Finite Element Analysis

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Lecture No. # 29

In the last class, we have seen a relatively simple method for deriving shape functions for higher order triangular elements. And also we looked at concept of isoparametric mapping for triangular elements.

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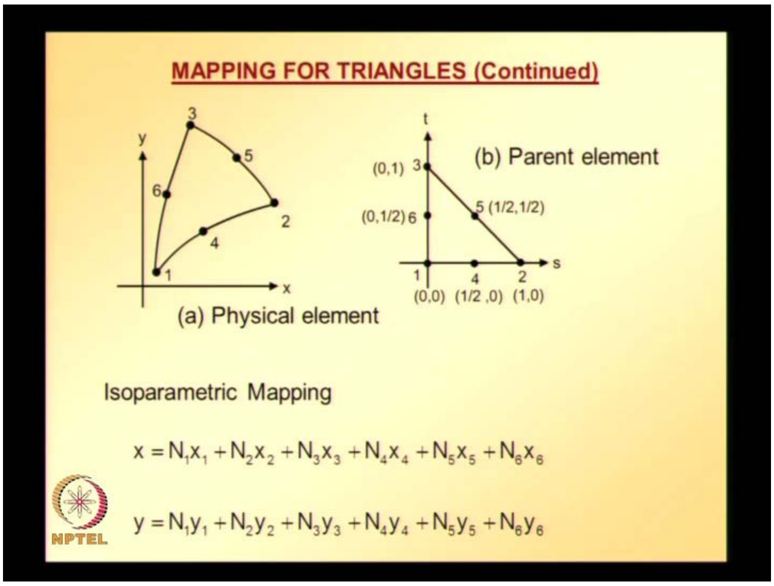

In today's class, we will continue with that. So, here a six nodded triangular element is shown, physical element. And also six nodded triangular element, parent element is also shown. Usually any arbitrary physical element, we are going to map on to a parent element as shown in figure b there. The relationship between x, y coordinates of the physical element and s and t coordinates of parent element is given by these relations isoparametric mapping; x as x is equal to N1 x1 plus N2 x2 all the way to the number of nodes that we have for that particular element. Similarly, using these two relations we get x and y coordinates of physical element in terms of s and t.

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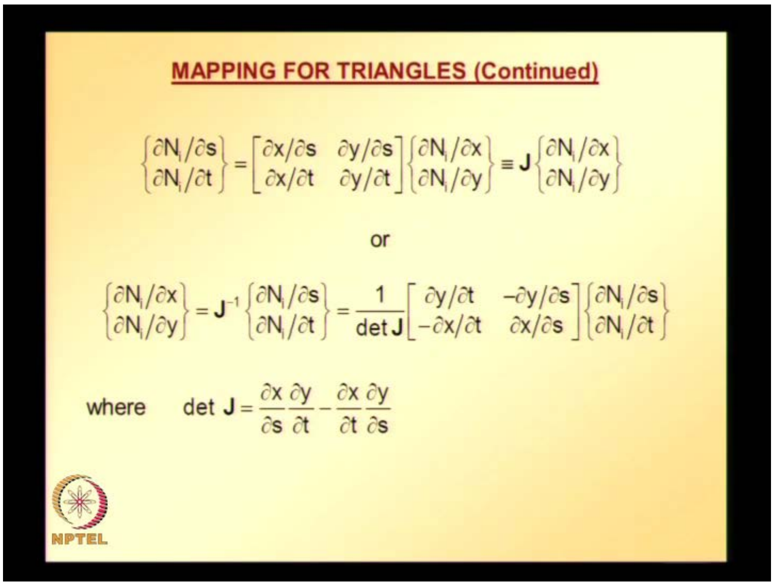

Once we have these relations, we can write the derivative of shape functions with respect to s and t can be expressed or they can calculated. Once we know derivatives of shape functions with respect to x and y. And this relation can be compactly written as J times where J is Jacobian, J times partial derivative of shape functions with respect x and y of the inverse relation. If we know derivatives of shape functions, with respect to s and t how to calculate derivatives of shape functions with respect to x and y given by this inverse relation . And one of the requirement is determinant of J should not be equal to 0. And determinant of j is defined like above.

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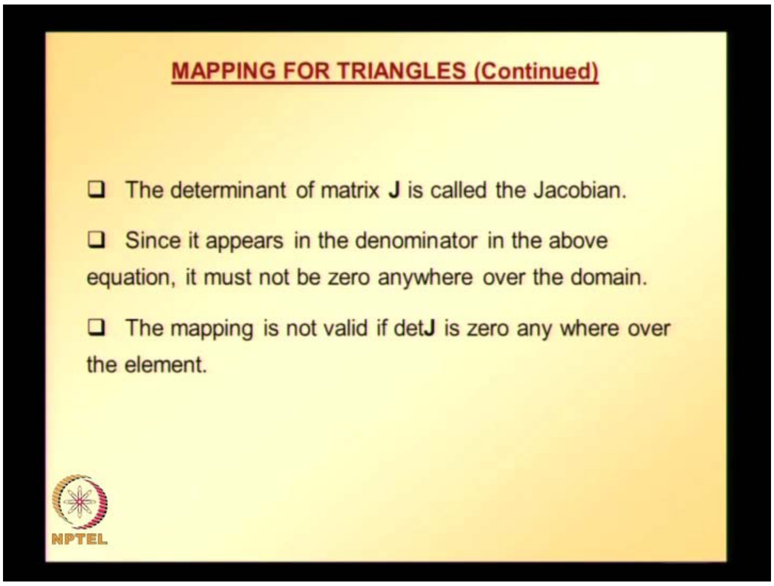

In the last class, we have discussed some of the requirements like the determinant. Since, it appears in the denominator of the above equation. It must not be zero anywhere over the domain. The mapping is valid, if determinant of J is not equal to zero or the mapping is not valid, if determinant of J is zero anywhere over the element. And this requirement is similar to what we have seen for even quadrilateral elements.

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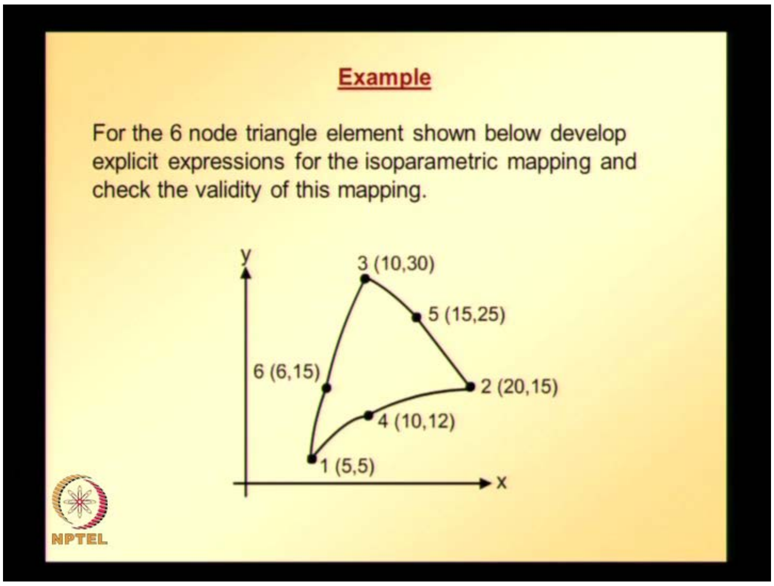

In today's class, what we will be doing is? We will take some examples. And check the validity of mapping for triangular elements .The first example for the six node triangle element shown above develop explicit expressions for the isoparametric mapping and check the validity of this mapping. The steps basically are same as what we did for quadrilateral element. So, first we need to write isoparametric relations of x and y in terms of s and t. And using those relations, we need to take partial derivative of x and y with respect to s and t. And find determinant of J and one way is to look at determinant of J expression. And see whether it is positive everywhere from s going over the entire domain. Here for triangular element s goes from 0 to 1 t goes from 0 to 1 or what we can do is? We can plot x y in terms of s and t over the domain of the parent element. And see whether the shape of the element is similar to what we started out or if there are any folds, then we can check for those. This is a six node triangular element. All the nodal coordinates are given x, y coordinates of all nodes are given.

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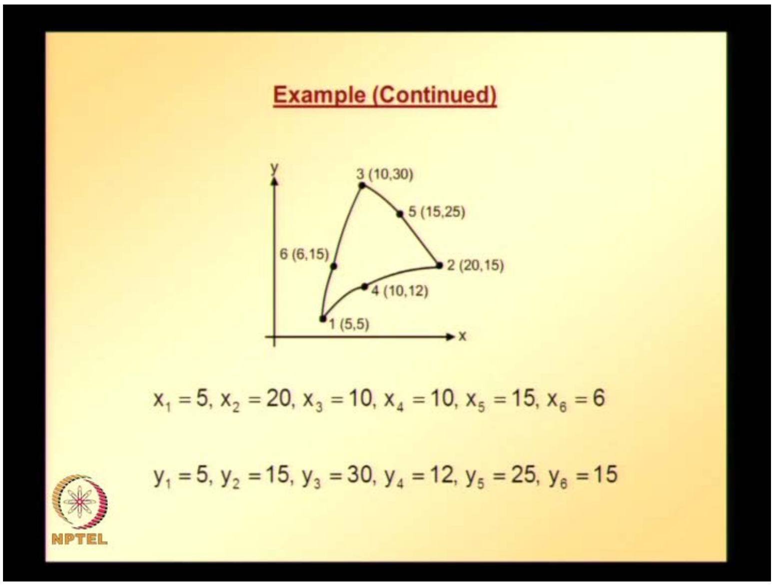

With this information, we can write the x coordinates and y coordinates of all nodes. And then once we have this information, we can write using isoparametric relations.

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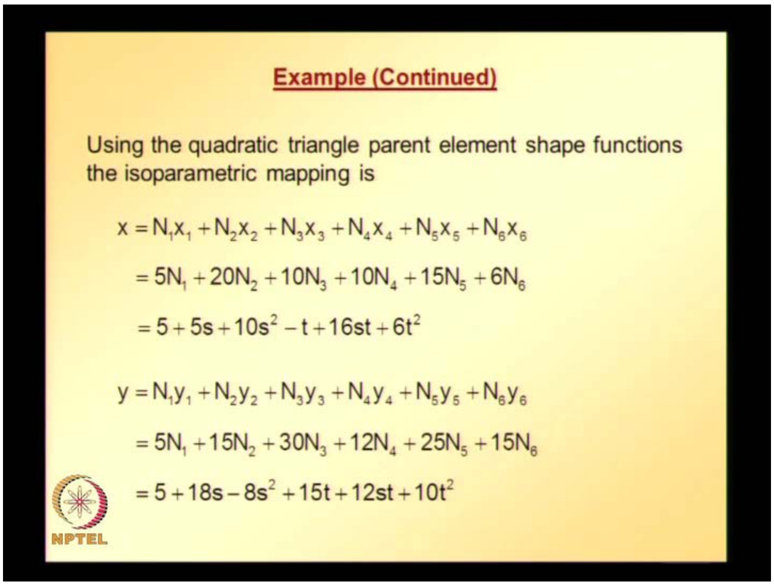

We can write x in terms of s and t, where shape functions corresponding to the parent element in terms of s and t. Substituting N1 to N6, which we already found or which we already derived in last class for six node triangle element. Similarly, y in terms of s and t, we can obtain using the shape functions for all the six nodes of the parent element in terms of s and t.

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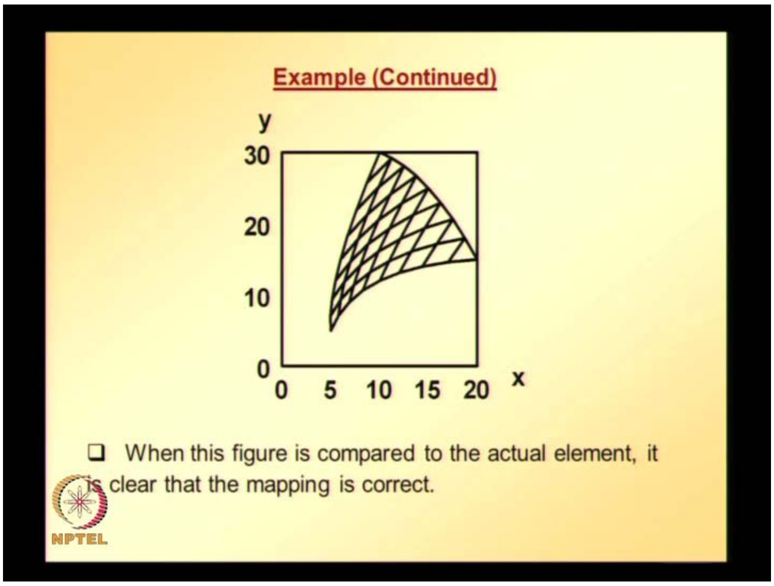

Once we have these relations, we can plot the equations of x and y for the range s going from 0 to 1, t going from 0 to 1. By giving different values of s and t, and plotting the equations corresponding to x and y; we get this figure. And this figure shows a triangle which is similar to what we started out with when this figure is compared to the actual element, it is clear that mapping is correct or we already know the relationship between x and y and s and t.

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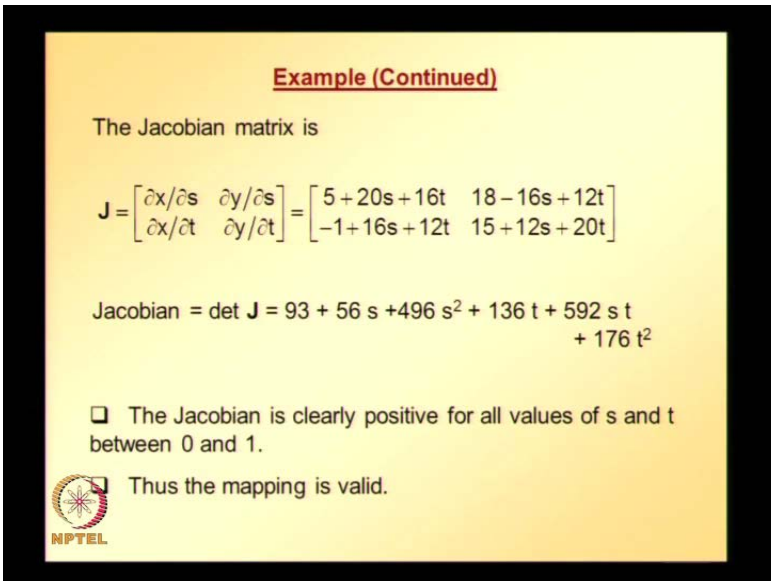

Using the information, we can even find jacobian matrix and determinant of Jacobian. And we can check whether this determinant of j is greater than zero for all values of s and t between 0 and 1. It can be easily checked that this is indeed positive for values all s and t between 0 and 1, so the mapping is valid. This is how we can check validity of mapping. And also this is how we can get isoparametric relations. Let us take this example again except that we move node number 4 and 6 little bit closer to node 1.

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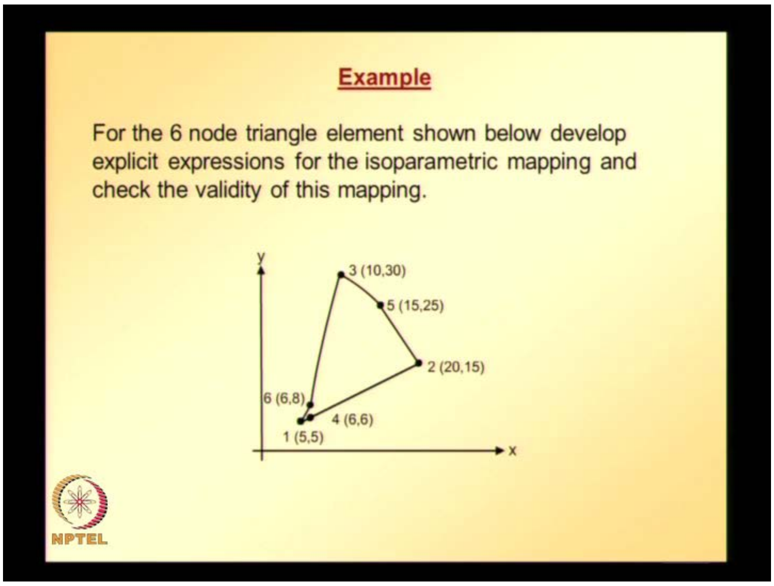

The location of nodes looks like this for the new element. For the six node triangle element shown above develop explicit expressions for the isoparametric mapping. And check the validity of mapping.

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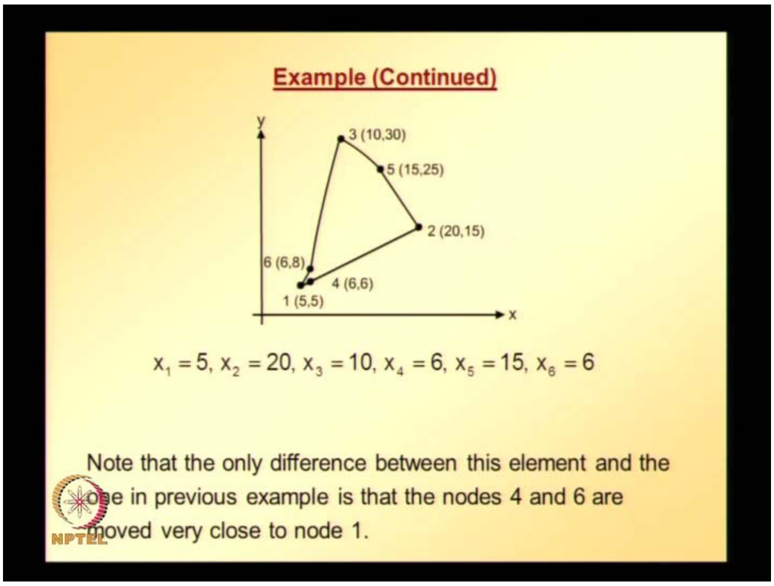

If you see what is the difference between previous element, and the above element. The only difference between this element, and the one in the previous example is that nodes 4 and 6 are moved very close to node one. We learnt earlier that the interior node should satisfy the condition that it should lie between L over 4 and 3 L over 3. Once that condition is violated that reflects in the mapping, which is what we are going to observe in this example.

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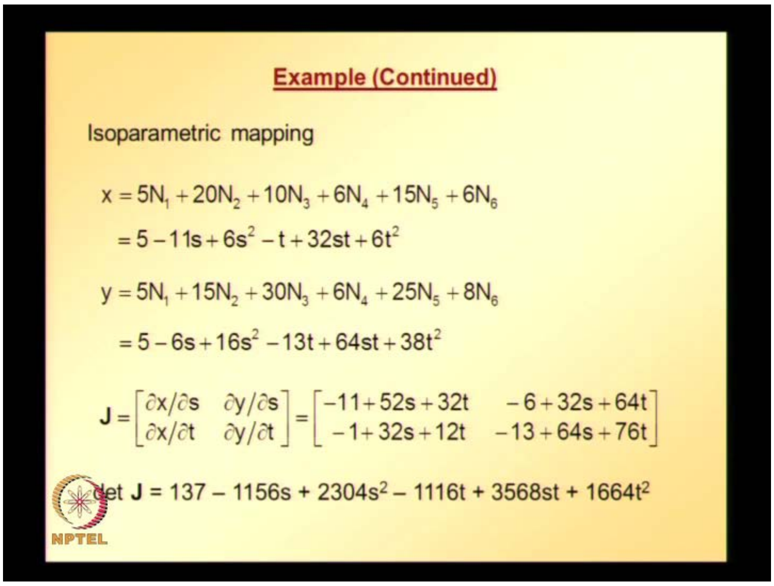

With this information of nodal coordinates, we can write isoparametric mapping expressions of x and y in terms of s and t. And once we have this, we can plot x and y for different values of s and t. And see how the plot looks like or we can also find determinant of J. And check whether it is greater than zero over the domain s going from 0 to 1, t going from 0 to 1. This is J, which is calculated from x and y determinant of J.

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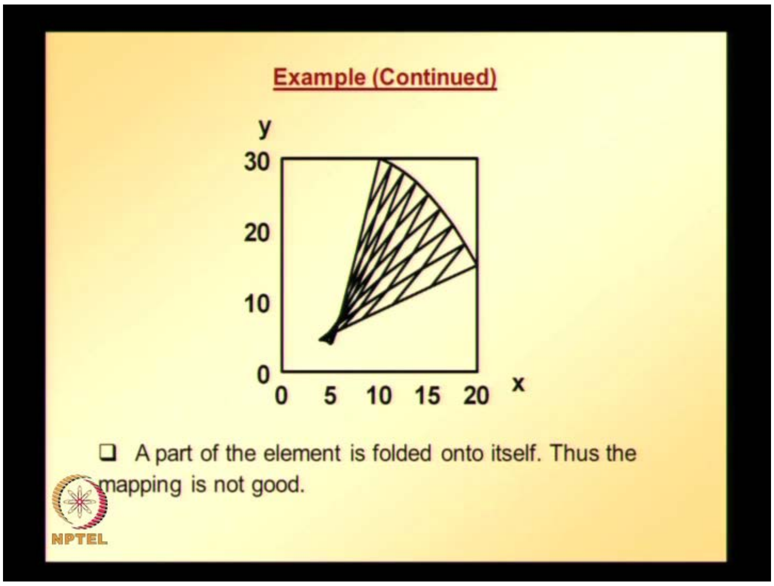

The is plot of x, y expressions in terms of s and t. And you can see here, a part of element is folded on to itself thus mapping is not good.

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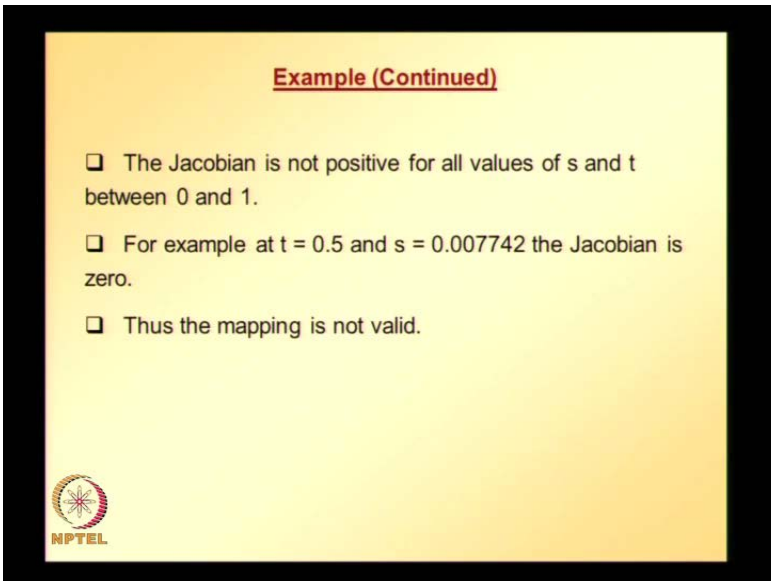

The above can also be verify by checking the jacobian or determinant of Jacobian, and it can be easily verified that the determinant of Jacobian that we just calculated is not positive for all values of s and t between 0 and 1. For example, at t is equal to 0.5 and s is equal to 0.007742. Jacobian is zero mapping is having some problem, it is not valid mapping. So, this is how we can check the validity of isoparametric mapping.

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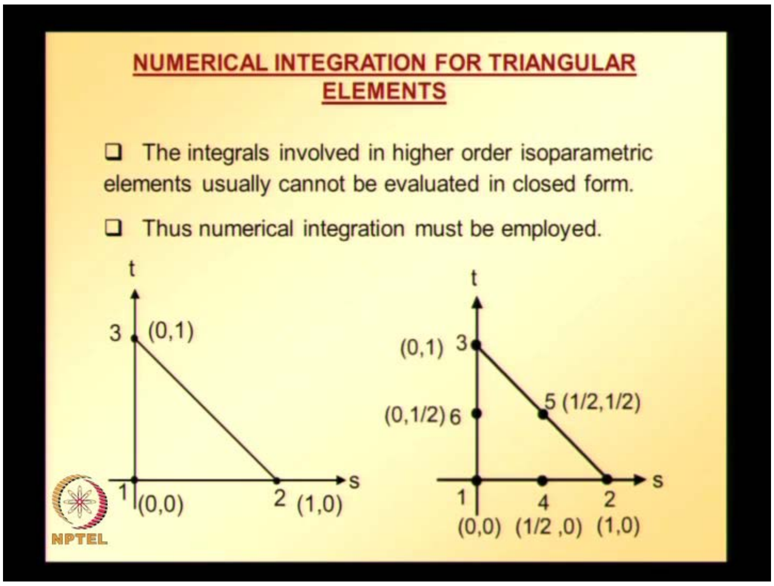

Now, let us go to the other concept that is numerical integration for triangles. When we have integrals especially for higher order elements the integrals that we usually encounter are not going to be very simple integrals. Sometimes, we need to adopt numerical integration we will discuss that now. The integral integrals involved in higher order isoparametric elements usually cannot be evaluated in closed form. Thus numerical integration must be employed. And recall that any arbitrary element whether it is three nodded or six nodded triangle element. The actual or physical element is going to be mapped on to parent elements like this. If it is three node triangle elements, it is going to be mapped on to three node parent element. If it is six node triangle elements, then it is going to be mapped on to six node parent element as shown in the second figure whether a particular triangle element is mapped on to either of these. You can easily see these two, the vertices of this two triangle elements are located at 0010, and 01 also s goes from 0 to 1 t goes from 0 to 1.

Equation of the inclined side 2 3 or equation of the side 2 3 for both the triangles is given by s plus t is equal to 1. If one wants to evaluate an area integral over this triangle element, then the limits of integration are going to be 0 limits for s are going to be 0 to 1 minus t. And limits for t are going to be 0 to 1 that is what is written. Let i be the integral which needs to be evaluated or this function needs to be evaluated over triangle area.

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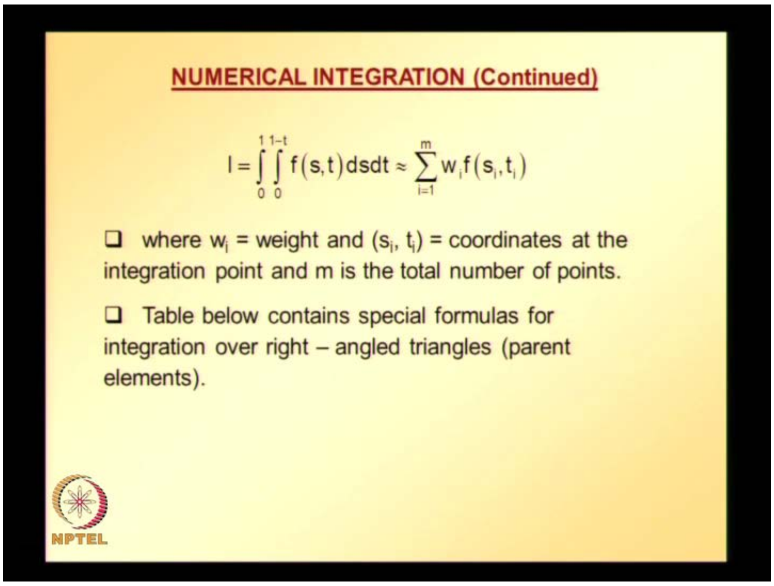

The limits of integration are 0 to 1 minus t, the limits of s and limits of t are 0 and 1, if one wants to evaluate this integral, it can be evaluated or it can be approximated in the way that is shown i goes from 1 to m. W i times integrant evaluated at each of the integration points. One needs to know if the order of integration is decided, one needs to know what are the weights. And what are the coordinates of that particular order of integration where W i is the weight, s i, t i are the coordinates of integration point m is the total number of points that are selected. The table above contains some information about the points and the weights .And also about the degree of accuracy of that particular order of integration.

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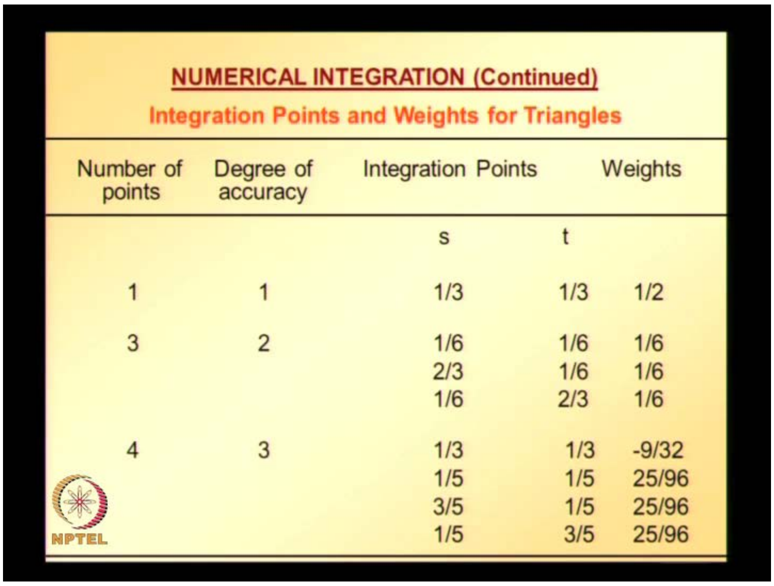

NUMERICAL INTEGRATION (Continued)

Integration Points and Weights for Triangles

Number of points	Degree of accuracy	Integration Points s	Integration Points t	Weights
1	1	1/3	1/3	1/2
3	2	1/6	1/6	1/6
		2/3	1/6	1/6
		1/6	2/3	1/6
4	3	1/3	1/3	-9/32
		1/5	1/5	25/96
		3/5	1/5	25/96
		1/5	3/5	25/96

Integration points and weights for triangles, if we decide to use one point integration then we need to evaluate that at s is equal to 1 over 3, t is equal to 1 over 3 multiply integrand evaluated at 1 over 3, 1 over 3 and multiply with weight half. If number of integration points is selected as 3, then we need to evaluate integrand at 1 over 6, 1 over 6 multiply with 8, 1 over 6 plus evaluate integrand at 2 over 3. And 1 over 6 multiply with weight 1 over 6 plus evaluate integrand at 1 over 6, 2 over 3 multiply with weight 1 over 6. And sum up all the contributions from all the three points then we get the final approximate value of those integrals. So, this is how we can evaluate. And similar procedure can even be repeated if the number of points is selected as four except that, we need to evaluate integrant at all the four points coordinates of which are have given in the table multiply with corresponding weight and sum it up.

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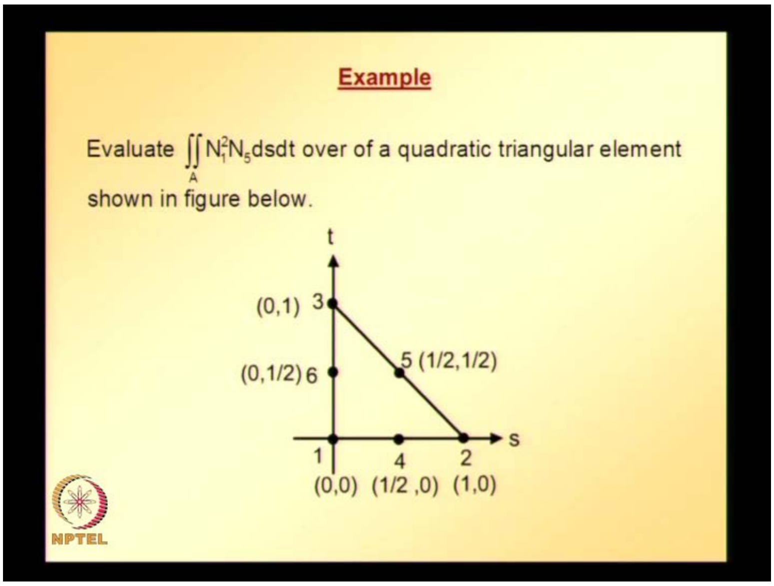

To illustrate this, let us take an example. Evaluate integral over triangular area N1 square plus N5 ds dt of a quadratic triangle element shown in figure above. For this kind of element, we already know this shape function expressions for N2 to N6. We can easily substitute what is N1 and N5? And simplify that integrand, and use apply numerical integration scheme corresponding to the triangle element and approximate that integral.

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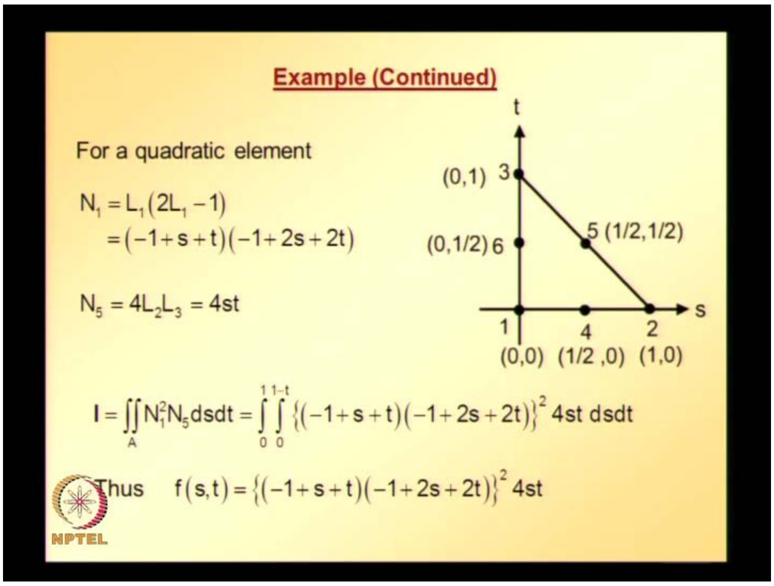

For this particular element, N1 is given by this and N5 is given by this, which we already derived in the last class. Substituting this information of N1 and N5 five into the given integral i simplifies to what is shown there. And as you can see integrand is complicated. So, we need to adapt numerical integration scheme. And we need to identify what is the integrand. This is f s f as a function of s and t which is given. There is integrand depending upon the number of integration points that we decide to use. We need to evaluate this integrand at so many integration points multiply with corresponding weights and sum it up.

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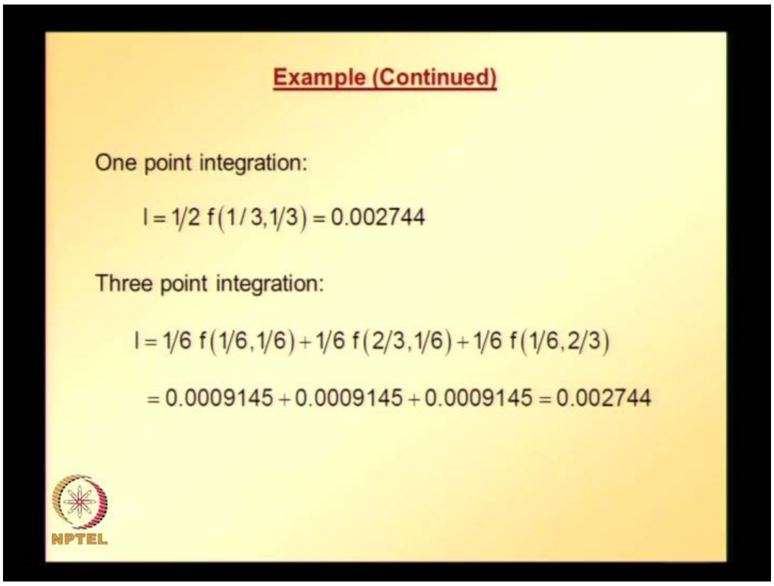

Let us start with one point integration. Integrand needs to be evaluated integration point coordinates for one point integration for triangle element is 1 over 3, 1 over 3. So, integrand needs to be evaluated at that point multiply with weight which is half gives approximate value of integral. Next is three point integration this is how integral is approximate, and evaluate the functions at the corresponding points simplify we get value to be this.

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Example (Continued)

Four point integration:

$$I = -9/32\, f(1/3, 1/3) + 25/96\, f(1/5, 1/5) + 25/96\, f(3/5, 1/5) + 25/96\, f(1/5, 3/5)$$

$$= -0.001543 + 0.0006 + 0.0018 + 0.0018 = 0.002657$$

It can easily be verified that the exact integral = 0.001587

You can even use four point integration except that integrand needs to be evaluated at four integration points multiply with corresponding weights and sum up. As you can see here the as we increase the order of integration we approach closer to the exact solution. This is about numerical integration over triangle element.

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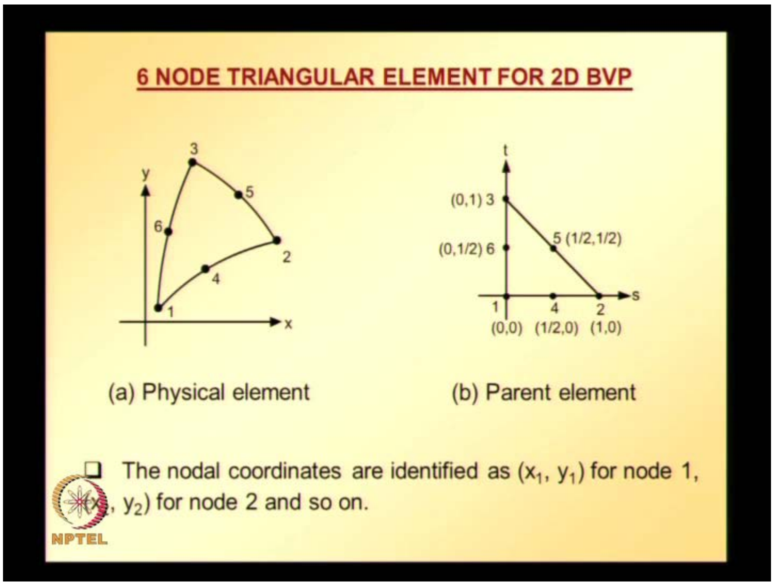

And now we will go, and discuss the how to solve two-dimensional; general two-dimensional boundary value problem using six node triangle element. For that we need to first develop the element equations, and the procedure for developing element equations for 2 D boundary value problem is similar to what we did for quadrilateral elements or even for that matter linear triangle element. A six node quadratic triangle element with curved boundaries, and it is parent element are shown here. Physical element and parent element of six node triangle is shown. As a nodal coordinates of the physical element are identified as x 1, y 1 for node 1 x 2 y 2 for node two and so on, and we can put all the x coordinates and y coordinates of parent element in a vector.

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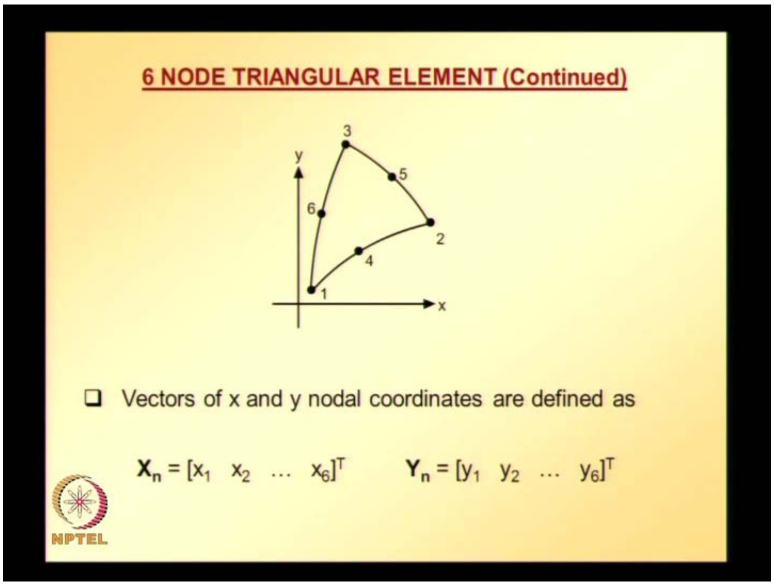

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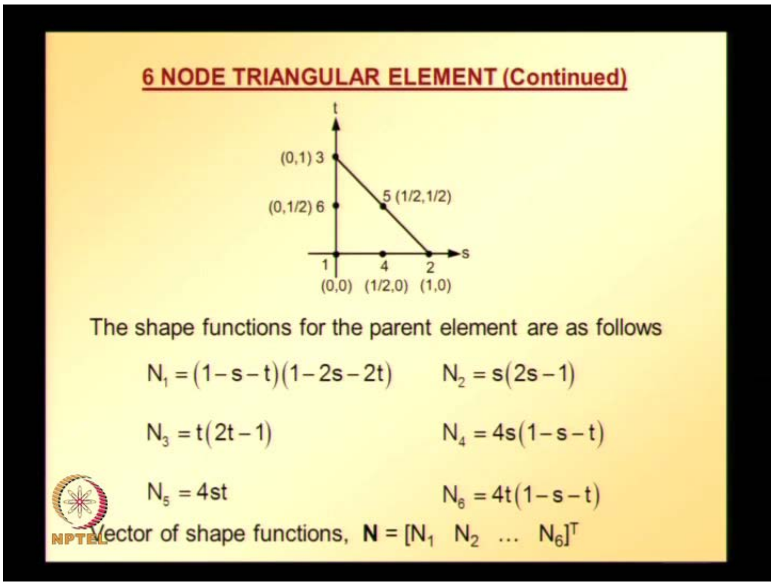

Vectors of x and y coordinates are defined like this. And for isoparametric mapping we require shape functions of six node triangle element in parent coordinate system or shape functions corresponding to parent element. This is six node triangle element in parent coordinate system the shape functions are as given here. We can put all the shape functions in a vector denoted with bold n.

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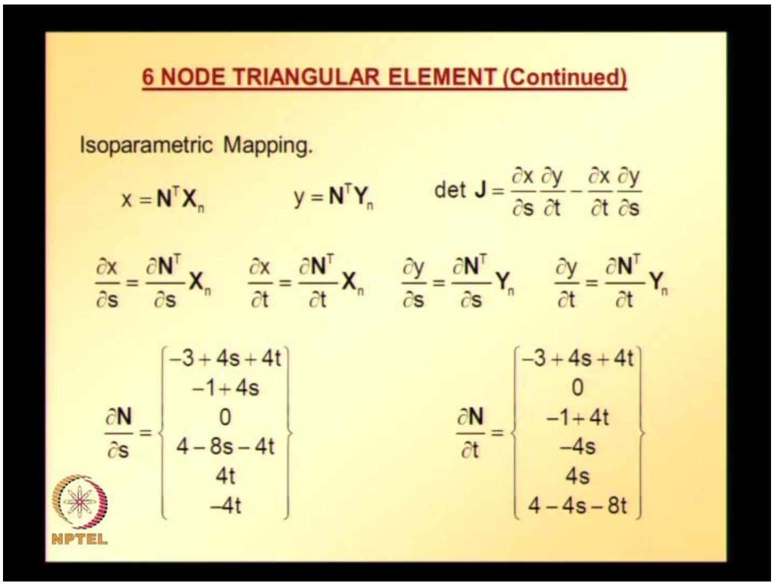

Isoparametric relations, using this we can find determinant of J. And once we have that we can find what is derivative of shape functions with respect to s and t. All these procedure is similar to what we are repeatedly doing for different kind of elements. Once we have this derivatives with respect s and t; we can find derivatives with respect to x and y derivatives with respect to x and y can be computed like this which are denoted with b x and b y.

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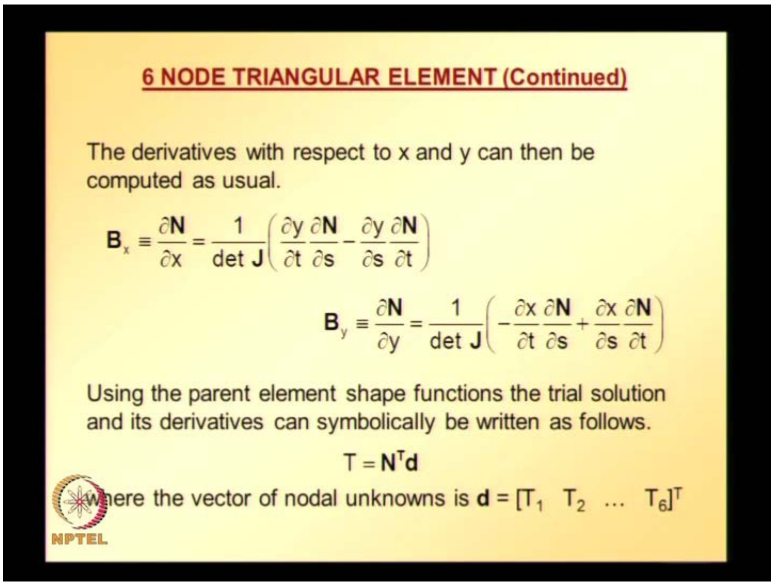

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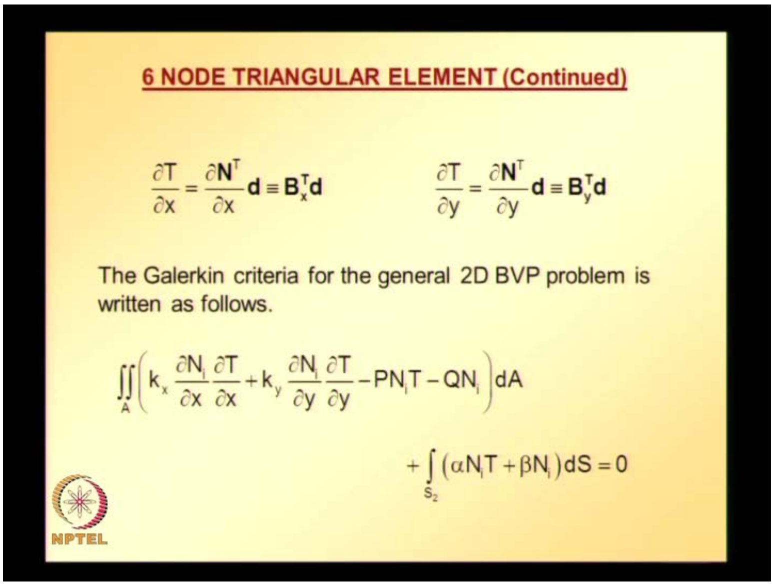

Using parent element shape functions the trial solutions and it is derivatives can be written as follows this is how p can be approximated. T is equal to N transpose d where N is a vector comprising of shape functions of all six nodes. And d is comprising of the nodal unknowns or nodal parameters six in number T1 to T6. And this is how we can get derivative of trial solution. Once we have all this, we can substitute into the Galerkin criteria of general two d boundary value problem. Now, let us look at what is Galerkin criteria? And we discuss this number of times earlier in conjunction with quadratic element sorry quadrilateral elements, and linear triangle elements this is the Galerkin criteria for 2 D boundary value problem; that we are looking at and now we have the trial solution and derivative of trial solution. We can substitute that information into this equation and get complete element equations.

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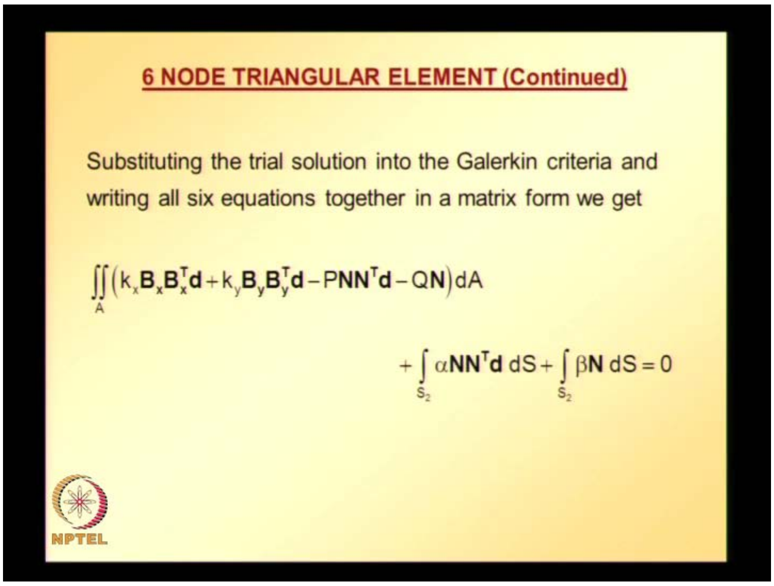

Substituting derivative of trial solution into this equation, and writing all six equations together in a matrix form we get this.

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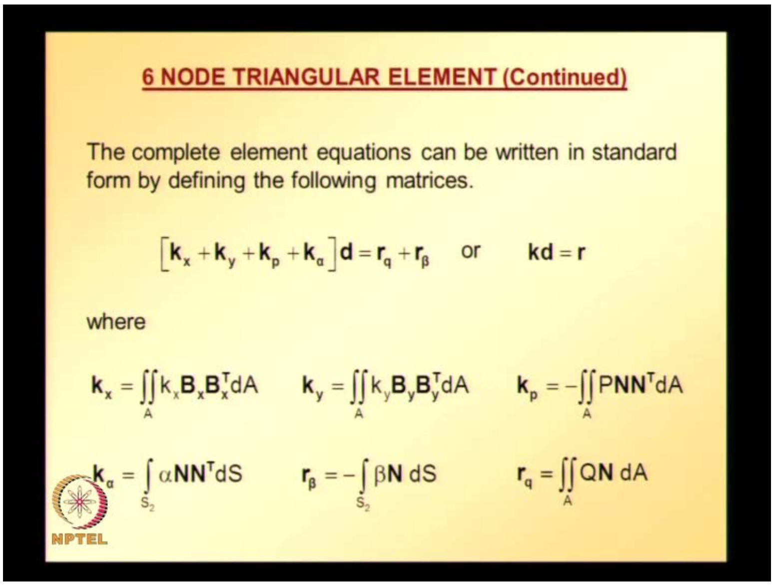

We can define k x, k y, k p, k alpha, r beta, r q to write complete element equations as k d equal to r. Similar, to what we did for quadrilateral elements. If you see here we have area integrals; that is integrand needs to be evaluated over triangular element area. And sometimes integrand needs to be evaluated over boundary of element. We need to discuss about integration similar to what we did for quadrilateral element for these triangle elements.

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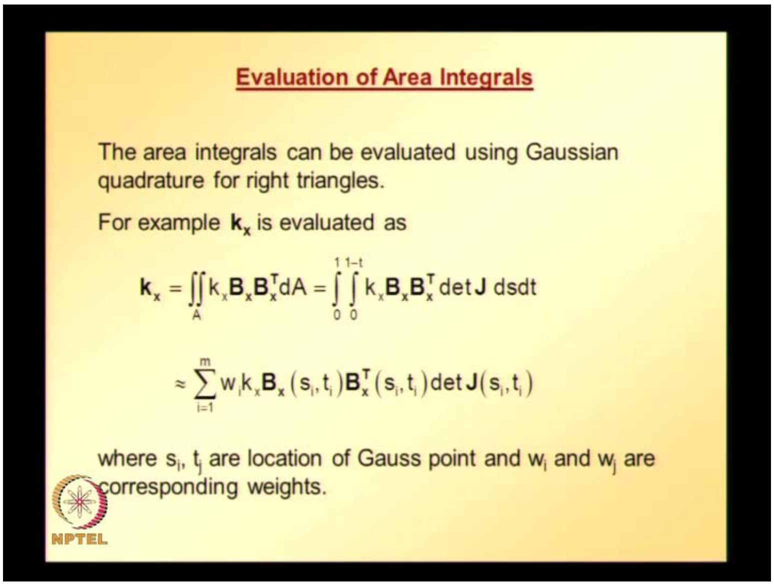

Evaluation of area integrals area integrals can be evaluated using Gaussian quadrature for right triangles. For example k x is evaluated as only thing is since we are dealing with triangle elements limits of integration for s goes from 0 to 1 minus t and for t goes from 0 to 1 with that understanding, we can proceed. And get the number or once we choose the once we decide the number of integration points we can get the points coordinates and weights from the table that is given. So, s and t are the locations of gauss point W i, w j are corresponding weights, so evaluating the integrand at the each of these points multiply with corresponding weight we can sum it up, and get the approximate value of integral.

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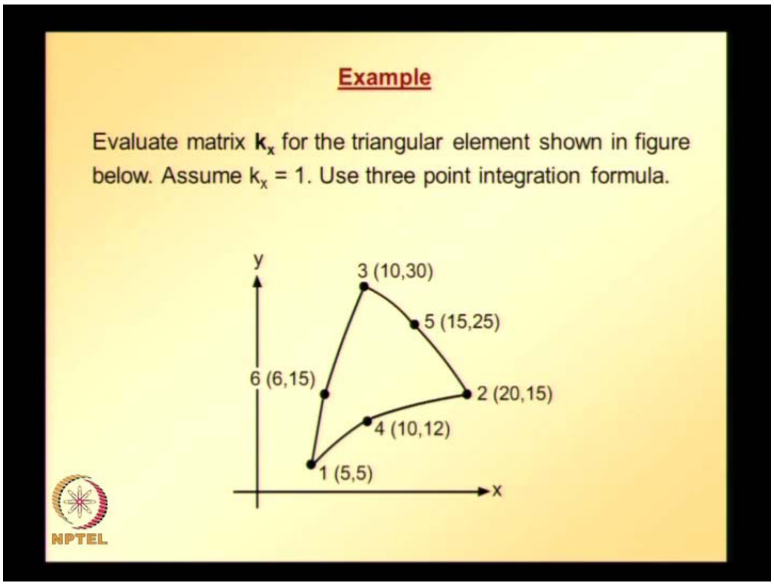

To illustrate, let us take an example. Evaluate matrix k x for triangle element shown in figure above. Assume the coefficient k x is equal to 1, and use three point integration formula .And this is the six node triangle element that is given, and from the information that is given in the figure we can easily figure out, what are the nodal coordinates? We can write all the x coordinates, and y coordinates in a vector form. And that is the starting point, and also it is suggested to use three point integration formula, and the computation details are shown for one integration point; similar procedure can be repeated for the other two integration points. And by summing up contribution from all the three integration points approximate value of this matrix k x can be obtained.

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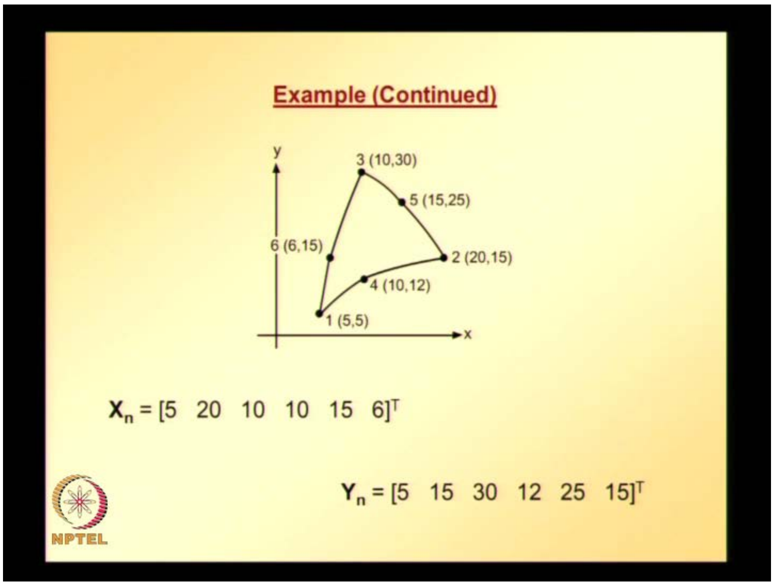

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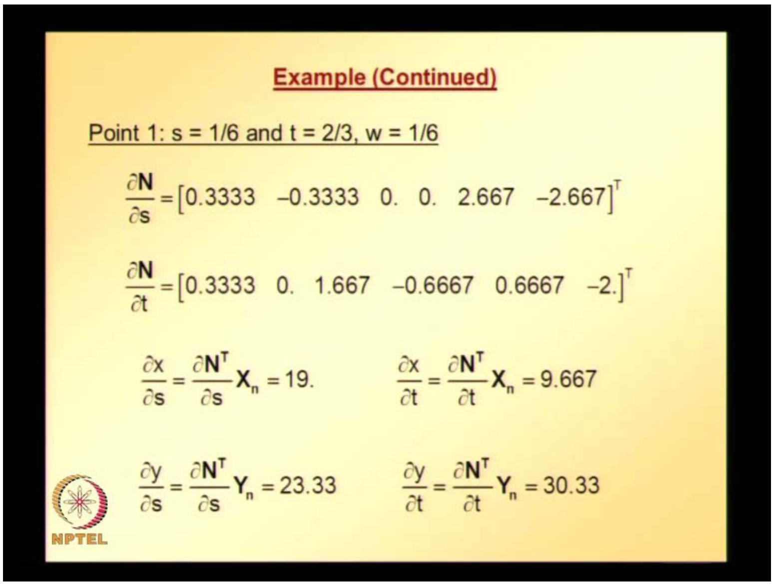

X coordinates, y coordinates and the workout the details - numerical details are shown here for one integration point; that is at s is equal to 1 over 6. And t is equal to 2 over 3 that is integration point that is selected. And weight corresponding to this integration point is 1 over 6. So, evaluate all the quantities at this point partial derivatives of shape function with respect s with respect to t, and then derivatives of x and y with respect s and t.

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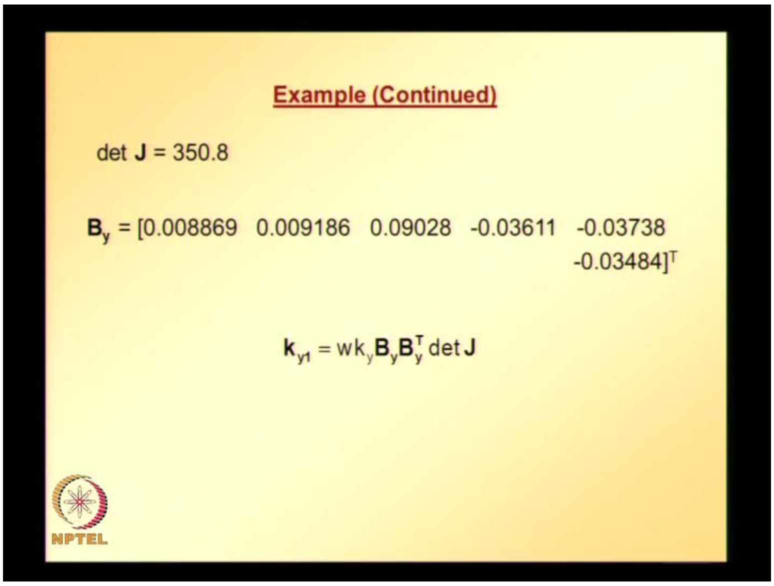

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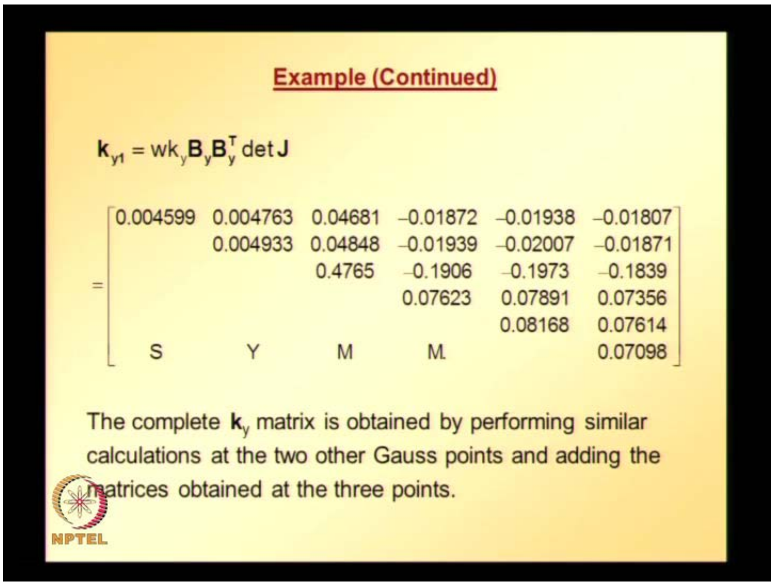

Then determinant of J, once we have determinant of J we can calculate what is b y. At this integration point what is the value of this matrix is given by this complete k y matrix is obtained by performing similar calculations at the other two gauss points adding matrices obtained at the three points. There is a small correction in the problem statement it is given as matrix k x, but it should be k y. And also coefficient is given as k x it should be k y so with that correction this is how we can proceed to evaluate these kind of integrals.

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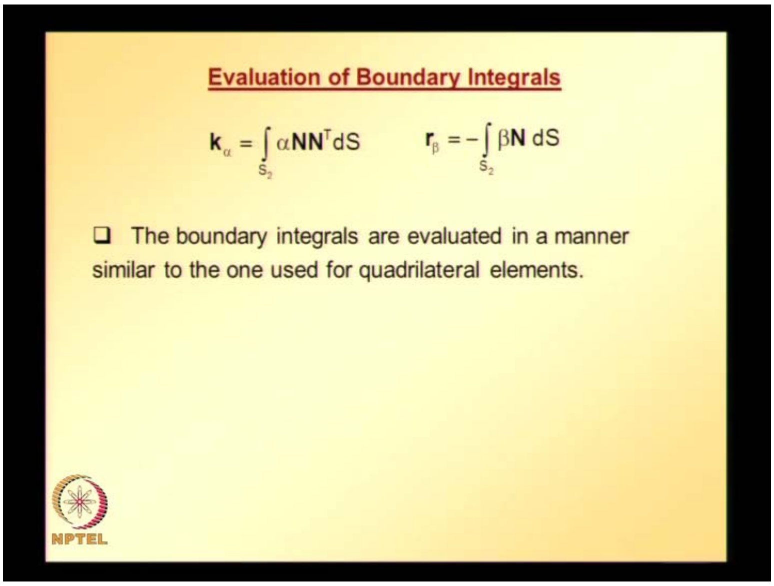

If you recall, these are the kind of boundary integrals that we need to evaluate along the element boundaries or element sides. Let us discuss about evaluating this kind of integrals over a triangular element edges. The boundary integrals are evaluated in a manner similar to the one use for quadrilateral elements

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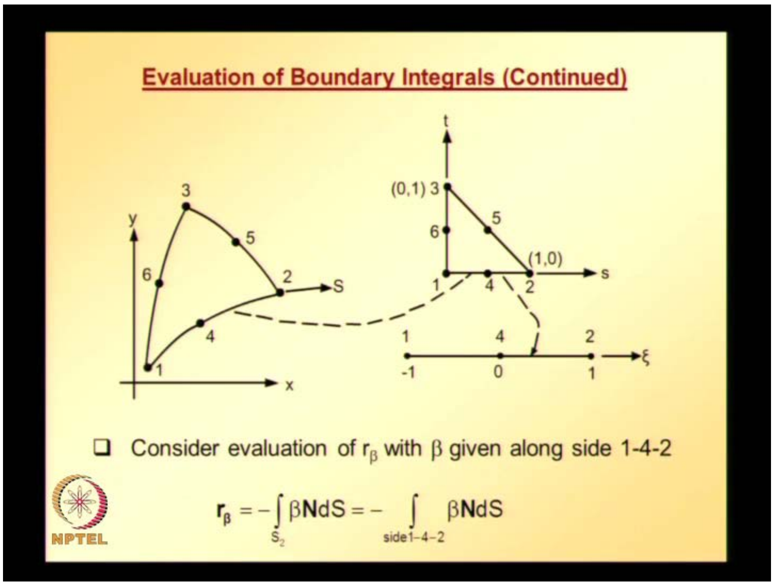

This is how we are going to map a six node parent element mapping is mapping to a six node sorry, six node physical element mapping to six node parent element is shown or is illustrated in the figure. So, edge or side 1, 2 of physical element is mapped on to side 1, 2 of parent element. And you can easily see that alongside 14 2 of parent element t is equal to 0 and s goes from 0 to 1 by substituting t is equal to 0 into the shape function vector. We can get the shape function vector along this side 14 2. And also, please note that where we want to use one dimensional Gaussian quadrature the limit should be from minus 1 to 1.We need to introduce another transformation or change of variable, and that will be illustrated as we proceed. Consider evaluation of this r beta alongside 14 2, r beta is given here we need to evaluate along this side 14 2. And that is the reason we are interested in finding the shape function vector alongside 14 2.

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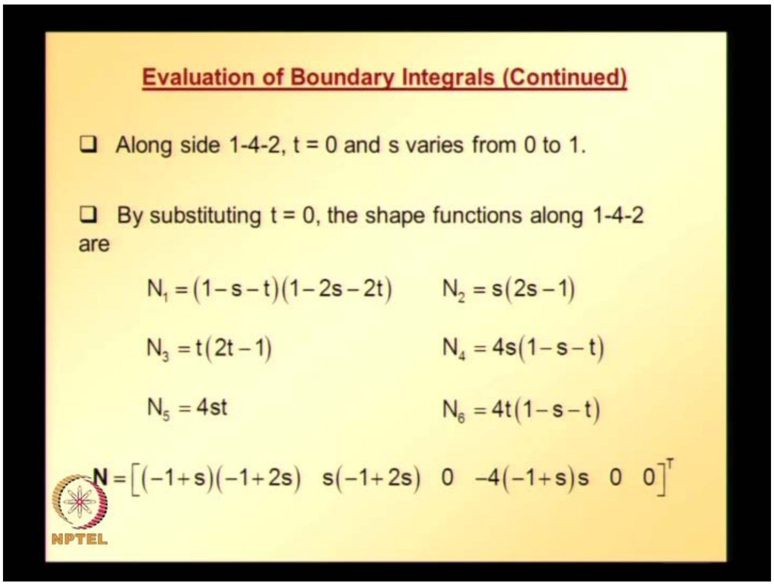

Alongside 14 2 t is equal to 0, s varies from 0 to 1 by substituting t is equal to 0. The shape functions alongside 14 2 or we need to substitute in these expressions of N1 to N; t is equal to 0 by doing that and putting in a vector form, we get this shape function vector. As you can see here, we need to here we can make another substitution or we can define another variable as a function of s and simplify this vector further.

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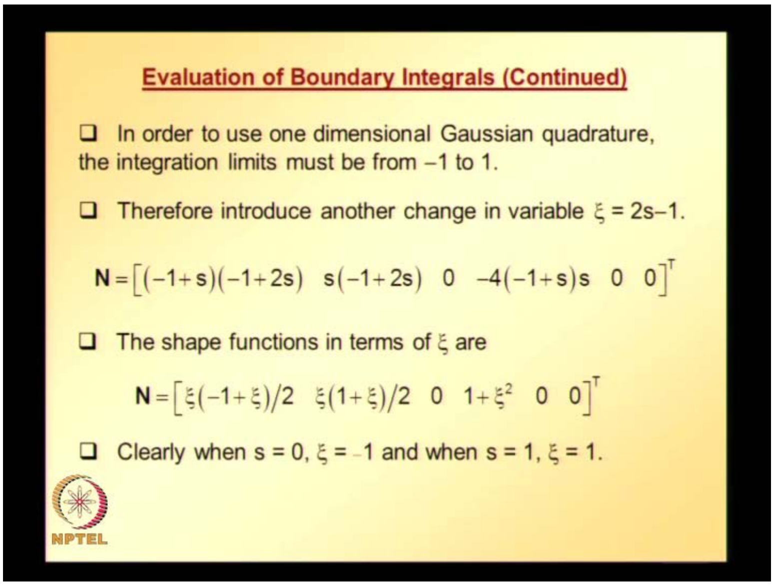

In order to use one dimensional gauss quadrature integration limits must be from minus 1 to 1 whereas s is going from 0 to 1. Instead of that if we define another variable, which goes from minus 1 to 1. We are introducing a change in variable psi, psi is defined as 2 s minus 1 with this change of variable. The previous shape function vector which is rewritten here when we make the substitution in terms of psi this becomes this one, where psi goes from minus 1 to 1 that can be easily verified clearly s is equal to 0 by substituting s is equal to 0 in the equation psi is equal to 2 s minus 1. We can easily check s is equal to 0 corresponds to psi is equal to minus 1, s is equal to 1 corresponds to psi is equal to (( )). The limits of integration which are the edges or the side 14 2 is going from 0 to 1. So, we change we used some kind change of variables and we get the limits from minus 1 to 1.

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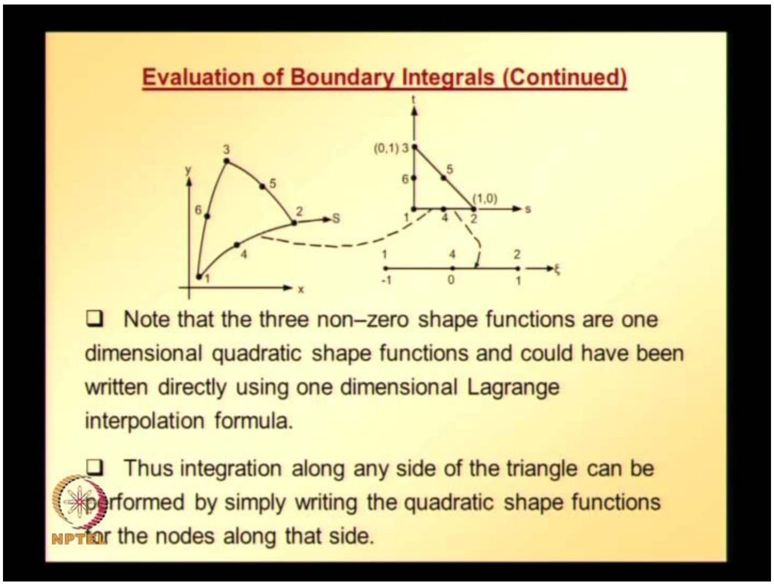

In terms of psi, we have the desired limits. And note that the three non-zero shape functions or one dimensional quadratic shape functions. And one could obtain one could have sorry and they could have been written directly using one-dimensional Lagrange interpolation formula, because if you see along each of the edges, the shape function of a node which is not part of edge that edge is going to be 0. The non-zero shape functions are going to be the shape functions which are lying on that particular side or edge. Those non-zero shape functions can easily be written using one-dimensional Lagrange interpolation formula. Thus integration along any side of triangle can be performed by simply writing quadratic shape functions for nodes along that side.

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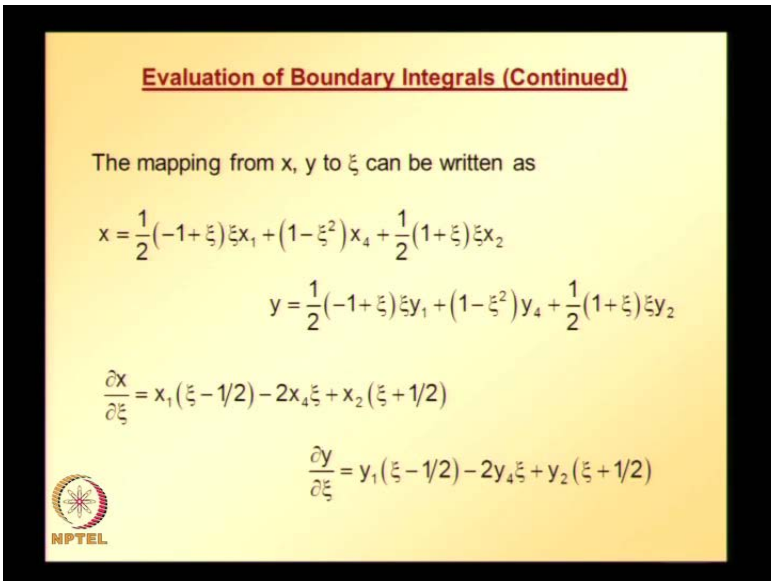

Now, this side 14 2 of physical element is mapped on to side 14 2 of the parent element. And this mapping in terms of psi can be written like this x is equal to N1 x1 plus N2 x2 plus N3 x 3 because rest of the shape functions are 0. Here it is a N1 x1 plus N2 x4 plus N3 x2 and N1, N2, N3 or N1, N4, N2 are substituted in terms of psi here. This gives us relation between x and psi. Similarly, we can write relationship between y and psi. And we can take partial derivatives of x with respect to psi and y with respect to psi. Also, we require the relationship between a differential element taken alongside 14 2 of the physical element .And how it is related to a differential element taken onside 14 2 of the parent element.

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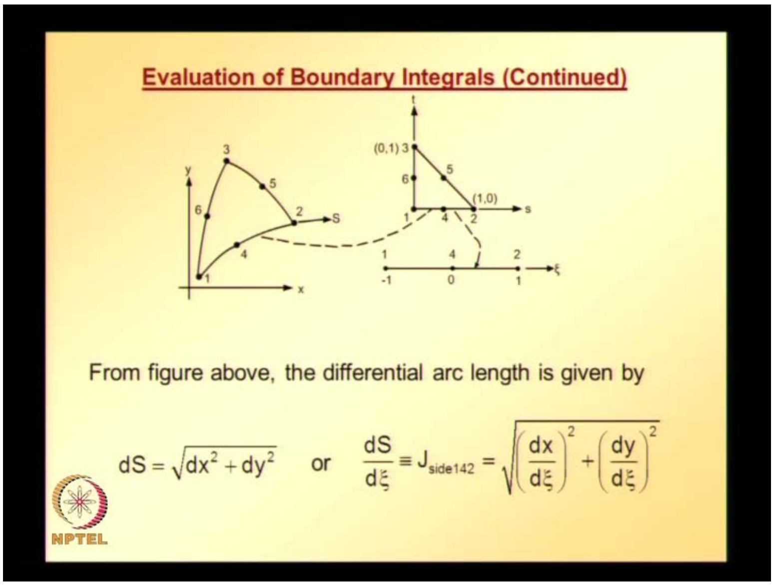

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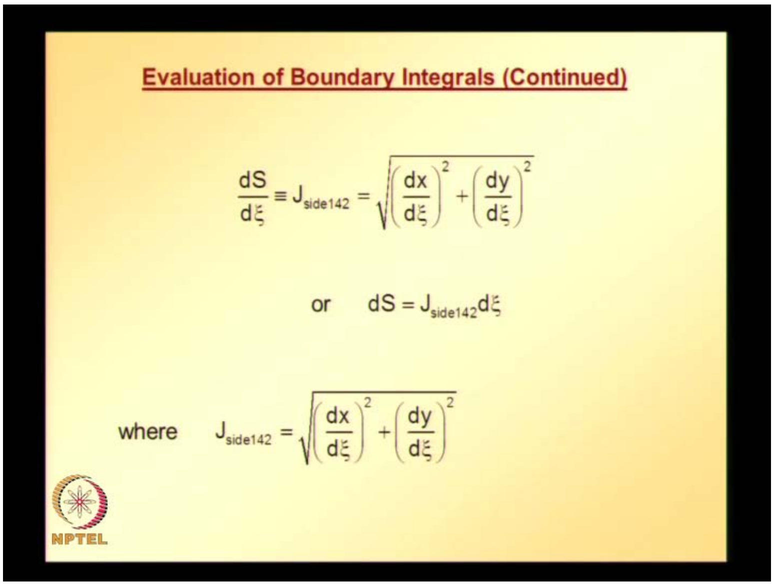

The differential arc length on the physical element is given by d S is equal to square root of d x square plus d y square and dividing on both sides with d psi we get the second equation. In which, we can substitute what is derivative of x with respect to psi and derivative of y with respect to psi. This gives us relationship between physical element and differential element in parent element, with this relation the boundary integral can be written in terms of psi, which goes from minus 1 to 1 before we use one dimensional Gaussian quadrature.

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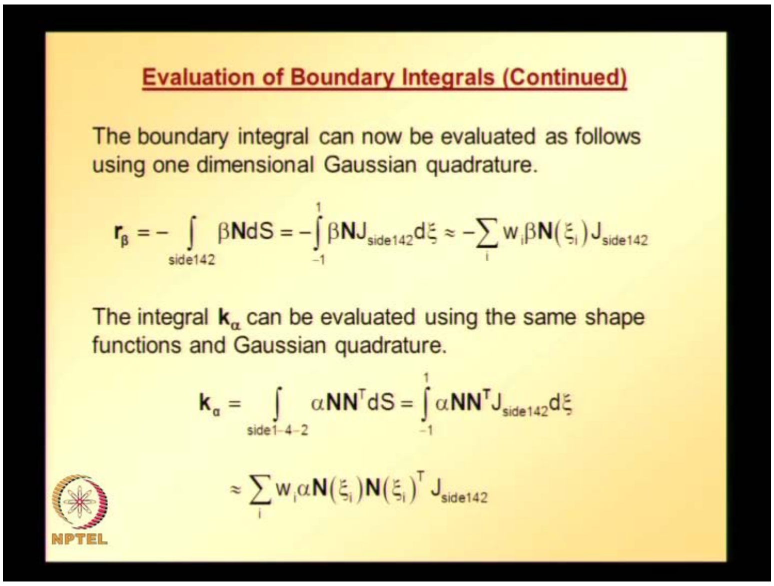

The boundary integral can now be evaluated as follows using one dimensional Gaussian quadrature. R beta needs to be evaluated alongside 1, 2 which is defined as beta times N d S. And using change of variables limits of integration becomes minus 1 to 1. And differential element of the physical element d S is replaced with J times d psi. And this integral can be evaluated or approximated using one-dimensional Gaussian quadrature by evaluating integrand that each of the integration points multiply with weight, and summing up contribution from all the integration points. This kind of procedure can even be repeated, if it is other integral. So, integral k alpha can be evaluated using same shape functions. And Gaussian quadrature only thing is it looks like this. And we discussed only alongside 1 4 2, and if it is some other side, everything is same except that different shape functions are going to be zero along different sides.

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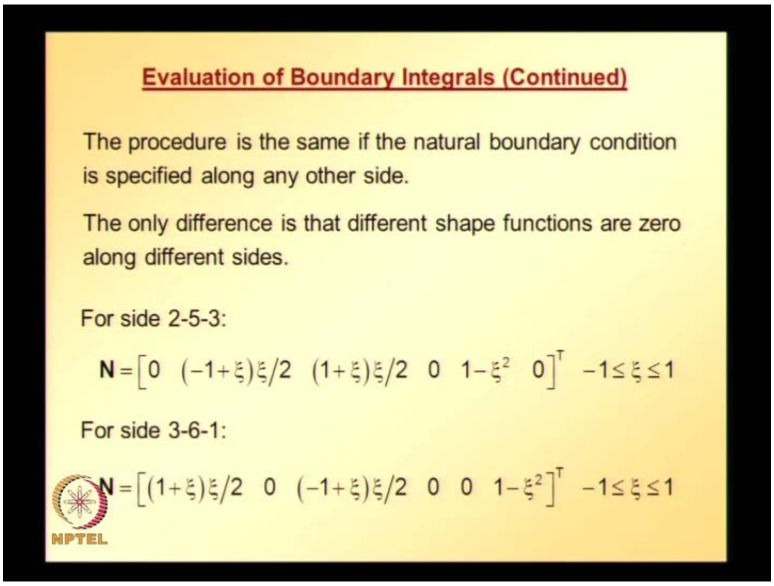

The procedure is same if natural boundary condition is specified along any other side, the only difference is that different shape functions are zero along different sides. For example, alongside 2 5 3 in non-zero shape functions in terms of psi or these, which can also be as I mentioned which can these shape functions can also be obtained using Lagrange interpolation formula. And the alongside 3 6 1 these are the non zero shape functions in terms of psi.

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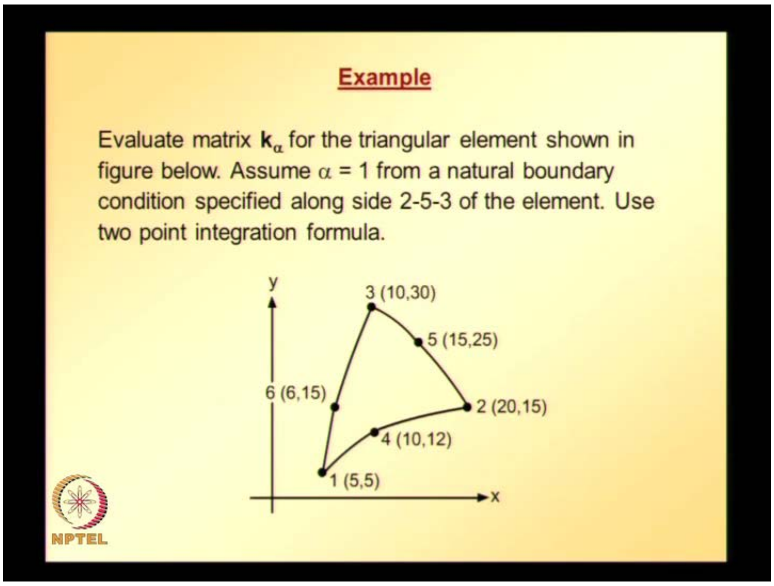

To illustrate to illustrate some of these details, let us take an example. Evaluate matrix k alpha for the triangle element shown in figure above. Assume alpha is equal to minus 1 for a natural boundary condition specified alongside 2 53 of element, use two point integration formula. This is the physical element that is given we need to evaluate this matrix k alpha alongside 2 5 3. And from the information that is given we can easily write what are the nodal coordinate vectors before we proceed.

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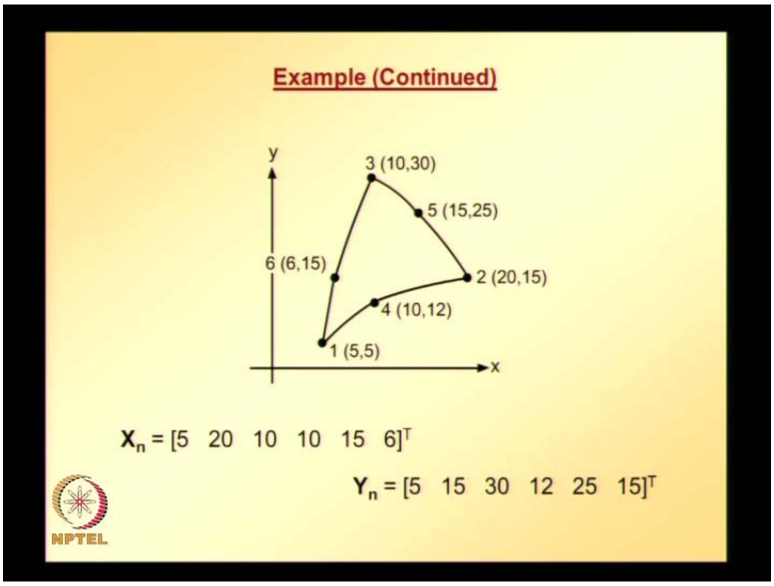

Since we need to evaluate this matrix k alpha alongside 2 5 3, we need to know the shape function vector alongside 2 5 3 of the parent element, which can also be or which can be obtained using Lagrange interpolation formula or which can be obtained using the shape function expressions that we have by substituting s is equal to 10, t is equal to 10 at nodes 2 and 3. And by substituting s is equal to half, and t is equal to half at node 5 or they can be easily obtained using Lagrange interpolation formula.

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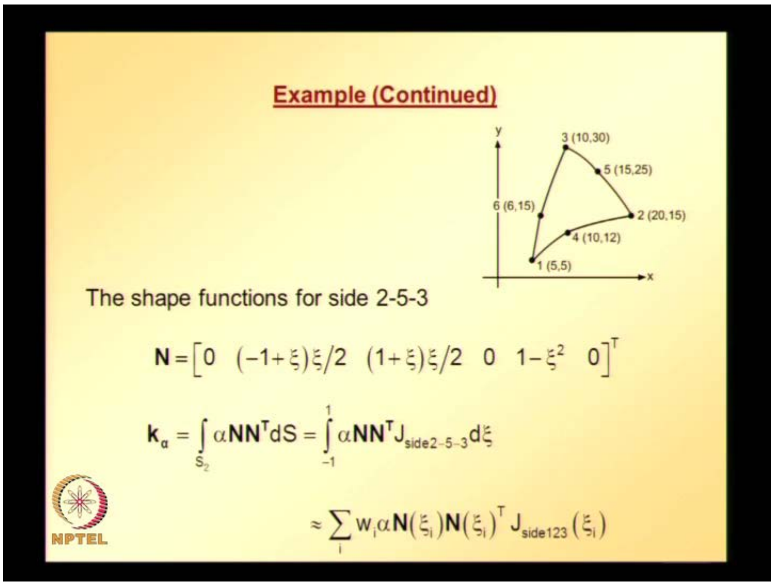

The non-zero shape functions alongside 2 5 3 in terms of psi are have given in this vector, so k alpha is defined like this alpha times N, N transpose needs to be evaluated alongside 2 5 3 .And this can be approximated with change of variables as minus 1 to 1 integral, minus 1 to 1, alpha N, N transpose jacobian times d psi, which can be approximated using one-dimensional Gaussian integration, as it is shown.

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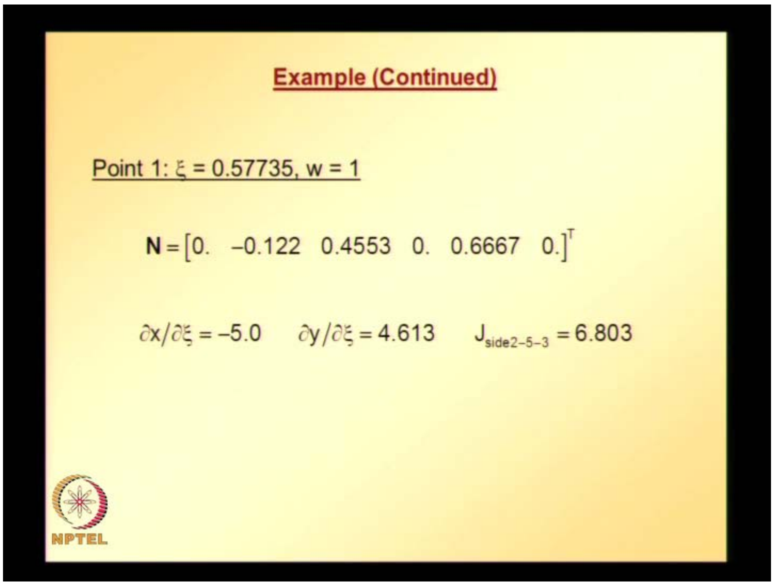

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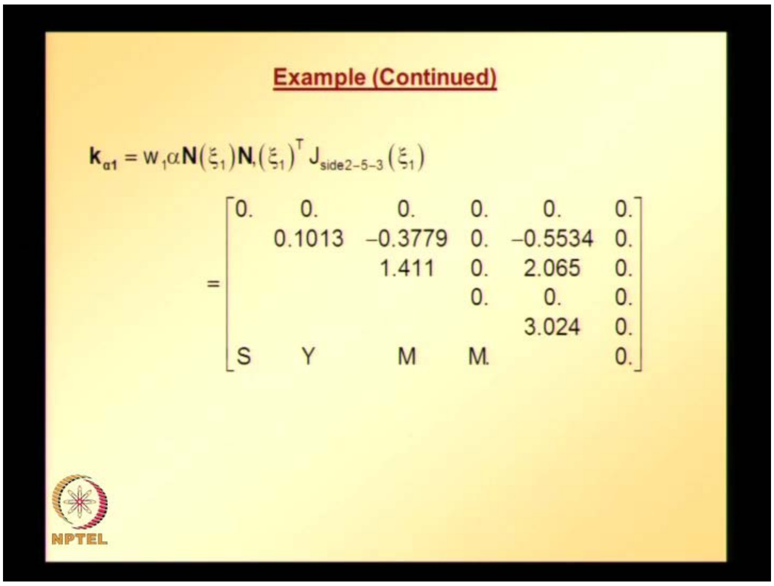

Now, this suggested to use two point integration and this is just one-dimensional integration. So, you can get the integration points and weights from the table; that is given earlier for one-dimensional integration. First integration point and weight are given. Evaluate the shape function vector at this integration point. And also rest of the quantities to calculate Jacobian. And then evaluate the contribution of this integration point to k alpha matrix by evaluating all the quantities in the in the integrand at this integration point multiply with weight. Here alpha is constant equal to 1. And finally, this is the contribution to k alpha from first integration point.

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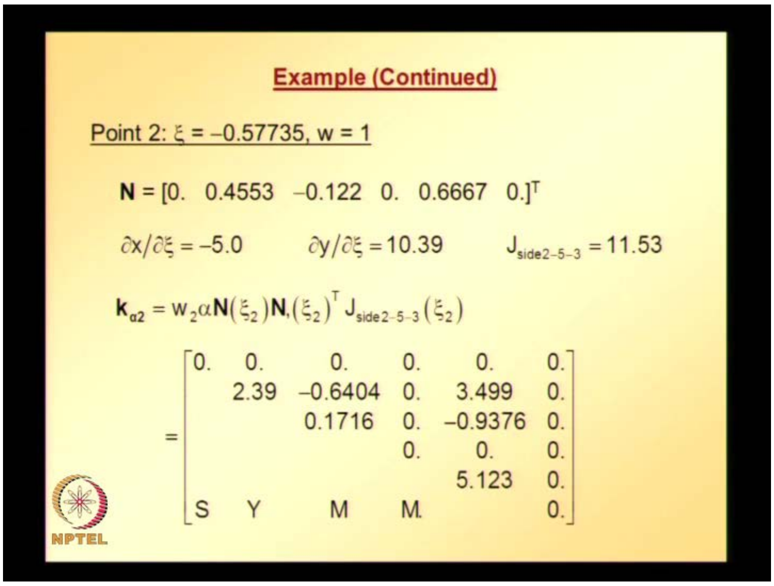

Second integration point coordinate and weight shape function vector. And once we get the all other information, we can easily evaluate contribution of second integration point two k alpha matrix. We got a contribution from the first integration point and we got contribution from the second integration point. We need to add these two contributions to get final value of k alpha. So, k alpha matrix is given by k alpha 1 plus k alpha 2. Finally, this is the solution for this problem.

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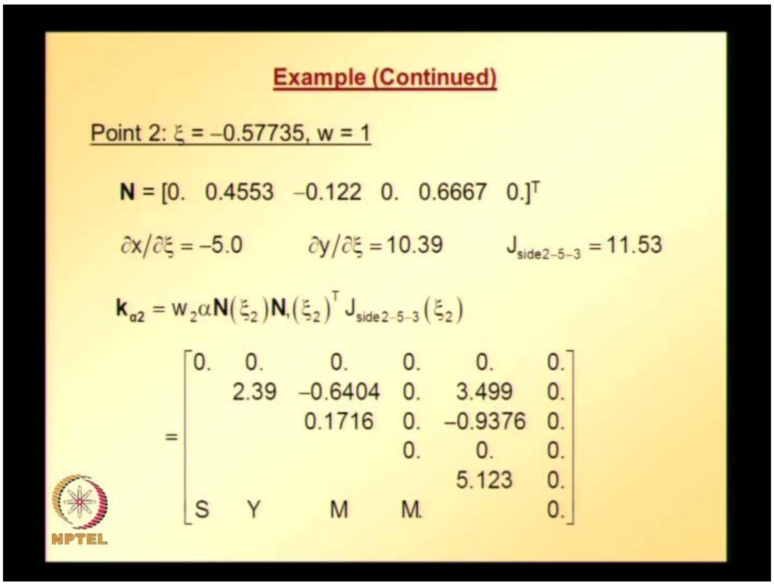

This is where we can evaluate these integrals using one-dimensional integration along the boundaries of a triangular element all the line integrals or the boundary integrals, and given any problem once we have the area integrals, and boundary integrals we can write complete element equations. And assemble the global equation system apply the essential boundary conditions, and solve for the nodal unknowns and then once we have the unknown sort we can come back to each of the elements. And do post processing to find trial solution at any point inside the element, and derivative of trial solution at any point inside the element. So, this completes higher order triangular elements.