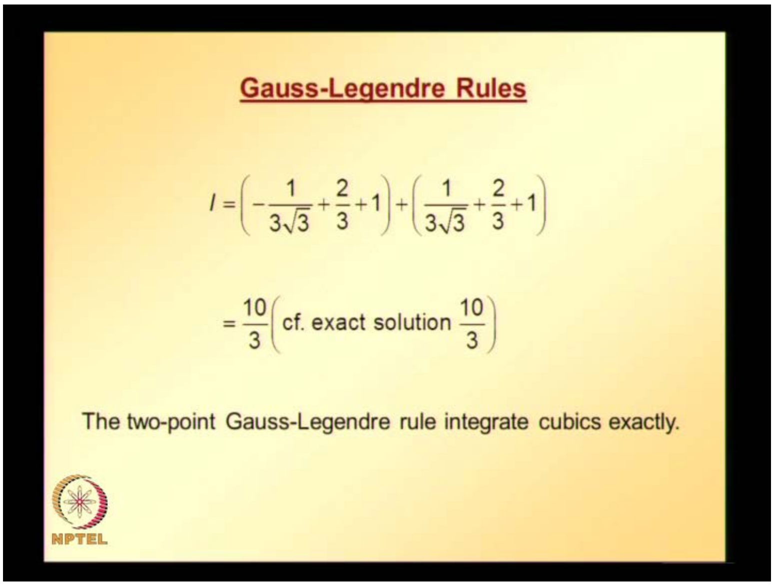

We need to evaluate function by substituting in the previous equation x is equal to minus 1 over root 3 and x is equal to 1 over 3, we get this. And it turns out that the approximate solution is same as the exact solution. What we can learn from this is? If we use Gauss-Legendre rule or if we adopt Gauss-Legendre rule, we can integrate a function of order two n Gauss-Legendre rule of n points can integrate a function of order 2 n minus 1 exactly. We can integrate a function up to order 2 n minus 1 exactly. If we adopt n number of points, so two Point Gauss-Legendre rules integrates cubic exactly a function cubic function exactly. This procedure can also be extended for more number of points.

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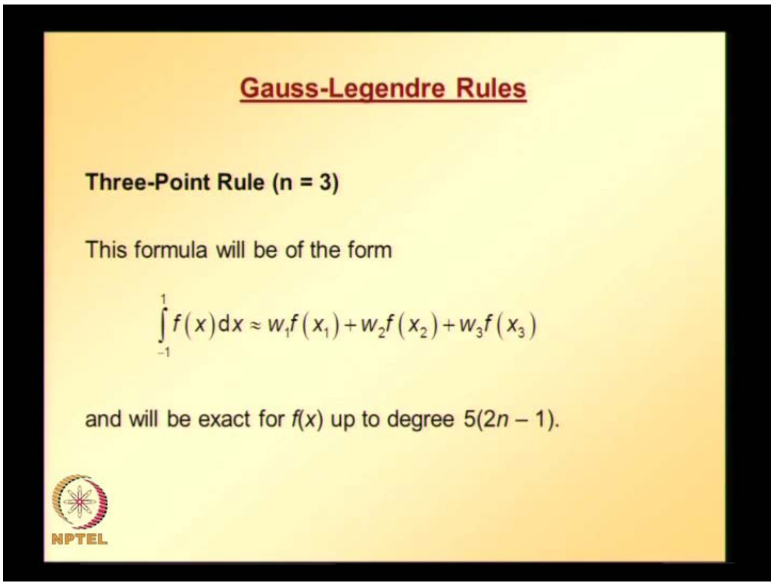

Now let us see three-point Gauss-Legendre rule. That is n is equal to 3 integral minus 1 to 1, f(x) d x is approximated as function evaluated at three points x1, x2, and x3 multiplied with corresponding weights W1, W2, and W3 and sum them up, this is how we can approximate this integral. We need to determine these weights and locations W1, W2, W3, and x1, x2, x3. What about a the accuracy? this formula will be exact for a function up to order 5 because a n point Gauss-Legendre rule is going to be exact in evaluating an integral having integrand of order 2 n minus 1. How to determine these weights and this locations, we need to adopt similar procedure that we already looked at. We need to get six equations to determine the six unknowns uniquely that is f (x) is equal to 1, f (x) is equal to x, f (x) is equal to x square, f (x) is equal to x cube, f (x) is equal to x power 4, f(x) is equal to x power 5. We impose these conditions.

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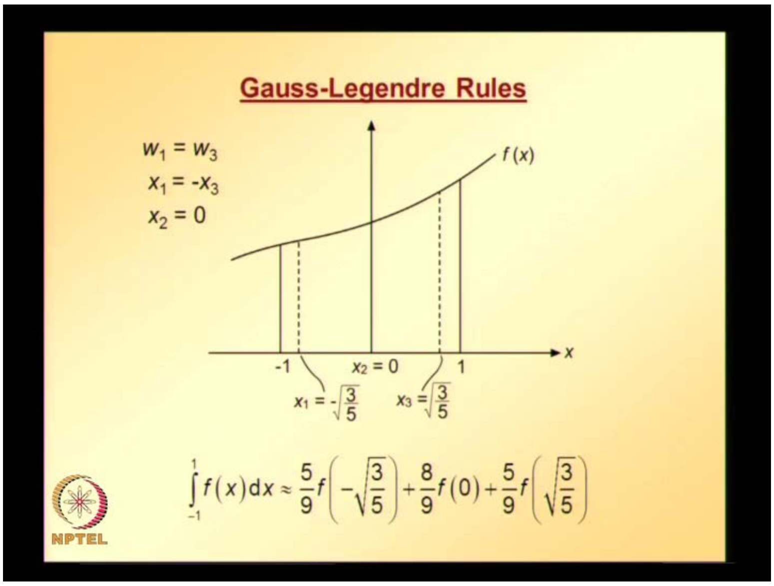

And we determine the six unknowns. And it turns out that the solution of six equations results in this that is W1 is equal to W3 x1 is equal to minus x3 and x2 is equal to. In the figure, the locations of these integration points is shown that is x1 is equal to minus root 3 over 5, x2 is equal to 0. And x3 is equal to root 3 over 5 integral f (x) between the limit minus 1 to one is basically approximated as 5 over 9 times function evaluated at function evaluated at minus root 3 over 5 plus 8 over 9 times function evaluated at 0 plus 5 over 9 times function evaluated at root 3 over 5. So, this is a procedure that we can follow even if one desires to find what are the locations, and weights for more points in Gauss-Legendre rules.

Basically, we have the weights and locations for this Gauss-Legendre rules for any number of points we can derive there is no problem. And also all these are well documented in any of the text books on finite elements or any numerical methods. The formulas or the weights, and locations are available for the limits minus 1 to 1. If limits are something else we need to discuss what are what will be the procedure changing limits of integration.

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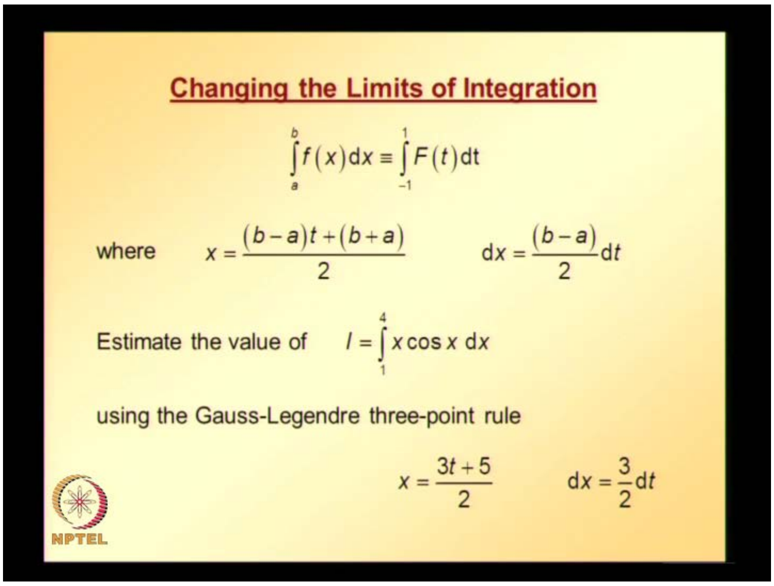

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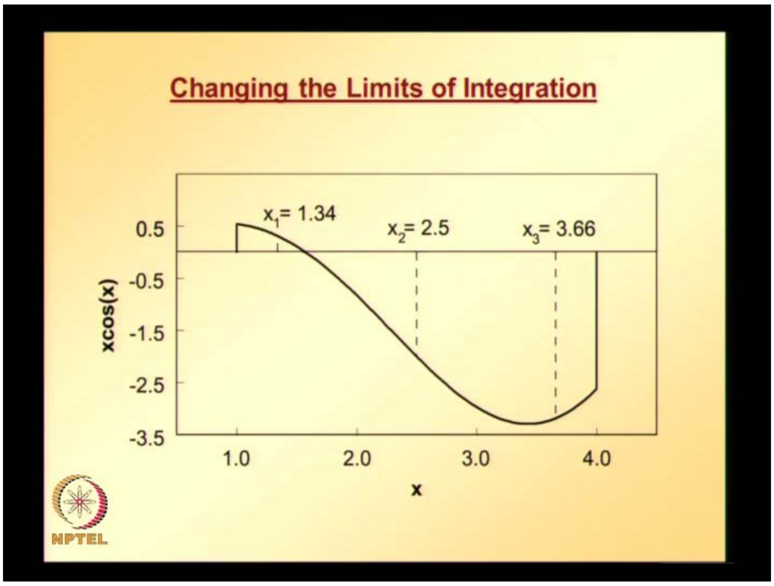

If one wants to evaluate this integral A to B, f (x) d x we need to somehow bring into the form minus 1 to 1 some integrand. The procedure is like this, we can use this formula to change the limits of integration that is limits A to B or changed to minus 1 to 1. Using this formula, x is equal to B minus A times t plus B plus A divided by 2, and when we take derivative of this equation on both sides. We are going to get d x is equal to B minus A over two times d t. So, this is how integral with limits other than minus 1 to 1 can be changed. Let us estimate but the rest of the procedure once we get into the form minus 1 to 1, integral minus 1 to 1 the rest of the procedure similar to what we already discussed. Now, let us estimate the value of integral 1 to 4 x cos x d x. First job is to change the limits of integration. And here also it is mentioned that we need to use three-point Gauss-Legendre rule. So, first job is to change the limits of integration.

We need to make a substituent x is equal to 3 t plus 5 over 2 and d x needs to be replaced with 3 over 2 dt. So, making these substitutions we can proceed as we already discussed by taking three points. We already know what are the locations? And what are the corresponding weights. This integral when we make this substitution of x in terms of t and d x in terms of d t this integral that is integral 1 to 4 x cos x d x becomes or the x cos x is plotted here as a function of x.

And the locations of the integration points are shown in this figure. So, we need to evaluate the function at these points x1, x2 that is t the integration point which is located at minus root 3 over 5, it corresponds to x1 is equal to 1.34. Integration point located at x2 is equal to 0 corresponds to the point x2 is equal to 2.55. Integration point located at root 3 over 5 corresponds to the integration point 3.66. Using the previous change of limits of integrations, we get these integration points. So, we need to evaluate function at these integration points.

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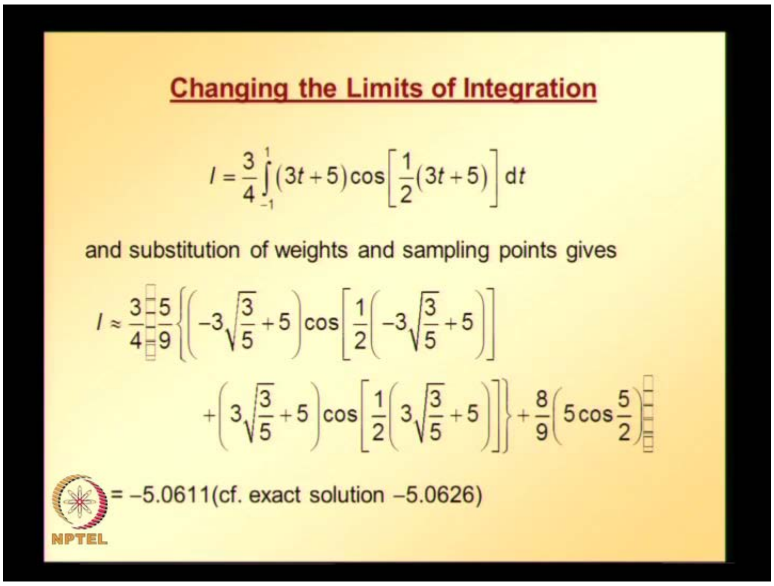

Substituting the change of limits into the previous equation of integral, we get this equation. And substitution of weights and sample points we get this approximation. Actually there is some problem with font again as brackets are shown as some blocks there. And when the integral is simplified in turns out that these integral values minus 5.0611 and we compare this with the exact solution, exact solution turns out to be minus 5.0626, which is very close to the exact solution.

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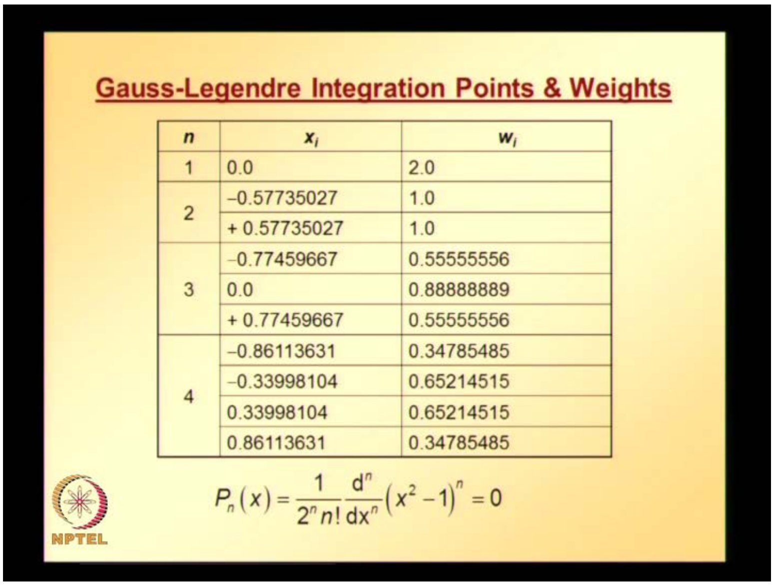

n	x_i	w_i
1	0.0	2.0
2	−0.57735027	1.0
	+ 0.57735027	1.0
3	−0.77459667	0.55555556
	0.0	0.88888889
	+ 0.77459667	0.55555556
4	−0.86113631	0.34785485
	−0.33998104	0.65214515
	0.33998104	0.65214515
	0.86113631	0.34785485

Now we can summarize Gauss-Legendre integration points and corresponding weights in a table like this. Here only up to n is equal to 4 is shown, but as I mentioned in any of the standard text book on finite element method or any numerical method. We can find integration point n is equal to a up to 10 or more. And also these values of location, and weights are can also be obtained in any of the commercial software like math lab. Why this word Legendre is? Actually these locations or functions of these are it can be shown that these locations that is exercise or roots of this Legendre polynomial that is why? It is called Gauss-Legendre integration rule or Gauss-Legendre rules or for that matter most of the finite element text books prefer this as gauss quadrature in a very simple form.