Finite Element Analysis

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Module No. # 01

Lecture No. # 03

Ok in today's lecture what we will do is we will take few examples and solve using Rayleigh Ritz method and those examples include Eigen value problems and also we will try with various trial solutions.

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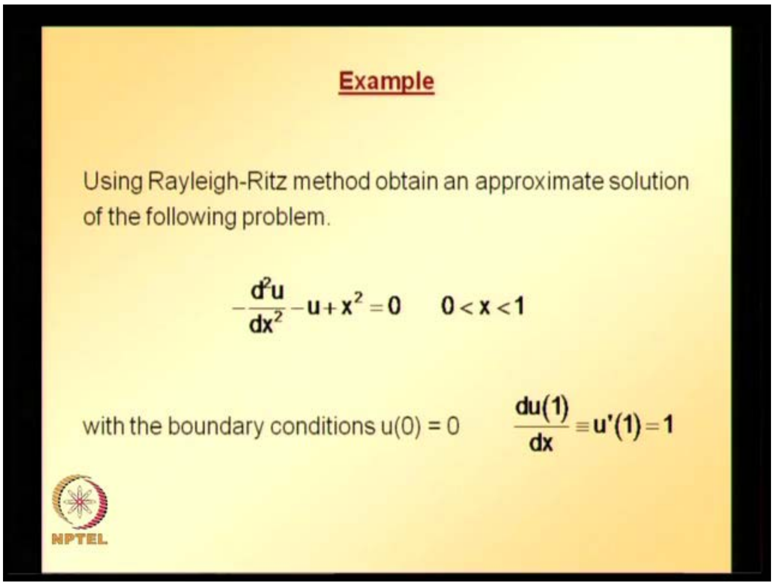

So first to start with let us take this differential equation or boundary value problem and solve using Rayleigh Ritz method to obtain the approximate solution and if you see this the problem statement it is a second order differential equation and the domain on which we need to solve that is zero to x zero to one, x goes from zero to one that is also given there. And since this is second order differential equation we require two boundary conditions to solve this one. So, the two boundary conditions given are u evaluated at x is equal to 0 is zero derivative of u or the first derivative of u evaluated at x is equal to 1 is 1.

And if you recall using the thumb rule that I already gave you earlier that is, if you have a differential equation of order two p those boundary conditions of order zero to p are essential boundary conditions. And those boundary conditions of order p two two p minus one are natural boundary conditions. Uh So, if you use that thumb rule the first boundary condition you can easily verify it to be essential boundary condition and the second boundary condition is what is called natural boundary condition. And also will see here in the today's lecture why they are called essential and natural boundary condition in a different sense. ok

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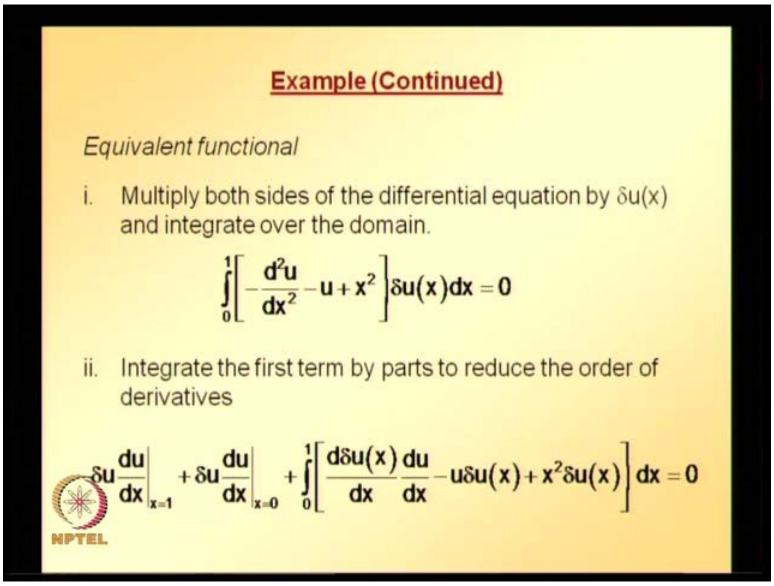

So let us start with solving this problem using variational method or Rayleigh Ritz method to obtain the equivalent functional. So the first step is the given differential equation we need to multiply with the variation of the quantity that we are looking for and integrate over the problem domain. Here, they differential equation that is given is is a, they are given in the square brackets and it is multiplied with the variation of u that we are interested in finding are u is what is the quantity that we have interested in finding. So you multiply the given differential equation with variation of u, integrate over the problem domain limits of this integration should be from the limits of the domain.

And the second step in obtaining the equivalent functional is integrate any term which has higher derivative, use integration by parts and reduce it to lower order derivatives. So the first if you see the this equation, the first term has second derivative of u. So, we can somehow reduce this second derivative of u to first derivative of u by use, the by applying integration by parts to the first term so that is what is done here. You just apply integration by parts to the first term you get this term plus this term and this term and these terms are as they are here, u times variation of u is already there and also x square times variation of u is already there ok.

And now this is a the first step or this is how you can reduce the order of differentiation because that helps us in putting less demand on the continuity of the approximate solution that we are trying to use or trial solution that we are trying to use.

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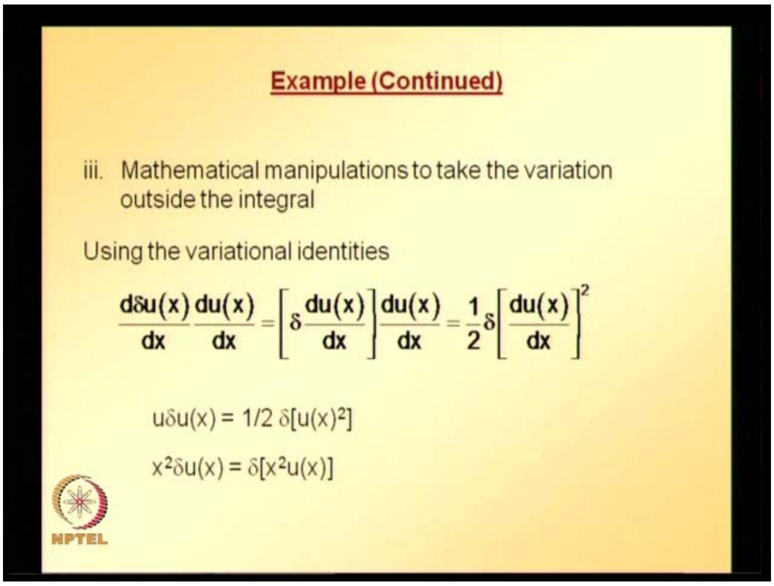

Now let see they the boundary conditions that are given. One of the boundary condition that is given is variation of sorry u evaluated at x is equal to 0 is 0 and these two terms; the first two terms can be we we can actually simplify using some mathematical manipulations and we will be using some variational identities here. the If you see the first term inside the integral you have this kind of quantity that can be rewritten in this manner.

And also if you see our equation; we have this kind of quantity u times variation of u that can be written as 1 over 2 variation of u square. Now, x square times variation of u; x acts like a constant it is not going to be function of any of the coefficients. So, x square times variation of u is it can be written as variation of x square u. So, using these variatonal identities, the equation that you have earlier can be rewritten in this manner.

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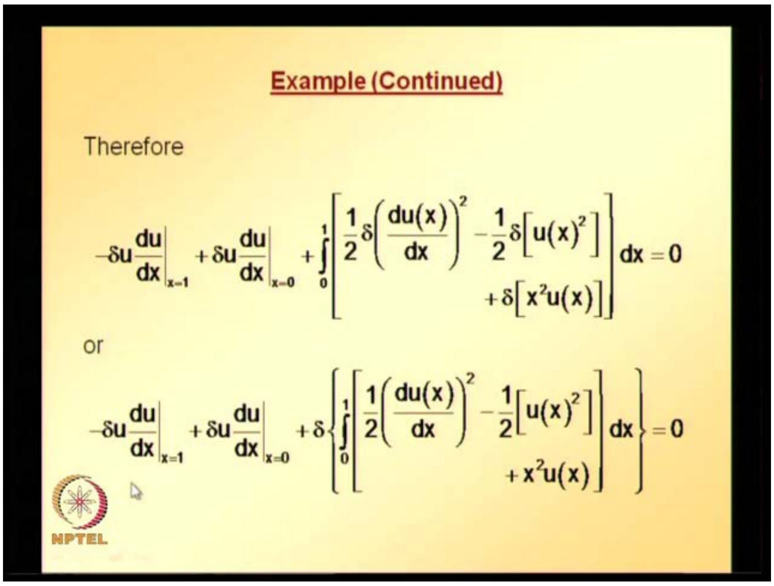

And if you see this equation somehow we need to eliminate this first two terms so that we can put to we can bring this into the form variation of some quantity inside the brackets is equal to 0. And this can be further simplified. Here you can see the inside the integral you have variation of second derivative of u square and variation of u square and variation of x square u and half is like a constant.

So, this is like variation of u plus variation of v plus variation of w and it is nothing but, variation of u plus v plus w. So, this entire thing can be put inside the variational operator and this this third part can be rewritten in this manner. Still we are left with the first two parts. Let see how to cancel these two terms first two terms.

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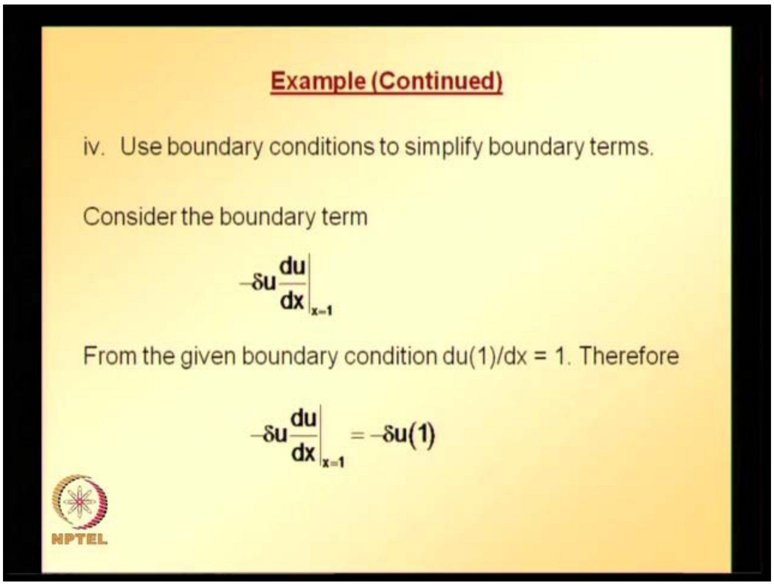

And to cancel the first two terms we use boundary conditions. And if you see the first term; this is the term that you have and already natural boundary condition is given at this point; that is x is equal to. So, substitute a derivative of u with respect to x evaluated at x is equal to 1 as is given in the natural boundary condition which is equal to 1. So, the first term can be simplified to when you submit the substitution it becomes variation of u evaluated at x is equal to 1 ok.

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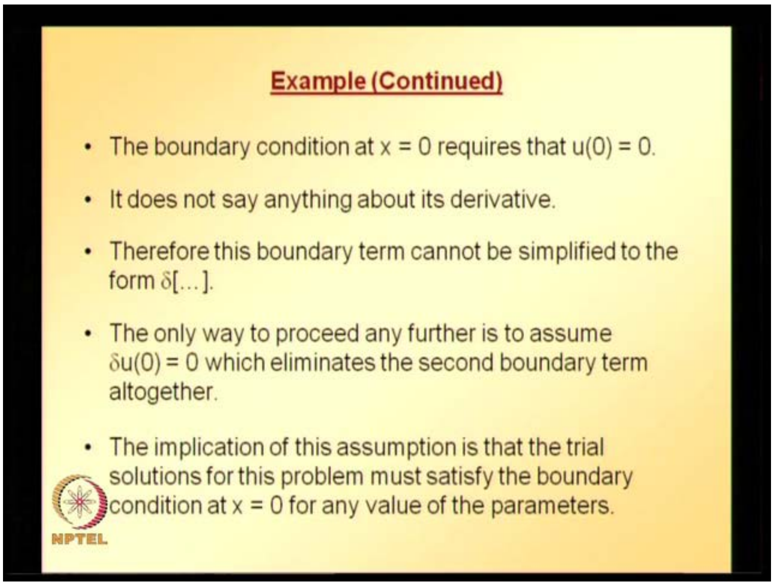

And if you see the other boundary condition that is, we need to evaluate at x is equal to 0 and already natural boundary condition is given that is, u evaluated at x is equal to 0 is 0. But, what the problem is, it does not say anything about its derivative and therefore, this boundary term cannot be simplified to the form variation of some quantity in side.

So, the only way to proceed further is to assume a variation of u evaluated at x is equal to 0 is 0 which eliminates the second boundary term altogether and we will see what is the implication of this. The implication of this assumption is that the trial solution for this problem must satisfy boundary condition at x is equal to 0 for any value of parameters. Please remember at x is equal to 0 essential boundary conditions is given. So, the implication is at any cost you need to satisfy essential boundary condition.

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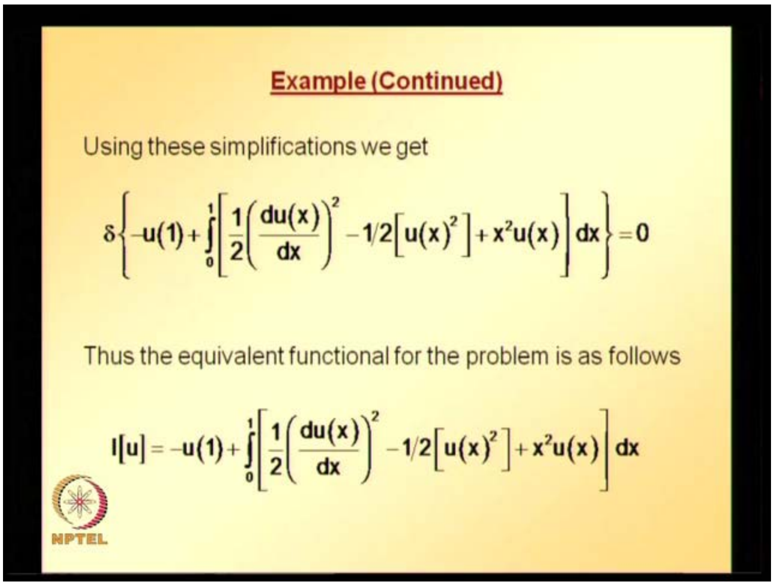

And so, with this reasoning we can eliminate the second term also. The first two terms get eliminated and you will be left with only the last term that is variation of some quantity is equal to 0. So whatever is there inside the bracket if you call that as I as a function of u because if you see whatever is there inside the the variational operator everything is function of u. So, I am defining a new quantity called i which is going to be function of u and that is what is called equivalent functional for this particular problem.

So given any boundary value problem with the boundary conditions, you can adopt these steps finally arrive at these kind of variational functional ok.

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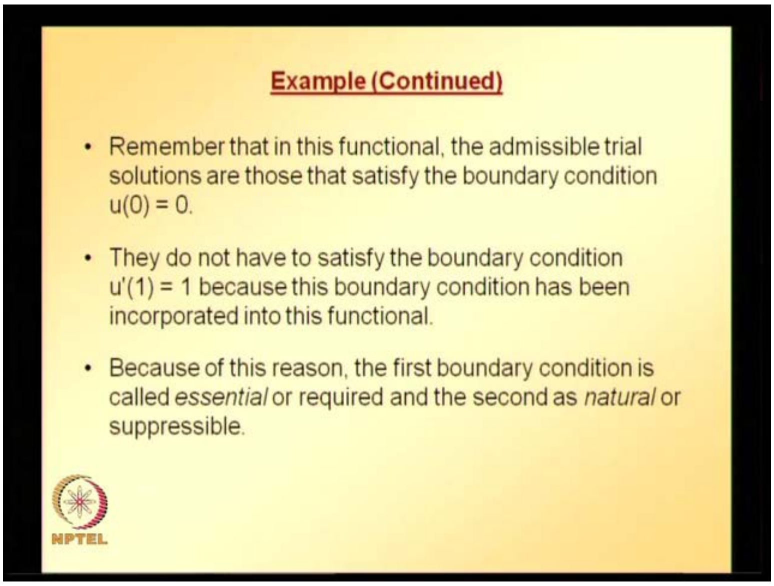

Remember that this that in this functional the admissible trial solution are those that satisfy boundary conditions that is u evaluated at x is equal to 0 is 0. They do not have to satisfy the boundary condition that is, natural boundary condition. That is derivative of u evaluated at x is equal to 1 is 1 because this boundary condition has been incorporated to the functional.

Because of because of this reason the first boundary condition is called essential. That is you need to satisfy at any cost. The trial solution has to satisfy at any cost. And the second boundary condition is called natural boundary condition of suppressible boundary condition. First boundary condition is also called required boundary condition. And you will understand better this is one once we look at one more problem you will understand these better.

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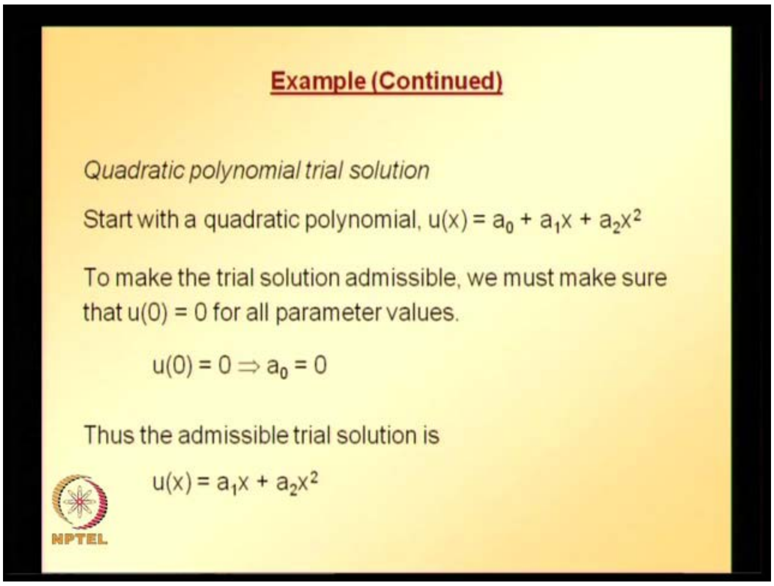

So now let us start with quadratic polynomial trial solution. So, let let us say that we take a polynomial quadratic polynomial trial solution as u is equal to a naught plus a one x plus a two x square. So now, first job is before we proceed further first job is to make this trial solution that we are starting with admissible trial solution. To make the trial solution admissible what we need to produce we have to make sure that the this satisfies the essential boundary condition that is given for this particular problem. If you recall the essential boundary condition that is given for this problem is u evaluated at x is equal to 0 is 0.

So now to make this trial solution admissible what you need to do is make substitute this that is u evaluated at x is equal to 0 is 0 that leads to a naught is equal to 0 ok. So the admissible trial solution becomes a one x plus a two x square. Now, our job is to find what is this a one, a two or what we will do is we will substitute. We will make the substituted we will make this trial solution admissible trial solution will substitute into the equivalent functional and apply the stationary conditions.

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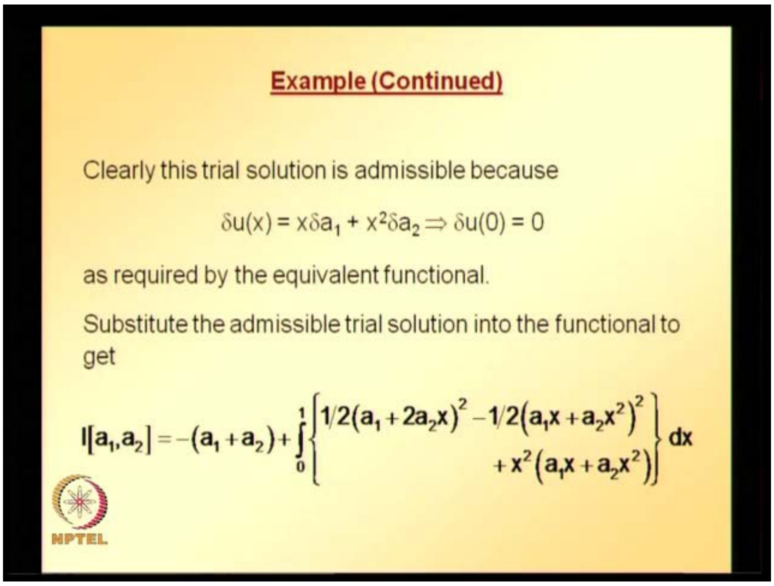

And before we proceed further we can check whether this admissible trial solution indeed satisfy the condition that variation of u evaluated at x is equal to 0 is zero or not because that is the condition that we use to eliminate one of the boundary term. So if you can just to check the given admissible trial solution; you take variation of it and then substitute x is equal to 0 then, it clearly satisfies the condition. And this is what is required for getting the equivalent functional.

So now, we have the admissible trial solution and we have the equivalent functional i as a function of u and now, u we are taking it as a one x plus a two x square. So, you substitute make this substitution of u into i. Then i are the equivalent functional becomes a function of two unknown coefficients a one and a two.

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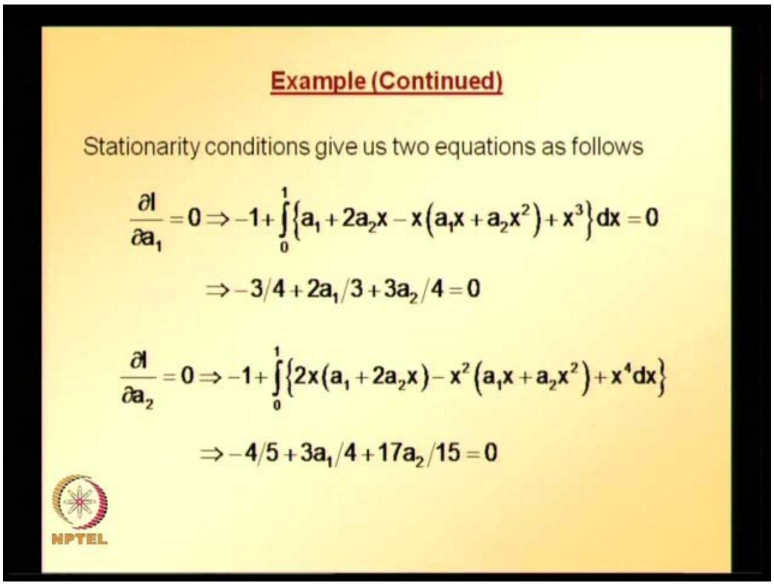

And what we have is variation of i is equal to 0. That is possible only if partial derivative of i with respect to the unknown coefficients that you have in I, the partial derivative of i with respect to the unknown coefficients is equal to 0 independently. If you have n number of unknown coefficients then, the partial derivative of i with respect to each of this n unknown coefficients should be independently equal to 0.

Here we have two unknown coefficients; a one a two. So the first stationarity condition is partial derivative of i with respect to a one is equal to 0 and you already have i as a function of a one a two you take a partial derivative of that with respect to a one and equate it to zero that gives u this first equation. And we have two unknown coefficients to be determined. So we need to get two equations. So how we get the second equation? Second equation can be obtain using the second stationarity condition that is, partial derivate of i with respect to a two is equal to 0 ok.

So, you got two equations both in terms of a one a two and you have two unknown coefficients to be determined. So you can solve these two equations simultaneous equation for a one a two.

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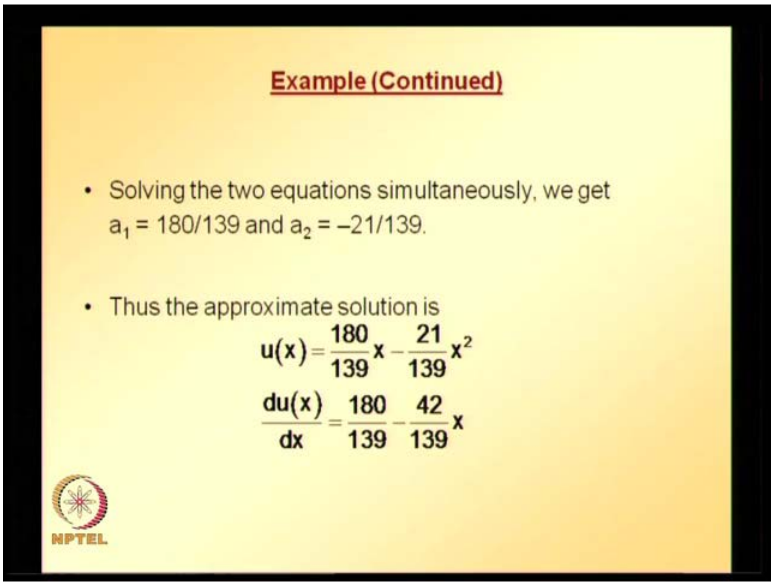

Solving the two equations simultaneously, we get the coefficients a one a two which are shown there. Now, our job is over we found what is a one a two and the admissible trial solution is, a one x plus a two x square. So, you substitute a one value and a two value you get approximate solution for u. And if you want you can take derivative of this because for this particular problem we know, we can also easily calculate or we can easily find what is the exact solution for u. And also from there we can find what is the exact solution for derivative of u.

So to make a comparison how approximate solution matches with exact solution. Here what is shown is approximate solution of u is obtained by substituting a one a two coefficients into the admissible trial solution, a one x plus a two x square and taking derivative of it we get the second equation.

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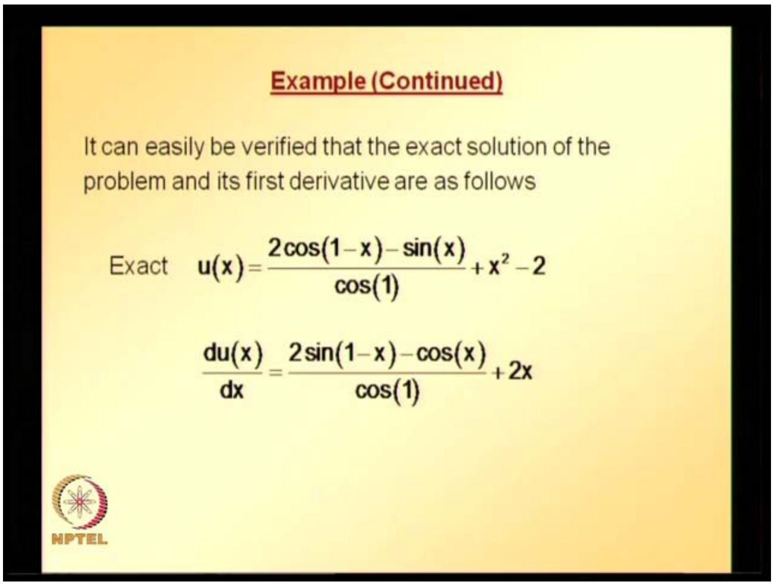

So now this is what I mentioned; the exact solution for this particular problem can easily be verified. This is exact solution and also from here if you take derivative of this with respect to x you will get the first derivative.

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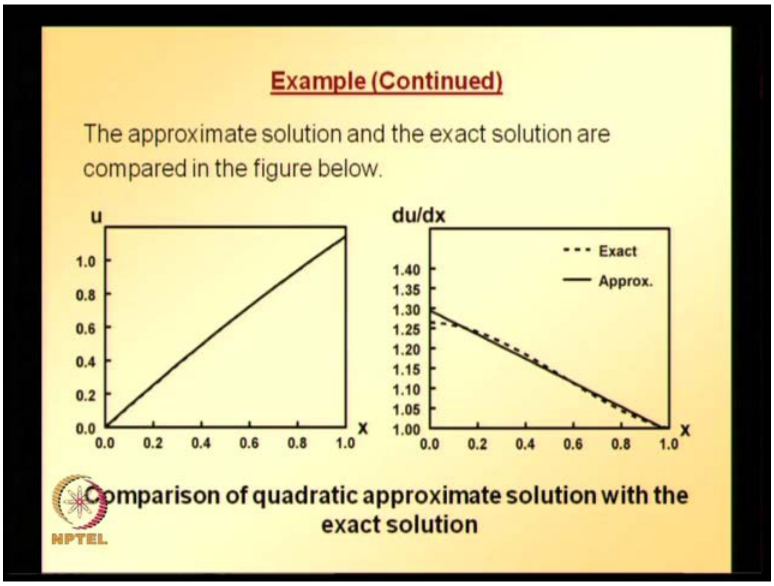

So now you have approximate solution for this problem and also exact solution for this problem and let us see how they match approximate solution and exact solution are compared in the figure below. And here, the approximate solution and exact solution are almost over lapping each other. So, that is why you are unable to make any, distinguishing between approximate and exact. But if you take the derivative we can clearly seen some error is there in the approximate solution when compared the exact solution.

And please remember whatever approximate solution that we got here is based on the quadratic trial solution that we started out ok. So if you want to increase the accuracy that means, here already you can see that u value approximate and exact are almost overlaying on each other. But, if you see the derivative of u, there is some error. if you want reduce this error you can go for higher order approximation ok

So you can start with cubic polynomial is a trial solution and follow the same process that is, you make sure the before you proceed anymore further once you make assumption of some approximate trial solution, make sure that it is admissible by substitute, making sure that it satisfies the essential boundary conditions and then using that admissible trial solution you plug that into the equivalent functional and apply the stationarity condition to get the unknown coefficients, solving the simultaneous equations that you get.

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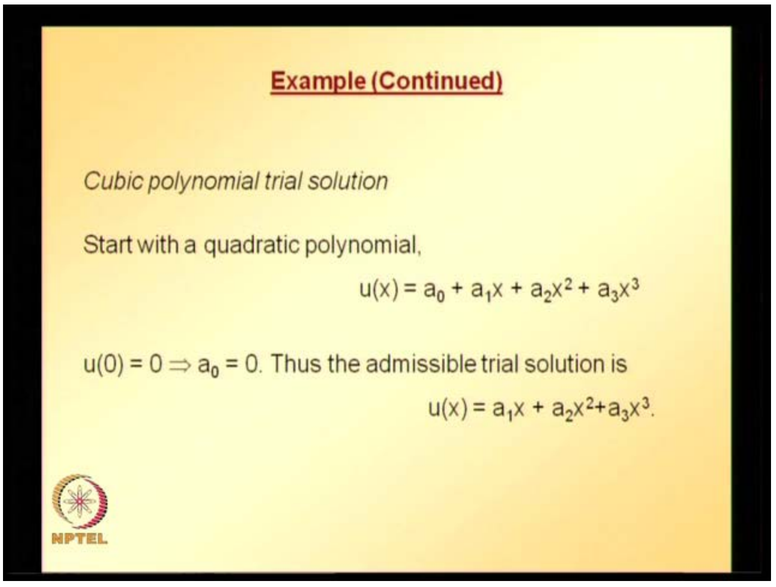

Now what we will do is; we will solve the same problem using cubic polynomial trial solution. Let us see whether the the derivative the error in the derivative small error that we observed in the derivative can further be reduced. So, let us start with cubic polynomial trial solution. You can easily guess that is nothing but, a naught plus a one x plus a two x square plus a three x cube ok.

So now first job is to make this admissible; admissible means it has to satisfy essential boundary condition and impose that, that is you take this trial solution substitute x is equal to 0 in. Then it leads to the condition that a naught is equal to 0. So the admissible trial solution becomes u is equal to a one x plus a two x square plus a three x cube ok. So now your job is to find what is a one a two and a three.

So what we will do is we will take this admissible trial solution and substitute into the equivalent functional that we already obtained using the various steps that are involved in variational procedure that is first step is, multiply the given differential equation with variational the quantity that you are interested, integrate over the problem domain, apply the integration by parts, reduce the order of the highest derivative appearing in the expression and that is and apply the essential boundary condition. That is wherever essential boundary conditionals are prescribed there variation of u is equal to 0; apply that condition and also you substitute the natural boundary conditions, and simplify using mathematical identities are sorry variational identities the bring it into the form, variation of some quantity inside the bracket is equal to 0. And whatever is there inside the bracket that is what is called equivalent functional. So, that process is same even for whether you choose cubic polynomial trial solution or quadratic polynomial trial solution. That is all same that is all not repeated here.

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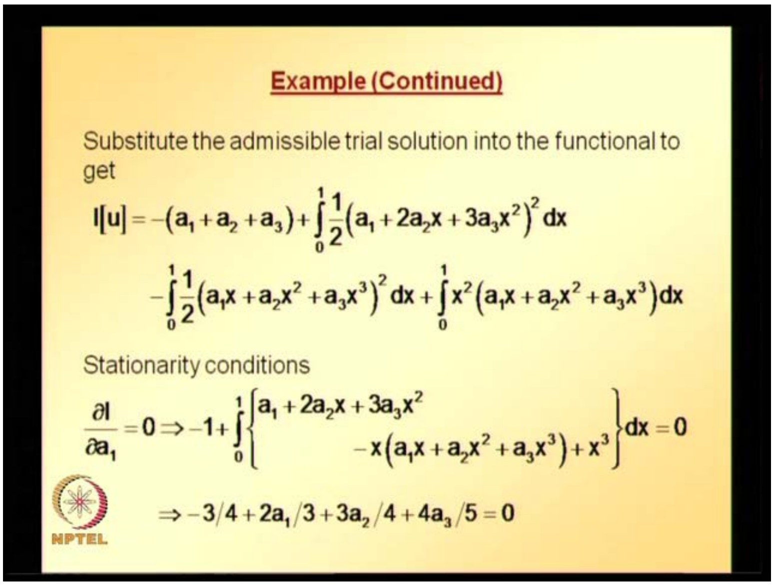

Sub directly taking that equivalent functional the admissible quadratic trial solution is substituted in the functional which we already have. Now, the functional becomes a function of a one a two and a three three unknown coefficients.

Function of function is called functional. That you just remember and now apply the stationarity condition. We got this equivalent functional; variation of i is equal to 0. That is the condition that is satisfied if each of this conditions that is partial derivative of i with respect a one is equal to 0, partial derivate of i with respect a two is equal to 0, partial derivative of i with respect a three is equal to 0 are satisfied.

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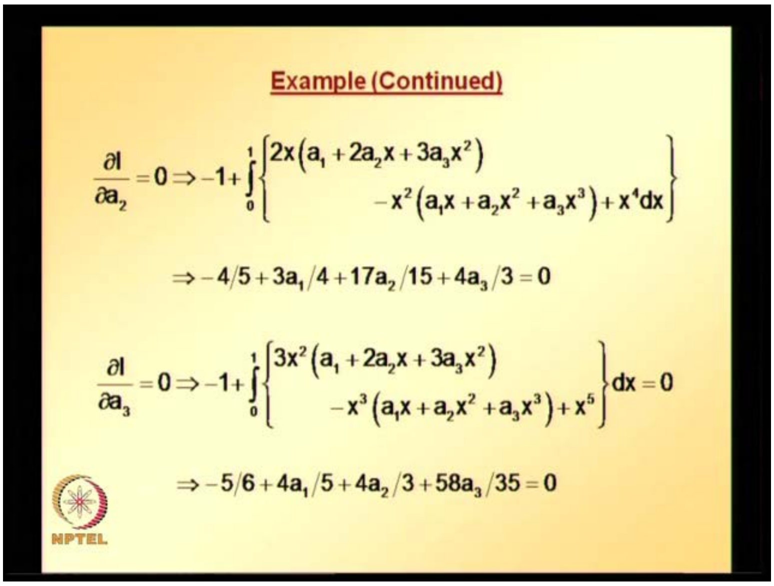

So here the first condition is applied we get one equation and this result in this equation. Similarly, apply the other conditions partial derivative of i with respect a two is equal to 0 you get the second equation and there are three coefficients unknown coefficients to be determine so we require three three equations you get the third equation by applying the third condition. So now you have the required number of equations for solving the three unknown coefficients. So, you solve these three equations.

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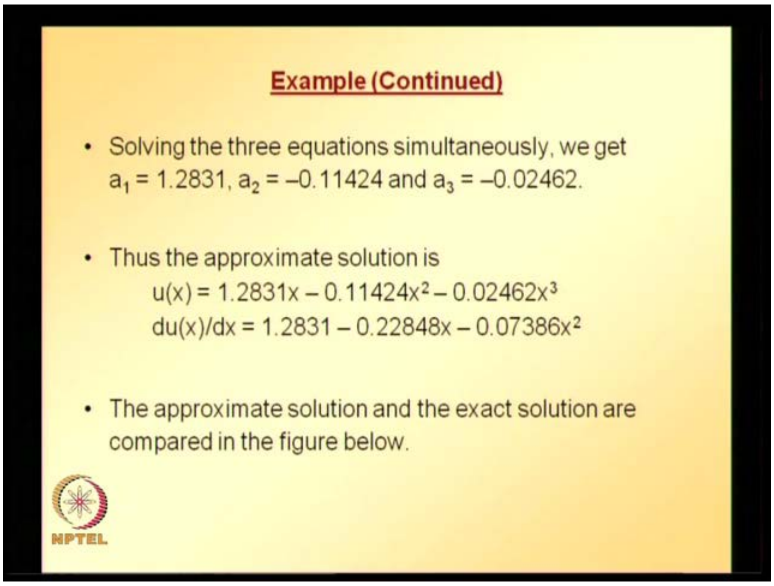

Solving the three equations simultaneously we can determine the unknown coefficients a one, a two, a three. The values are given there now next step is to substitute this a one, a two, a three into the admissible trial solution is started out with, then you get the approximate solution and by taking derivative of this. You get the second equation that is given there.

And then to see how the solution matches approximate solution matches with exact you can over lie the exact solution plot with this approximate solution. The approximate solution and exact solution are compared in the figure below. And now, if you see, if you compare this with what you have seen earlier that we obtained using quadratic trial solution, the first derivative of solution is now much closer to the exact solution than with quadratic approximate solution.

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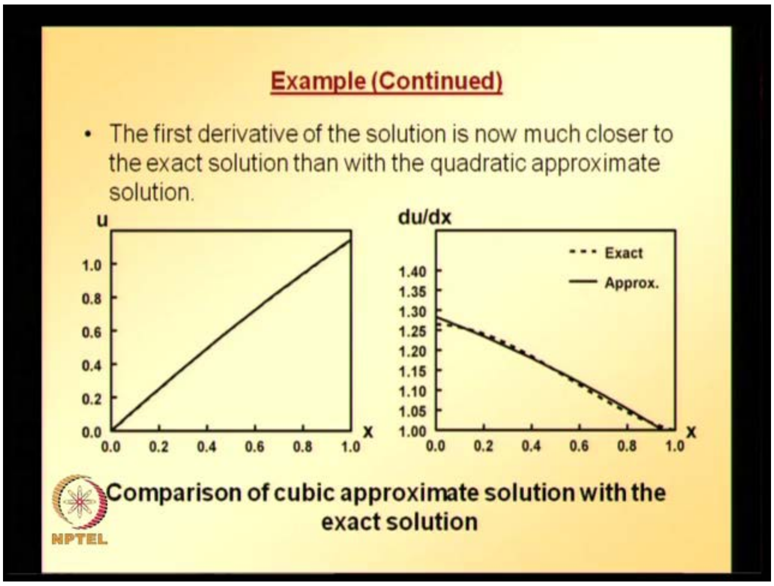

So, this is how you can increase the accuracy and the derivative quantity if you are interested. But, it comes with some expense that is you need to spend more or you need spend more computational effort or it involves more calculations.

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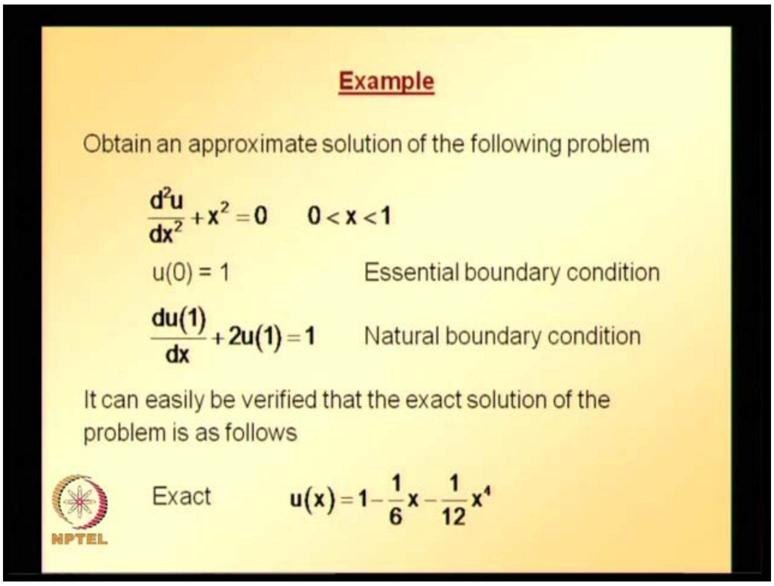

So now, let us take another example. Here what we will do is, this example is similar to the previous example. But, here what we will do is we will try using quadratic cubic and quartic trial solution that is, fourth order trial solution fourth order polynomial trial solution. So, the boundary value problem that I am choosing is this one. That is second derivative of u with res[pect]- with respect to x plus x square is equal to 0 that condition needs to be satisfied over the domain zero to one.

Again this is second order boundary value problem. So, we required two boundary conditions and here mixed boundary conditions are given as given in the previous example that is you have a combination of essential boundary condition and natural boundary condition. And also as in the previous example, you can easily find exact solution for this problem you may ask me why I am, you once we know the exact solution why we are trying to find approximate solution? The logic is we use various order of trial solutions and see how they match with exact solution so, that we can get an idea about the efficiency of this method. But, the whatever technique we are learning here that can be applied for some of that problem where you do not have exact solution. That is the basic idea.

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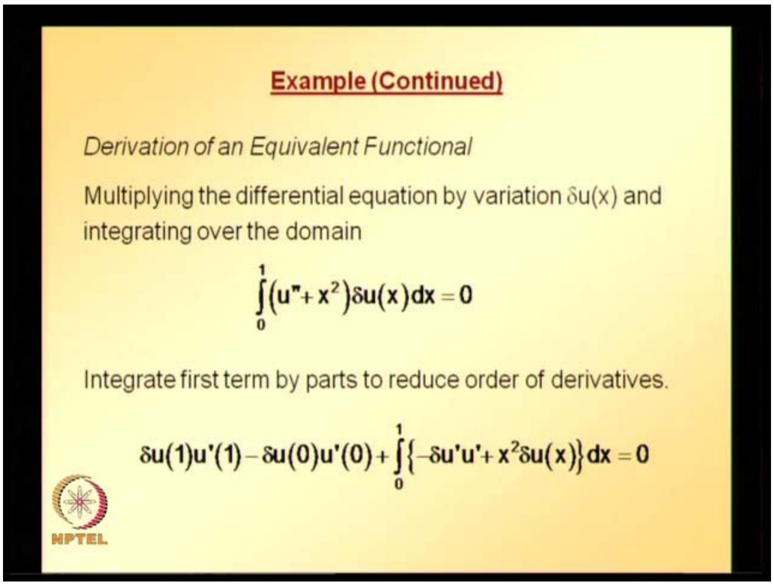

So the first step is derivation of equivalent functional. So, in that what we are need to do is the given differential equation we need to multiply with variation of the quantity that we are interested that is u and integrate over the problem domain that is zero to one. Ok. And here a new notation is used I hope you are familiar with this. This is u double prime it is nothing but, second derivate of u usually this kind of notation is used to write the equation in a compact manner.

And now highest derivative term is second order derivative. So, what we can do is we can reduce the order of derivative here u double prime to u single prime. That is first derivative of u by using integration by parts over the first term. Do not disturb the second term to apply the integration by parts for the first term then that leads two this one. And once we get this, we need to think how to manipulate this using variational identities and how to eliminate the boundary terms using the boundary conditions that are given. If not, if we cannot eliminate the boundary terms what we can do is we can somehow manipulate them using variational identities. So, that we can bring it this entire thing into the form variation of some quantity is equal to 0.

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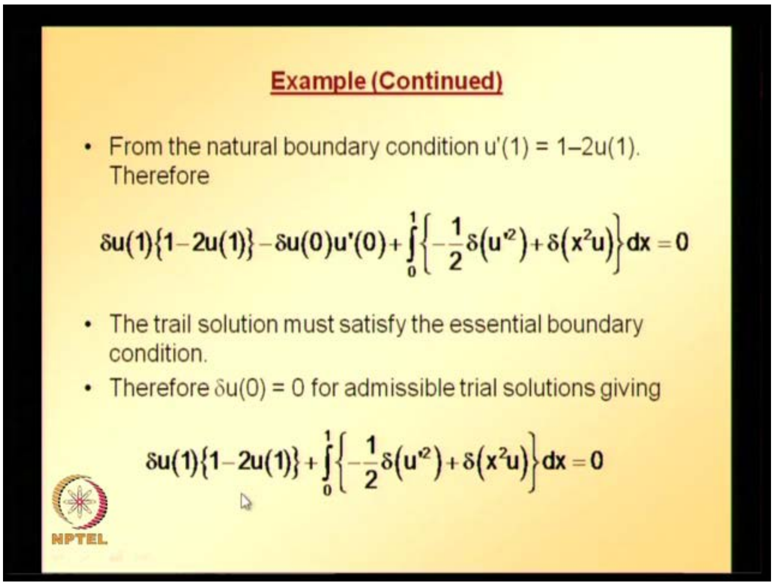

We will see those things. So, first thing is first term that is from the natural boundary condition that is given, you have first derivative of u evaluated at x is equal to 1 is equal to one minus two times u evaluated at x is equal to 1. And what we can do is, we can substitute it directly this quantity into the previous equation that results in the equation that is given here.

And also, we have the other condition that is trial solution must satisfy essential boundary condition. What it means? It means wherever essential boundary condition is specified, at that point variation of u should be equal to 0. Therefore, variation on of u evaluated at x is equal to 0 is 0 for admissible trial solution. So, wherever essential boundary condition is prescribed, at that point variation of u is equal to 0. Here in this particular problem essential boundary condition is prescribed at x is equal to 0 so variation of u evaluated at x is equal to 0 is 0.

So the second term appearing in this equation becomes zero, you will be left with first term and the third term that is what is shown here in this equation. We cannot further more we cannot eliminate any of the terms because no more essential boundary conditions are given but, only thing is we can use some variational identities to simplify this. Already if you see the first term sorry the second term; if you see the second term half is a constant. So you can this is the first term is like variation of u second term is like variation of v. So, you can take the variational operator out. Variation of u plus variation of u. So, there is no need of much manipulations required for this term. Only thing is we need to look into this.

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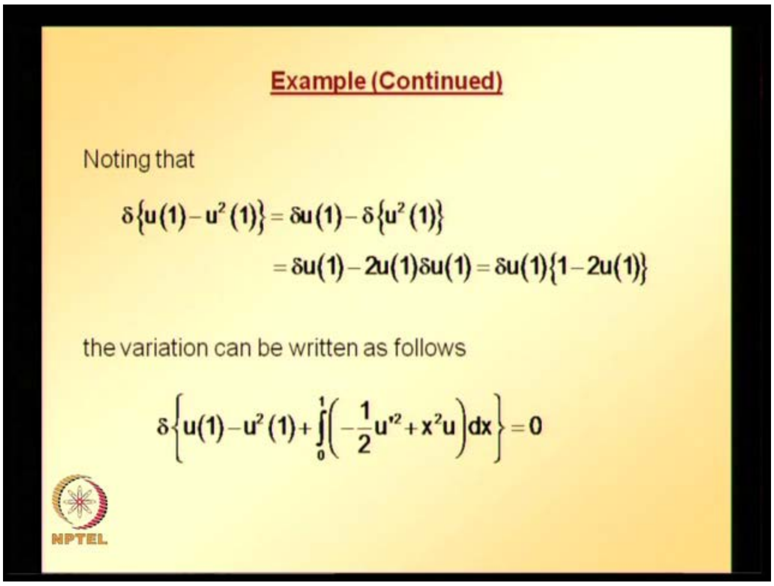

So let us look how can we write this one and note that variation of u evaluated at x is equal to one minus u square evaluated at x is equal to 1. Variation of this entire thing is nothing but, variation of u evaluated at x is equal to one minus variation of u variation of u evaluated at x is equal to one square ok. And this can be further written in this manner and which is equal to this. So if you see the previous equation this is a term you have so what i am doing what I am going to do is replace this term with this because both are equal. So then the previous equation becomes this.

So know you have what you want. What you want is, you want to bring the given problem into the form variation of some quantity is equal to variation of some quantity inside bracket is equal to 0. So that is what exactly you got it. So, whatever is there inside the bracket that is nothing but, where equivalent functional and everything in the bracket is function of u. So, you can denote it with i as a function of u that i is given by this is equivalent functional for this problem ok.

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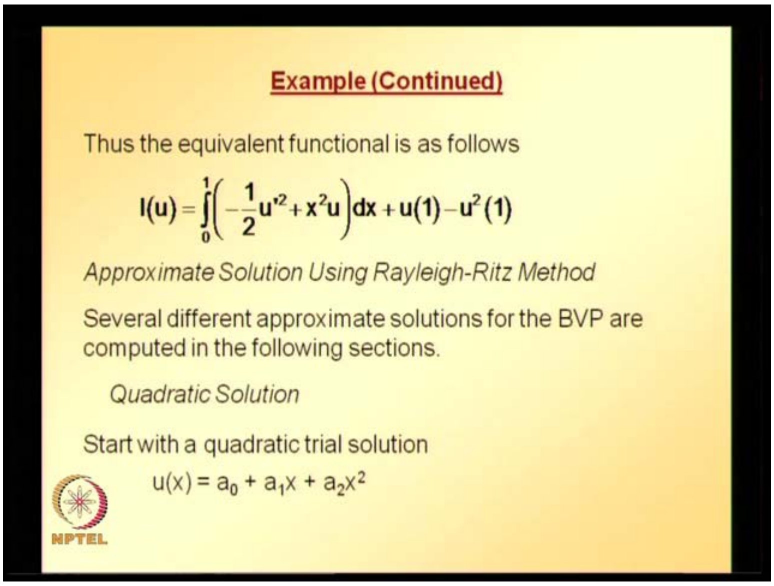

So now once we get equivalent functional what is the next step? Next step is we need to assume some trial solution here for this particular problem. What I will do is, I will take three types of trial solutions; one is quadratic trial solution cubic trial solution and quartic trial solution and show you the procedure.

So, several different approximate solutions for the boundary value problem are computed in the following sections. So first one is quadratic trial solution. So, what I will do is I will start with quadratic trial solution that is a naught plus a one x plus a two x square and now by this time you can guess what is the next step.

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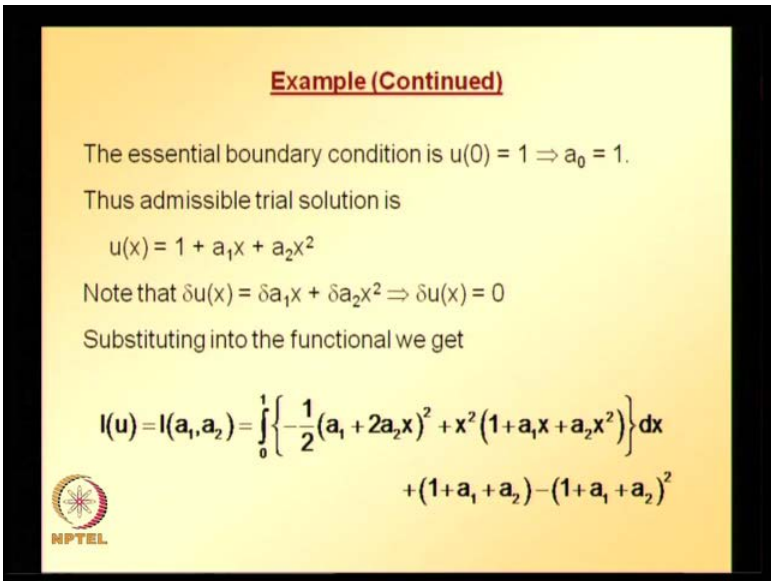

Make this trial solution admissible. To make this trial solution admissible what you need to do? You need to substitute the essential boundary condition that is given for this particular problem. So this is the reason why you should be good at identifying if a boundary value problem is given with boundary conditions you should be good at identifing which boundary condition is essential which boundary condition is natural because we need to to make the trial solution admissible. We need to substitute the essential boundary condition before we proceed further.

So the given quadratic trial solution that is u is equal to a naught plus a one x plus a two x square in that you substitute the essential boundary condition that is u evaluated at x is equal to 0 is 1. That leads to the condition a naught is equal to 1. So, your admissible trial solution becomes one plus a one x plus a two x square. Again here, your job is to find what is a one a two? If you find what is what are these unknown coefficients a one a two you can back substitute once you go through all the process, once you find the values you come back to this equation back substitute what is a one, a two into it and you will get the approximate trial solution approximate solution for this problem.

And before you proceed further, you can a check whether earlier what we did is to get the equivalent functional we cancelled out the term variation of u is equal to 0. Variation of u evaluated at x is equal to 0 is 0. So, to check whether that whatever we did is correct or not know what you can do is now you have the admissible trial solution, you take variation of it and substitute x is equal to 0 then, you get variation of here variation of u evaluated at x is equal to is 0 is 0 ok.

So, that is how you can verify. So, the substituting the admissible trial solution into the equivalent functional, the equivalent functional becomes function of two unknown coefficients a one and a two. All these, this is all similar to what we have done in the previous problem.

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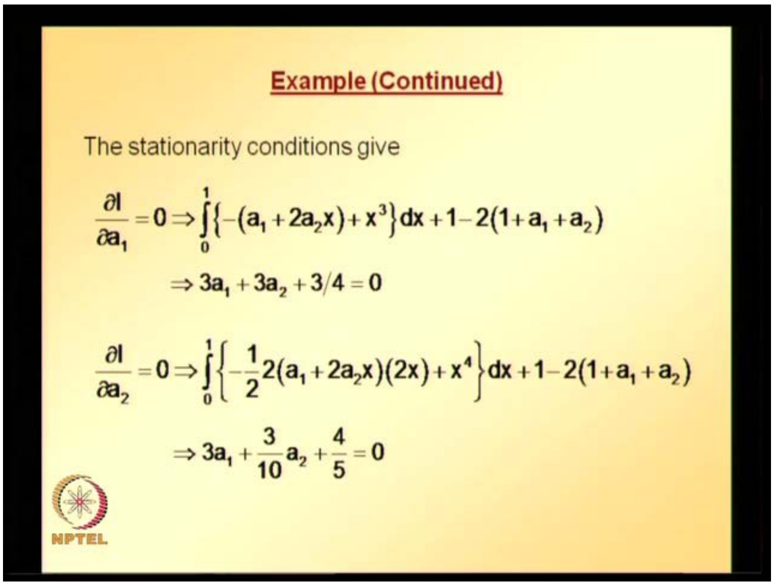

So now, variation of i should be equal to 0. That is possible only if partial derivative of i with respect to the unknown coefficients is equal to 0. These are the stationarity conditions so apply that condition. You get the first equation that is partial derivative of i with respect to a one is equal to 0 and apply the second condition that is partial derivative of i with respect to a two is equal to 0, you get a second equation. And you got two equations two unknowns a one a two you can solve for these two unknowns.

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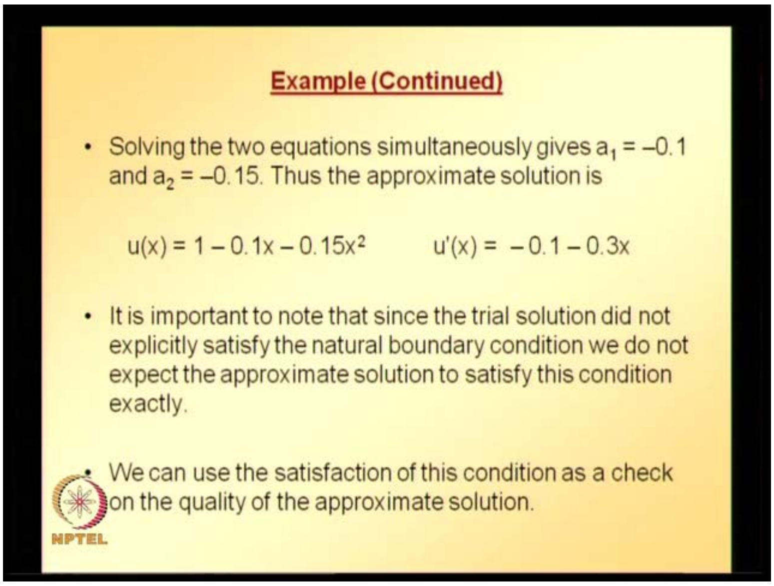

Solving the two equations simultaneously gives us a one a two and what you can do is, once you get a one a two; You go back to the trial, admissible trial solution that is one plus a one x plus a two x square in that we substitute this coefficients, coefficient values you get approximate solution. And by forcefully taking derivative of it you get the derivative of approximate solution.

And one more thing why the derivative quantity is not matching well with when when we compared with exact solution. The reason is it is important to note that since trial solutions did not explicitly satisfy natural boundary condition. Whenever we are assuming, what we did is we started with a quadratic trial solution and we made that admissible by imposing the essential boundary condition but, we never imposed the natural boundary condition. So the point here is it is important to note that since trial solution did not explicitly satisfy natural boundary condition, we do not expect the approximate solution to satisfy this condition exactly.

So, there will be some error when you take this, if you see the derivative quantity. That is what we observed even in the previous example our u value is exactly matching with, very accurately matching with exact solution where as derivative of u, we have some error. Why it is so? Because we have not the trial solution did not explicitly satisfy natural boundary condition ok.

So now, how can we measure our accuracy, the solution accuracy? We can use this condition to check whether the the quality of the approximate solution is good or not. So, that is what it is mentioned there they can use this we can use the satisfaction of this condition. Satisfaction of this condition refers to the natural boundary condition as a check on the quality of the approximate solution ok.

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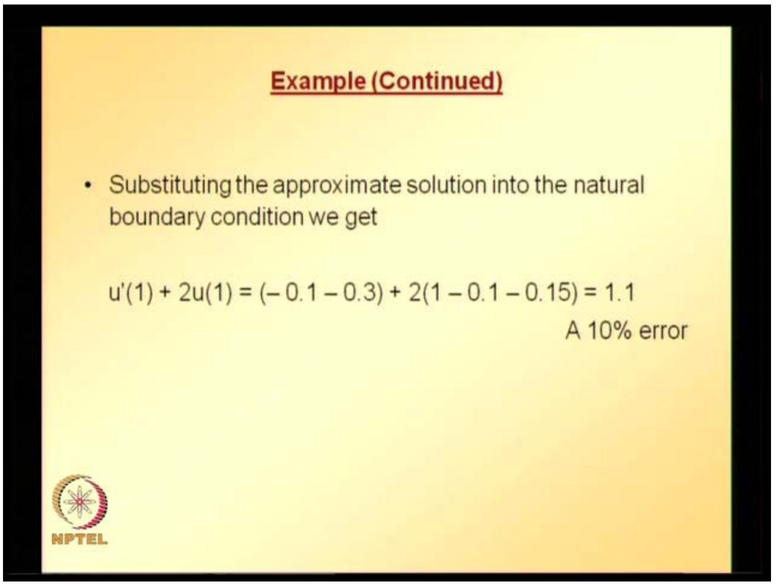

So what we can do is, we have the derivative of u here we evaluate this derivative of u at x is equal to 1 and the natural boundary condition that is given is, derivative of u evaluated at x is equal to 1 is 1. We see whether we get one or not. Substituting approximate solution into natural boundary condition we get 1.1. Actually we should get one means what we got ten percent error.

We got ten percent error. How we got it? We started with a quadratic trial solution. If you want to reduce this error further what you can do is you can start with higher order trial solution. You can go with, you can start with a cubic trial solution or a quartic trial solution.

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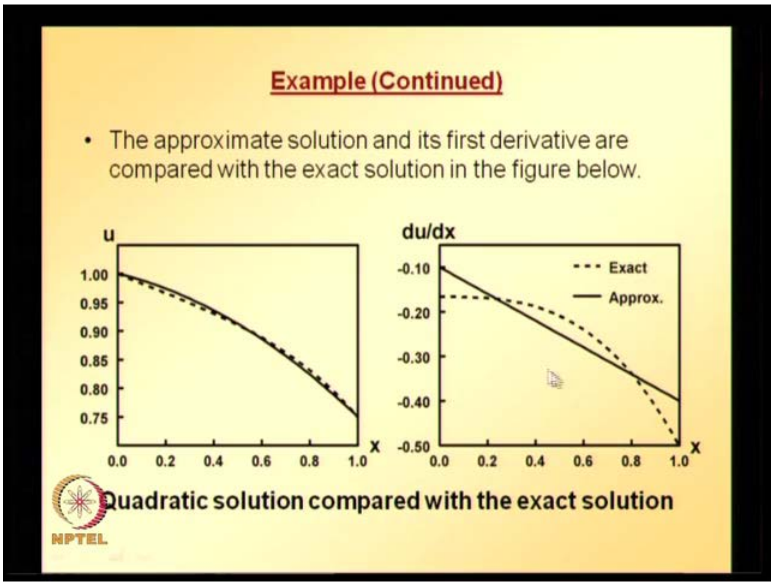

So now what we will do is we will solve the same problem using cubic trial solution and see by how much percent this error reduces. this And as we did earlier we will just see how the approximate solution and the exact solution matches for the u and also derivative of u that comparison is shown here. Here, this is where ten percent error is coming now.

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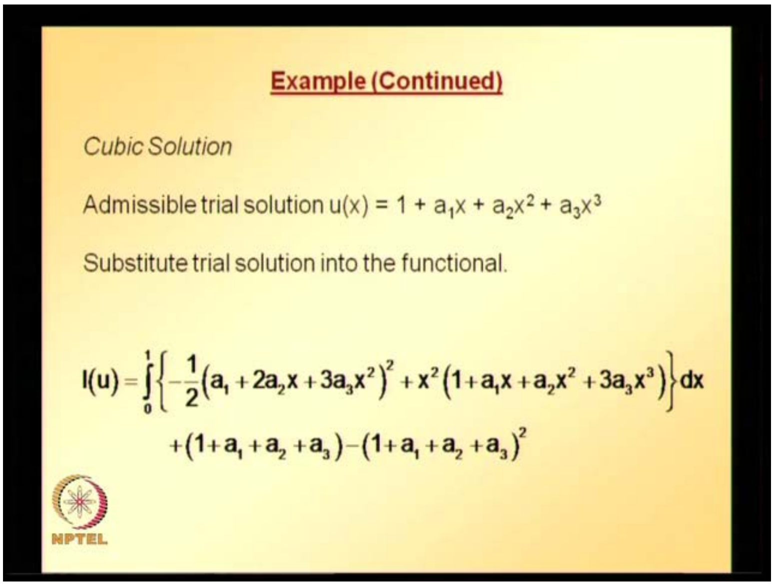

Let us solve this same problem using cubic trial solution. Cubic trial solution is a naught plus a one x, a two plus, a two x square plus a three x cube and substitute the essential boundary condition a two it you get the admissible trial solution which is one plus a one x plus a two x square plus a three x cube ok. Now, you substitute this admissible trial solution into the equivalent functional. The equivalent functional remain same whatever you already have that is equivalent functional for this problem. You do not need repeat it again only thing is we are changing the starting trial solution.

So, we have to whatever trial solution you start out with you have to make sure that it becomes admissible by substituting it the essential boundary condition, make it admissible and once you have the admissible trial solution; you substitute into the equivalent functional then you get i as a function of the unknown coefficient.

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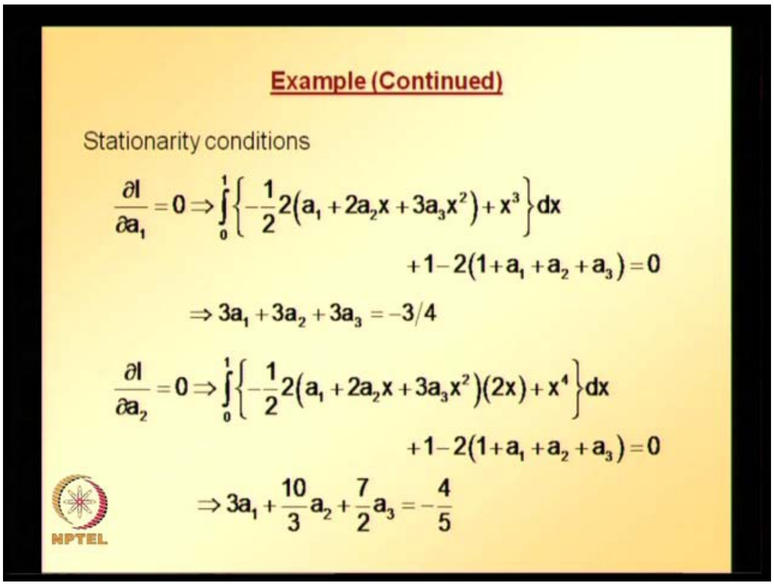

See here you have three numbers of unknown coefficients; that is a one, a two, a three and we need to apply the three stationarity conditions. I think by this time you know what are the stationarity conditions you apply this stationarity condition that is partial derivate of i with respect to a one is equal to 0 you get one equation. Partial derivate of i with respect to a two equal to 0 you get other equation. Partial derivate of i with respect to a three is equal to 0 you get the third equation so, straight forward.

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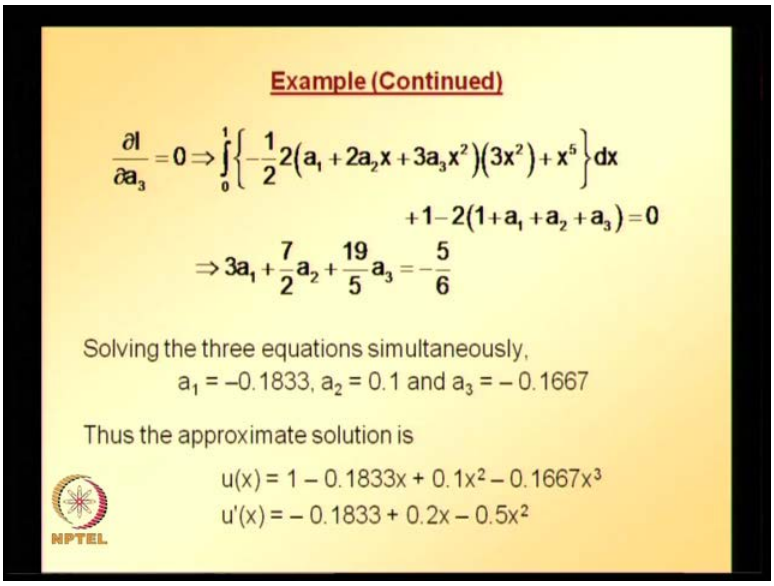

We have three equations three unknown coefficients. So, we can solve for the three unknown coefficients a one, a two, a three. And once we get this the values of a one, a two, a three; you can go back to your cubic admissible cubic trial solution and substitute in place of a one whatever value you got here, a two and a three and then you get the approximate solution. And you can take you can forcefully take derivative of it you get the derivative of approximate solution ok.

And as I mentioned a few few minutes back away one way of checking the accuracy of solution is may just check whether natural boundary condition is satisfied are not ok. So we got derivative of u. Expression is given here so, what we can do is we can substitute at in this equation x is equal to 1 and see what value we get whether that value is how close it to one so whatever the difference in the closeness that indicates the error in the approximation.

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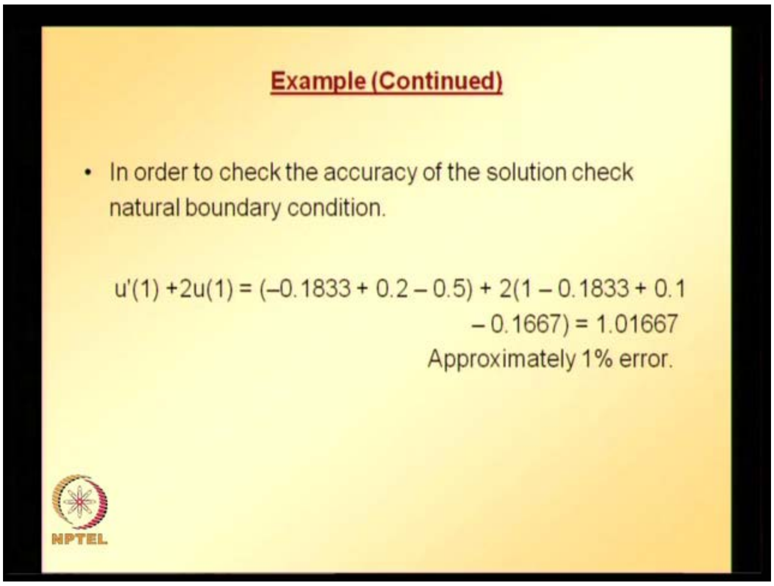

In order to check the accuracy of solution; check natural boundary condition. You can see as expected cubic trial solution results in a less error. Whereas quadratic trial solution you have ten percent error whereas cubic trial solution error became one percent ok.

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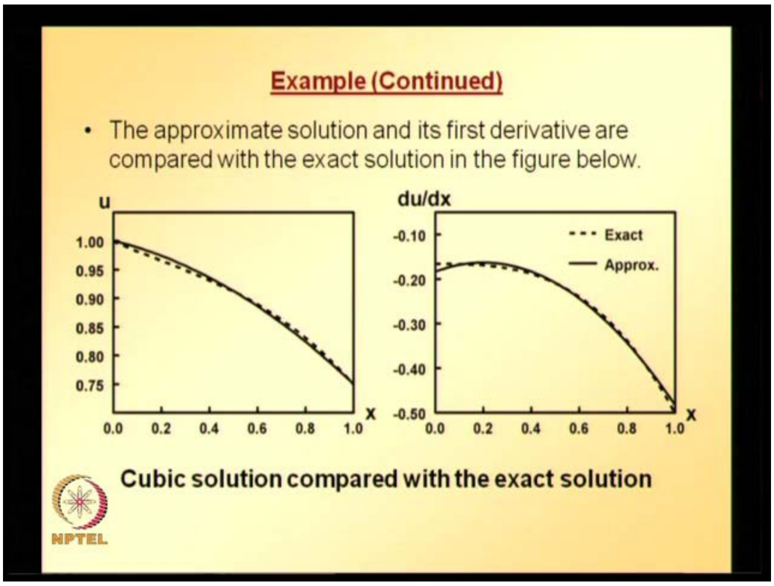

And we can further follow the same step and you can do the higher order trial solution approximation also. Here we the plot is shown how the approximate solution first derivatives are compared with exact solution for cubic trial solution. If you recall earlier with quadratic trial solution there is lot of error here at this location and at this location now the error got reduced.

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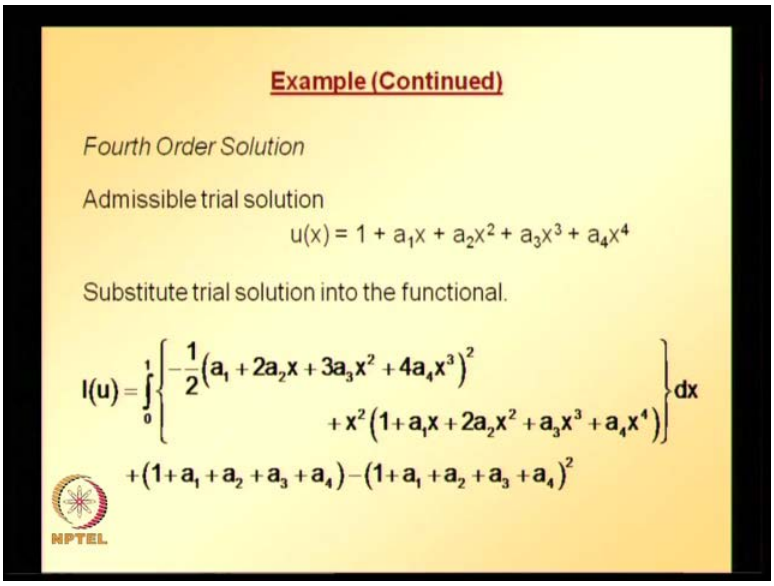

So now what we will do is; we will repeat this problem with a fourth order polynomial or quartic trial solution. Again the story is same that is, you start out with some polynomial. Here we decided to go with fourth order polynomial that is, u is equal to a naught plus a one x plus a two x square plus a three x cube plus a four x power four and when you substitute the essential boundary condition that is, u evaluated at x is equal to 0 is 0; u evaluated at x is equal to 0 is 1 sorry then that results in a naught is equal to 1.

So then, substitute a naught is equal to 1 you get the admissible fourth order trial solution. So, here you need to determined four coefficients; unknown coefficients a one, a two, a three, a four you substitute this into the equivalent functional. You get this equivalent functional became a function of four unknown coefficients you know, you can guess easily guess what is the next step? Apply the stationarity conditions.

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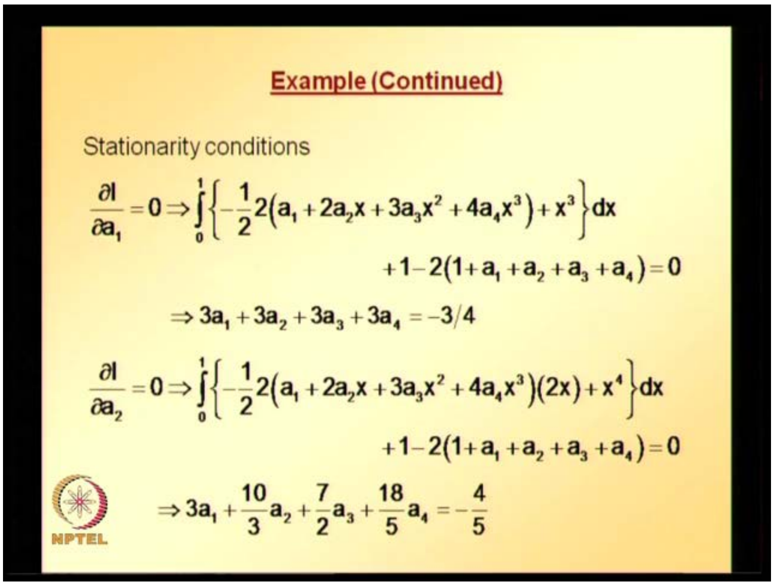

Partial derivate of i with respect to a one is equal to 0 you get one equation, partial derivate of i with respect to a two is equal to 0 you get the other equation and similarly, you get two more equations because we require in total four equations to solve for these four unknown coefficients.

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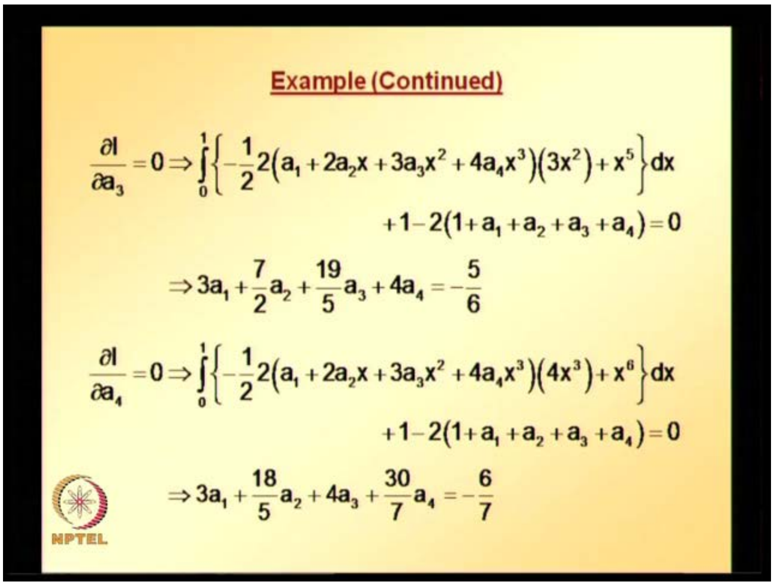

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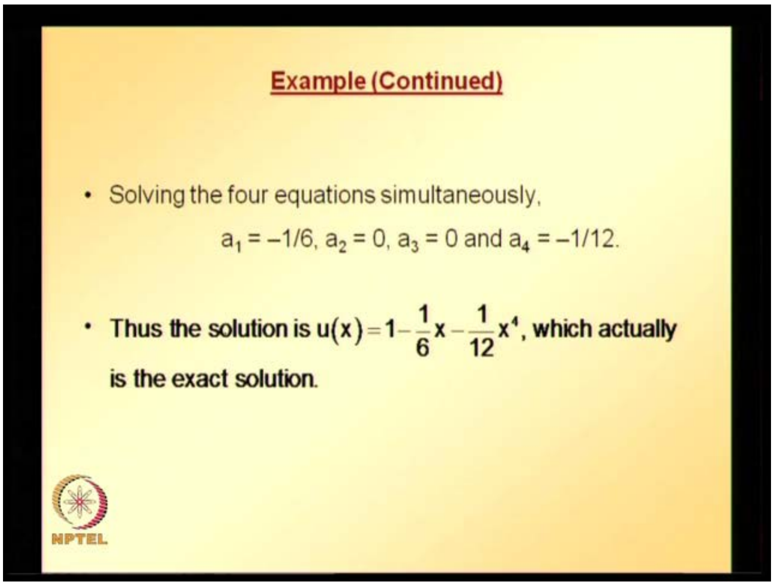

Third equation, fourth equation and what you can do is you can solve the four equations simultaneously to get a one, a two, a three, a four and what we need to do now is we need to substitute this coefficients into the trial solution. That is one plus a one x plus a two x square plus a three x cube plus a power a four x power four and surprisingly what you observe is whatever approximate solution that you get is same as exact solution.

So the in summary what we can say is, as we increase the order of the trial solution, we approach the exact solution but, at the expense of additional computational effort. Whenever you have more coefficients what you are basically you need to solve more number of equations to solve for these unknown coefficients.

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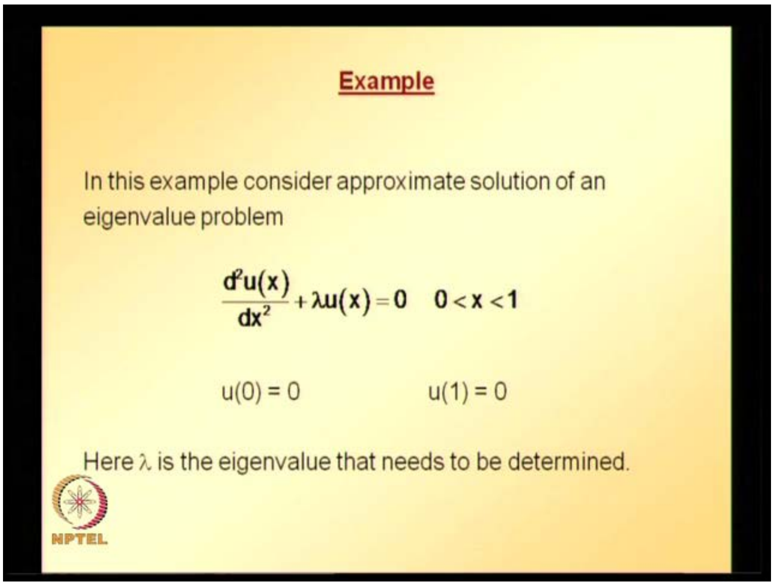

And what I will do is I will show you next example an Eigen value problem and this is an Eigen value problem and why it is an Eigen value problem there? If you see here this differential equation you have one quantity lambda is there because of that it becomes an Eigen value problem and this differential equation needs to be satisfied over the domain zero to one and as you can guess this is a second order differential equation.

So we require two boundary conditions the two boundary conditions for this problem are given here and you can go back and check these two boundary conditions are essential boundary condition because this is a second order differential equation and here lambda is the Eigen value that needs to be determined ok.

So what we do is, do not get scared with this lambda appearing here. You just follow the regular steps to obtain the equivalent functional and then you assume some trial solution, substitute the trial solution, apply the stationarity conditions, get the equations solve, for the unknown coefficients. All those steps are the same.

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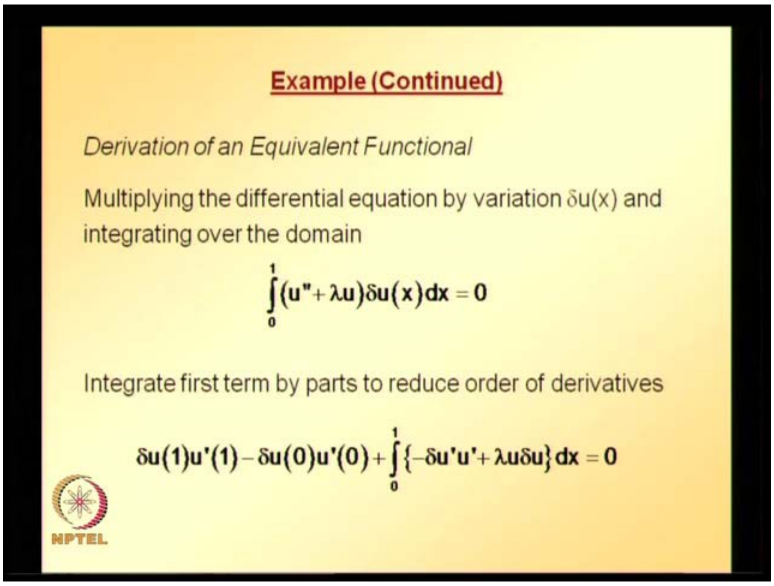

So the derivation of equivalent functional proceeds in this manner; multiply differential equation by variation of the quantity that you are looking for. That is variation of u. Integrate over the problem domain and second order derivative is appearing. The first term has second order derivative. So story is same; integration by parts. You need to apply, reduce the order of derivative and here we are very fortunate because at x is equal to 0 and x is equal to 1 essential boundary conditionals are prescribed. Essential boundary conditionals are given at x is equal to 0 and x is equal to 1.

So automatically the first two terms, boundary terms get cancelled. Wherever essential boundary condition is given variation of u at that point is equal to 0 or at least the variation of admissible solution at that point is equal to 0. So the first two terms gets cancelled because of that reasoning.

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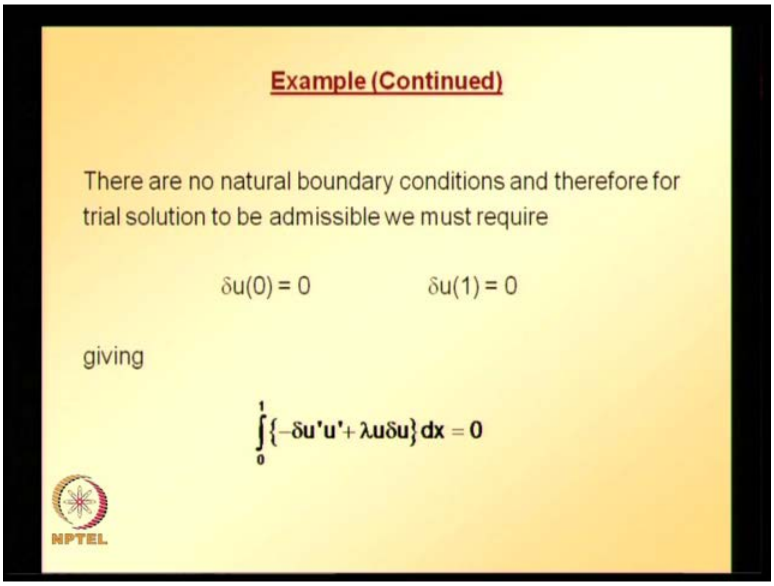

So that is what is mentioned here. There are no natural boundary condition therefore, for trial solution to be admissible we must require variation of u at x is equal to 0 is 0, variation of u at x is equal to 1 is 0.

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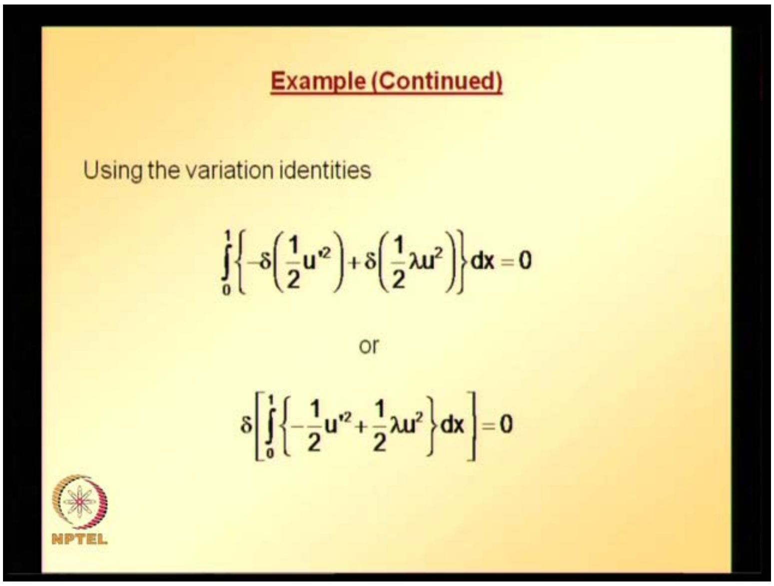

So with that reasoning the first two terms gets cancelled and you will be left with this. And here also you can apply variational identities and bring the variational operator out. The variational identities you can use are that is what is mentioned here. Using variational identities the first term can be written in the manner given here, the second term can be written like this. Now, it became like variation of u plus variation of v. So, we can bring the variational symbol operator out variation of u plus variation of e is nothing but, variation of u plus v. So the equivalent functional big is nothing but, whatever is inside the bracket.

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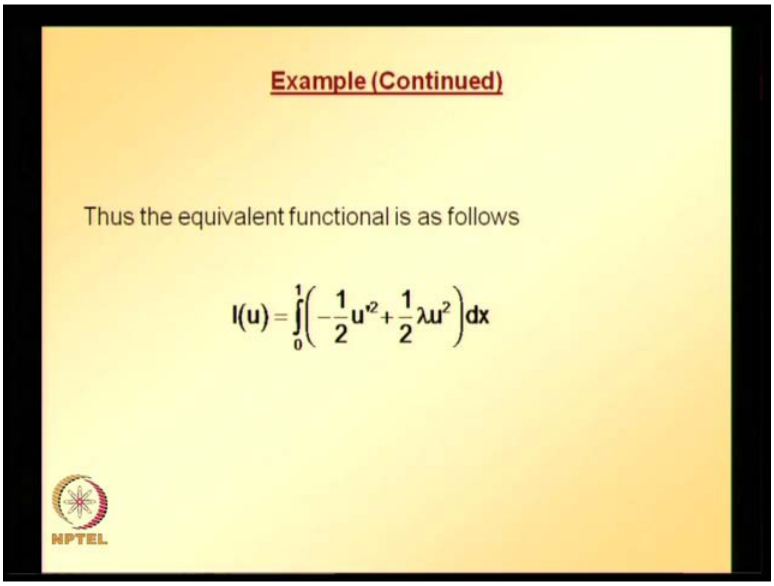

So the equivalent functional is here. Equivalent functional is again it is looking similar to what you have seen earlier except that lambda is, the additional quantity is lambda but, do not get scared. It is it becomes an Eigen value problem.

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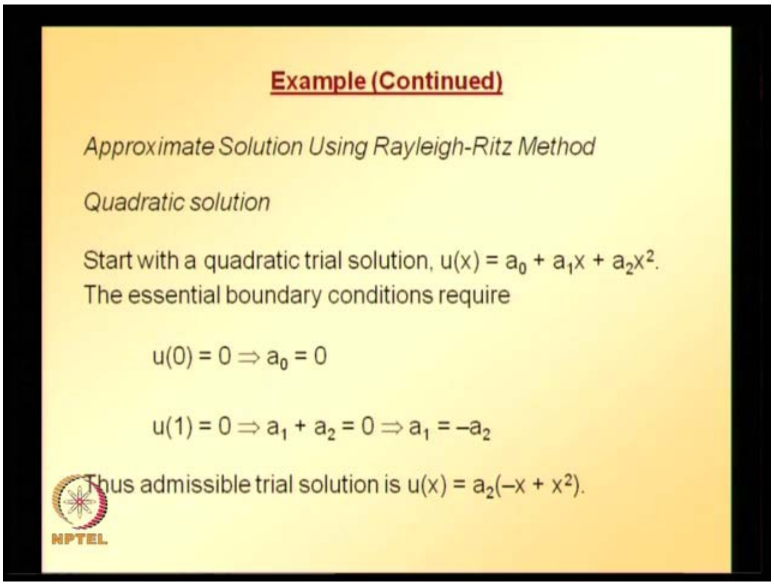

So we will go with quadratic trial solution and before we proceed further, what we need to do is, we need to make this admissible, substitute the essential boundary condition and then that results in a naught is equal to 0 and a one turns out to be minus of a two. So, the admissible trial solution becomes a two times minus x plus x square ok.

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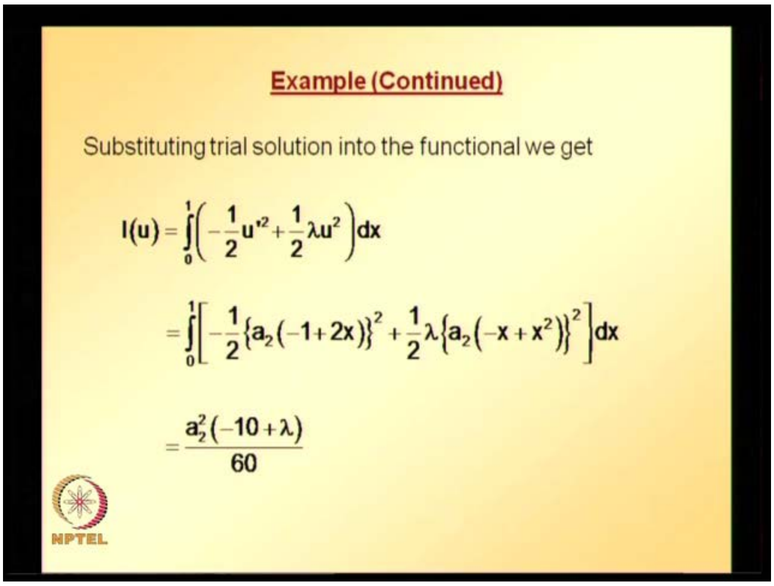

So now what you need to do is, you take this admissible trial solution and substitute into the functional. Functional becomes a function of only one coefficient, unknown coefficients a, only one unknown coefficient a two. And what is this stationarity condition? Partial derivate of i with respect to a two is equal to 0.

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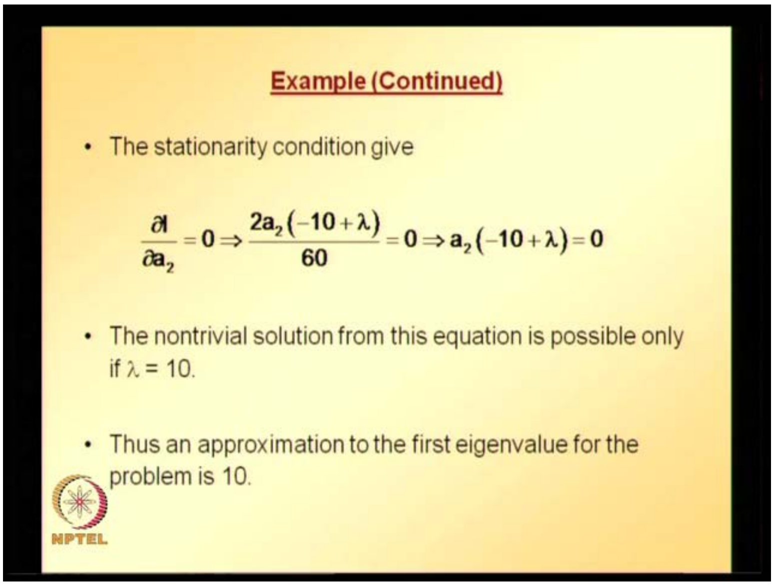

That results in one equation one unknown. And if you see this equation a two times minus ten plus lambda is equal to 0. This is what is this is what an Eigen value problem is. Here a two cannot be zero because if a two is zero; the actually all these coefficients a, a naught, a one, a two are all non zero coefficients. And a two cannot be zero and only way this this equation can be zero is lambda should be equal to 10. And a two is equal to 0 is a trivial solution but, we are looking for nontrivial solution so nontrivial solution leads to the thing that lambda is equal to 10. So the approximate first Eigen value for this problem is ten ok.

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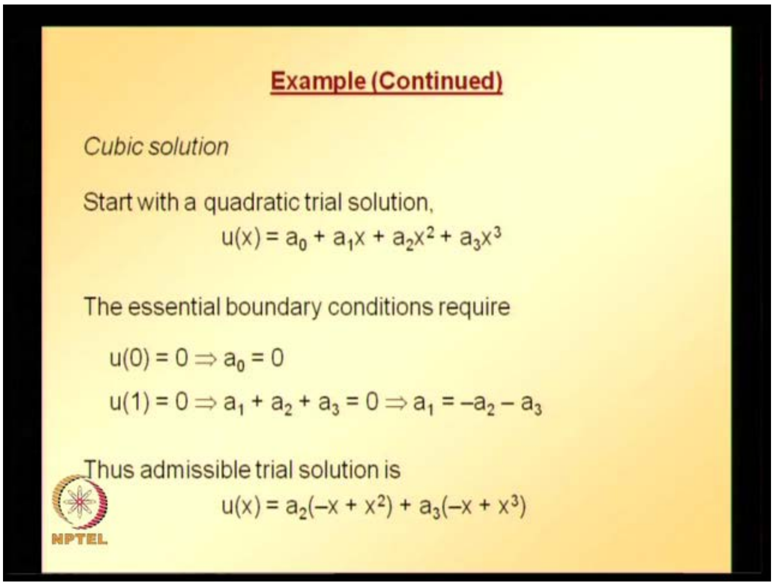

And you can, as we did for the previous problems what we can do is we can start out with a cubic trial solution. We will go through this process quickly, start with a quadratic sorry here it should be cubic. Start with a cubic trial solution a naught plus a one x plus a two x square plus a three x cube and substitute the essential boundary conditions and that leads to a naught is equal to 0 a one is equal to minus a two minus a three. So the admissible trial solution becomes a two times minus x plus x square plus a three times minus x plus x cube.

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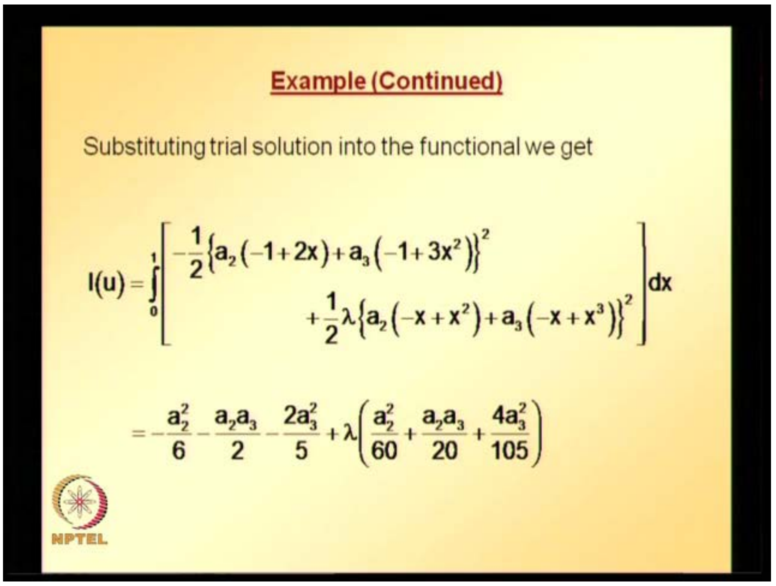

And substitute this admissible trial solution into the functional. Functional is going to become a function of unknown coefficients a two and a three ok.

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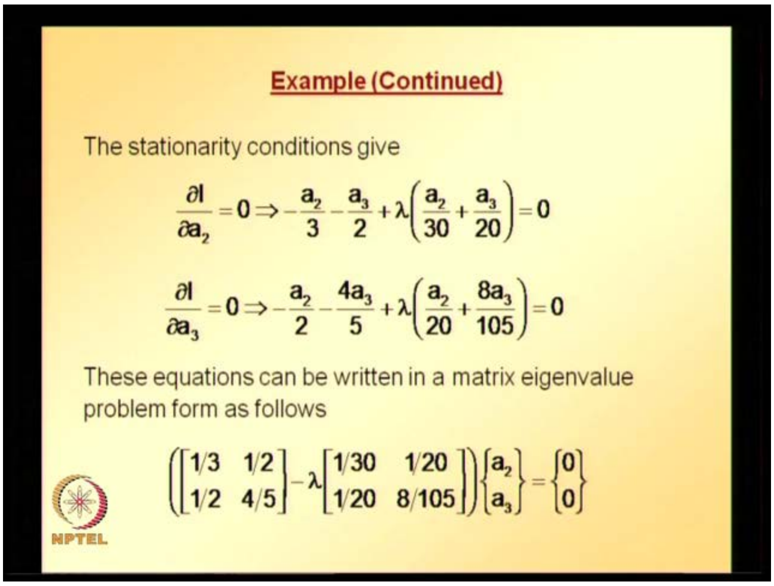

So you got this, as a function of a two and a three. Next step is, apply the stationarity conditions; two equations, two unknowns. You can put this in a a matrix form these two equations can be put in a matrix form and if you see this equation system, how it is looking? It is looking like a minus lambda b times x is equal to 0 which is a Eigen value problem. For nontrivial solutions that is, non zero value of a two, a three means non zero value of that vector a two. One of them can be zero but, both of them cannot be zero. So, for non zero value of that a two a three vector the condition is for nontrivial solution a minus lambda times b determinant of that should be equal to 0 which results in a a characteristic characteristic equation of order if it is a a two by two matrix system; you will get a characteristic equation of order two which when you solve you get two roots and those roots gives you a first Eigen value, second Eigen value of lambda ok.

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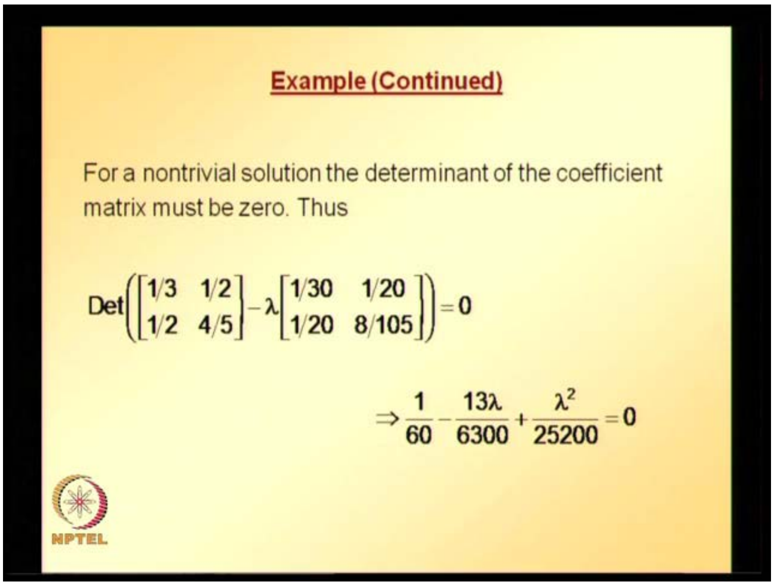

So that is a condition that what we are going to apply here. For nontrivial solution; determinant of the coefficient matrix must be zero. That is this one. So, this is what I am saying characteristic equation and depending on the the magnitude or size of the size of the matrix that you have the order of the characteristic equation depends on that. Here you get a quadratic characteristic equation which you can solve for lambda.

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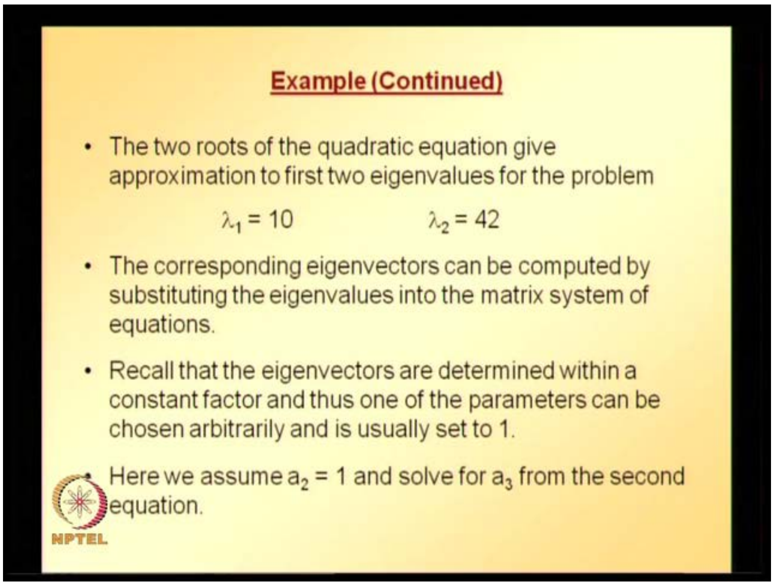

And you get lambda you get two roots lambda one lambda two and once you get the two roots; lambda one, lambda two. The corresponding Eigen vectors can be computed by substituting these Eigen values back into the matrix system of equations. i I hope you are familiar with this Eigen value problem solving. Recall that eigenvectors are determined within a constant factor and thus one of the parameters can be chosen arbitrarily and is equal and usually set to 1 ok. So what we will do is we will take lambda is equal to 10 and we assume a two is equal to 1 and solve for a three. the We get first Eigen vector by doing that and we are solving this equation system.

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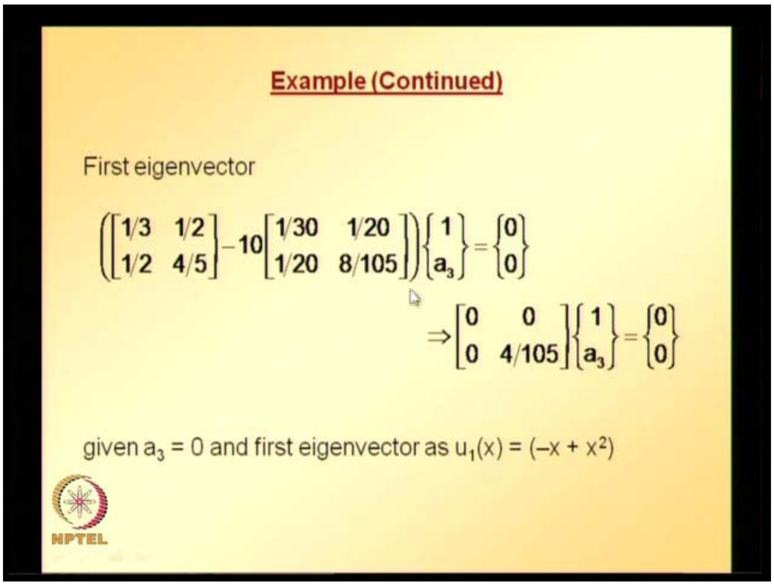

We get a three is equal to 0 but, a one is equal to a two is equal to one. So, the first eigenvector is given here and similarly, you can take lambda is equal to 42 and repeat this process you get the second Eigen vector.

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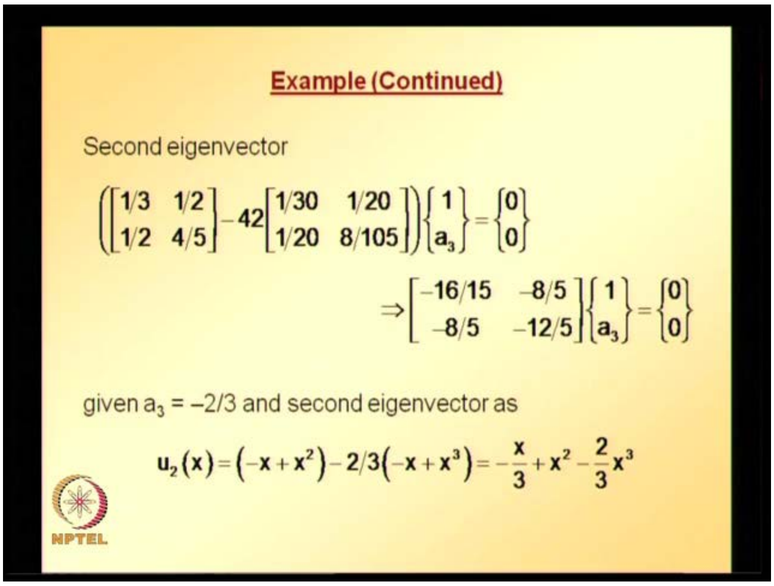

And the second Eigen vector is this. So, we will continue in the next class.