Finite Element Analysis

Prof. Dr. B. N. Rao
Department of Civil Engineering
Indian Institute of Technology, Madras

Lecture No. # 28

In the earlier lectures, we have seen formulation for 3 node linear triangular element. In the next two lectures, we will be looking at formulation for higher order triangular elements.

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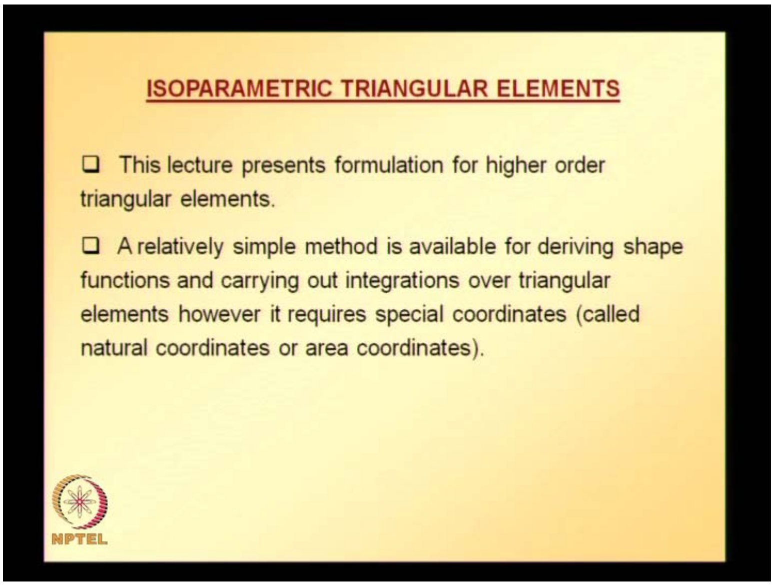

A relatively simple method is available for deriving shape functions for higher order triangular elements. And that method requires what is called special coordinate system, natural coordinate or area coordinate system. So we need to learn that before we proceed with formulation for higher order triangular elements for deriving shape functions or subsequently for deriving the element equations.

In the next two lectures, we will be seeing, what are this natural coordinates or area coordinates, and formula for deriving shape functions for triangular elements. This formula you can even use for 3 node linear triangular element. And also it can be used for deriving shape functions for higher order triangular elements. And then as usual similar to what we did for quadrilateral element, we look at isoparametric mapping concept for triangular elements, and then numerical integration to derive area integrals or to obtain area integrals, and boundary integrals, when we adopt triangular elements for discretization. And also at the end, we look at quadratic triangular element with curved boundaries.

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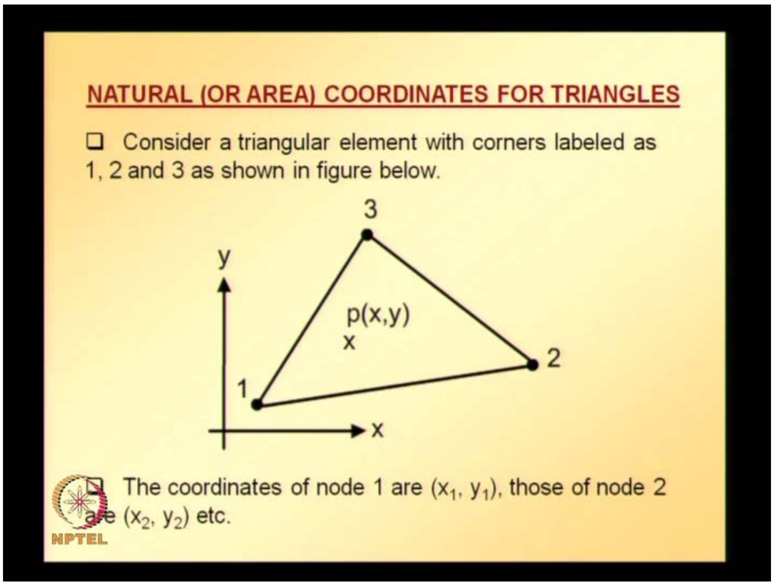

Let us start with understanding this natural or area coordinates for triangles. This is what I mentioned. Consider a triangular element with corners labeled as 1, 2 and 3 as shown in figure; x, y coordinate system is shown; and the three nodes are numbered as 1, 2 and 3. And the coordinates of node 1 in x, y coordinate system are x 1, y 1; similarly coordinates of node 2 are x 2, y 2. And for node 3, they are x 3 and y 3.

Instead of expressing coordinates or positions of these nodes 1, 2 and 3 in terms of x, y coordinates. We can also express in terms of what are called natural or area coordinates. To understand that let us consider a point P inside the triangular element like the way is shown in the figure. If the point P is connected with each of these vertices 1, 2 and 3, and based on area opposite to node 1, area opposite to node 2 and area opposite to node 3, we defined what are called natural or area coordinates. If you take any point, this point need not be inside the triangle, it can even be on the boundaries of the triangle. You can easily verify that this is area opposite to each of these nodes is going to be unique for any point inside or on the boundary of the triangle.

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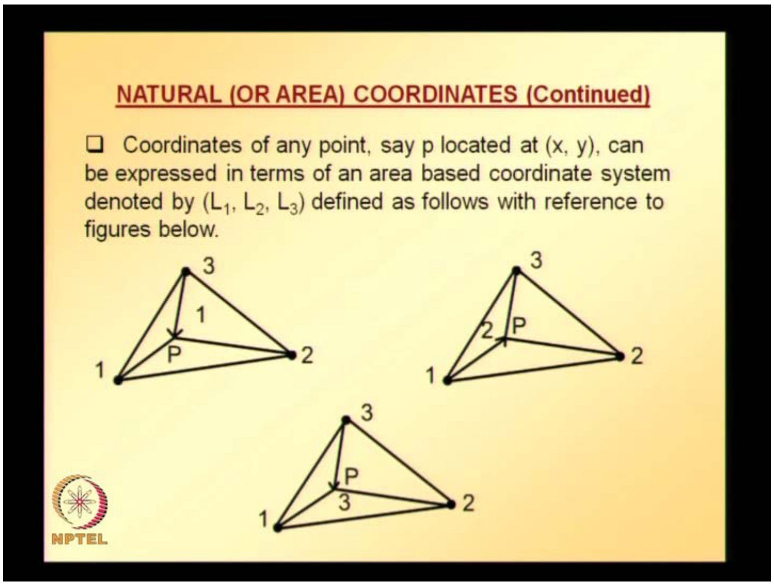

Based on that, we define what are called area coordinates. Coordinates of any point P located at x, y can be expressed in terms of area. Based on coordinate system denoted by L1, L2, L3 defined as follows with reference to the figures below. The triangle which is shown earlier is reproduced here, except that point P is connected with each of the vertices by a line. And area opposite to node 1 is denoted with 1. Now let see, how L1 is defined. I hope you understood, what is area opposite to node 1, which is clearly denoted with 1 in the figure? And similarly, area opposite to node 2 is denoted with 2. And area opposite to node 3 is denoted with 3. Let us see how L1, L2, L3 are defined. L1 is nothing but ratio of area opposite to node 1 divided by or ratio of area opposite to node 1 to total area or area opposite to node 1 divided by total area.

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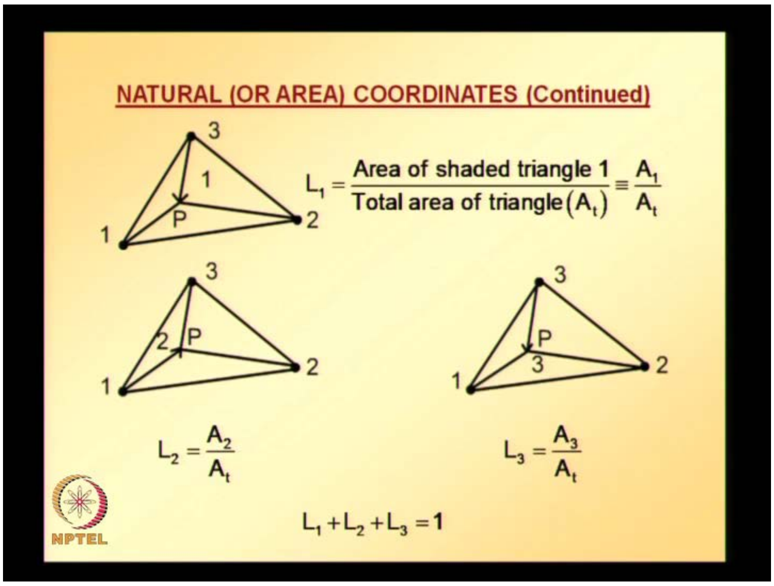

L1 is area of shaded triangle. Here it is not shaded, but it is denoted with 1. That is area opposite to node 1 divided by total area denoted with A t. So, area opposite to node 1 is denoted with A 1. And total area is denoted with A t. L1 is defined as A 1 divided by A t. If you consider any point this L1 is going to be unique, because once the point location changes point P location changes, area opposite node 1 automatically changes. And it is not going to be unique sorry it is going to be unique for every point. And it is not going to be same for any two points. Similarly, L2 is defined as area opposite to node 2 divided by total area. And L3 is defined as area opposite to node 3 divided by total area .This is what a coordinate system based on area. And instead of describing location of a point with respect x, y coordinate system; we can as well express the location of point in terms of L1, L2, L3 because L1, L2, L3 values are going to be unique for any point. And it can be easily verified if you sum up L1, L2, L3. It is going to be A1 plus A2 plus A3 divided by total area whereas A1 plus A2 plus A3 is equal to total areas. So, L1 plus L2 plus L 3 is going to be 1.

Also, it can be easily verified that if point P coincides with node 1, that means we are trying to write L1, L2, L3 for node 1. In that case area opposite to node 1 is going to be total area. You can easily see that by imagining point P coincide with node 1. In that case area opposite to node 1 is going to be total area. And in that case L1 is going to be 1. And L2, L3 are going to be 0. Similarly, if point P coincides with node 2, L1 is going to be 0, L2 is going to be 1, L3 is going to be 0. Also, when point p coincides with node 3, L1 is going to be 0, L2 is going to be 0 and L3 is going to be 1.

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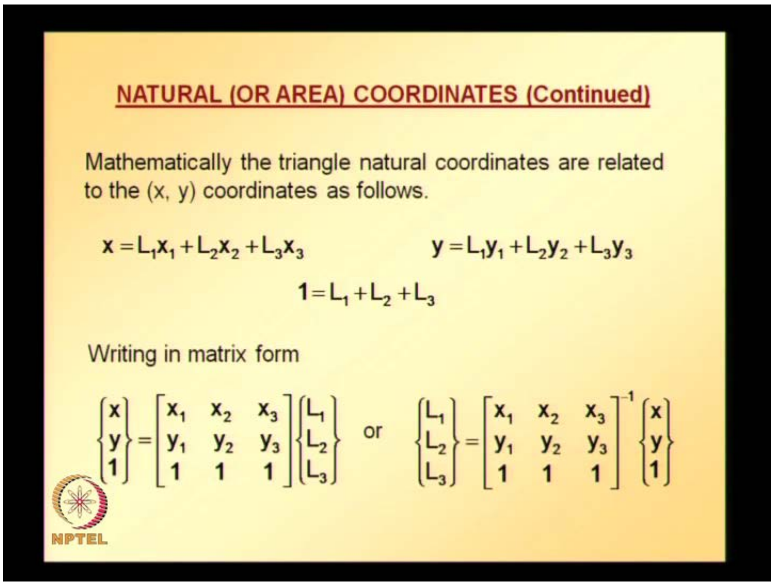

With this, understanding mathematically the triangular or triangle natural coordinate systems are related to x, y coordinate system as follows; x is equal to L1, x1 plus L2, x2 plus L3, x3. Whether this relation is correct or not, it can be easily verified by making x coincide with any of the nodes. Suppose, this point coincides with x, y coordinates coincides with coordinates corresponding to node 1.In that case L1 is going to be 1, L2 is 0, L3 is 0; and this equation perfectly correct. Similarly, y is equal to L1 y1, L2 y2, L3 and y3 this can also be verified. We have three relations. One is L1 plus L2 plus L3 is equal to 1. The second relation is x is equal to L1, x1 plus L2, x2 plus L3, x 3. The third relation is L1, t1 plus L2, y2 plus L3, y3 sorry y is equal to L1, y1 plus L2, y2 plus L3. And we can write all these three equations in a matrix form. We get x, y 1 putting in a vector is equal to matrix consisting of x1 y1 1, x2 y2 1, x3 y3 1, and L 1, L2, L3. Using this relation, we can back calculate what are these L1, L2, L3 for a particular point x, y having coordinates x, y by taking inverse relation.

Since, we know the coordinates of all three nodes, node 1, 2, 3 are the vertices of the triangle. We can easily figure out what are this x1 y1 x2 y2 x3 y3. We can easily calculate or we can easily find what is this L1, L2, L3 in terms of x, y for any point having coordinates x, y.

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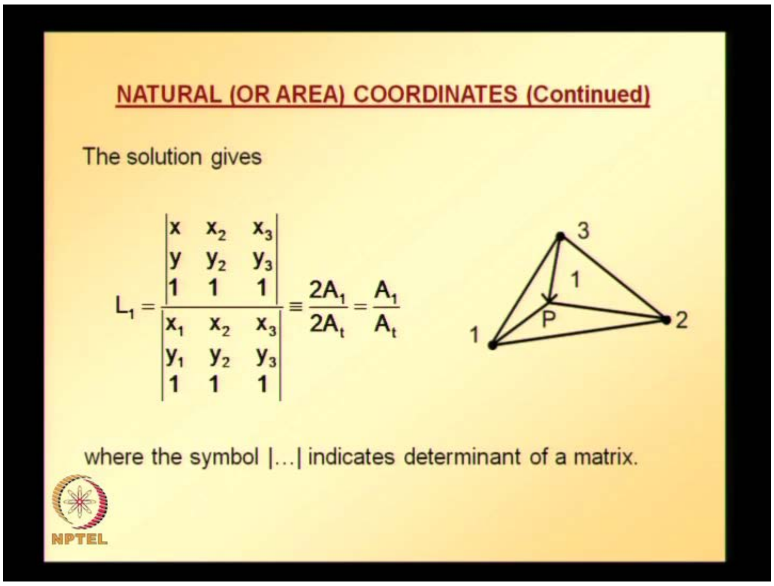

Solving previous equation for L1, L2, L3; L1 is given by this, where the symbol indicates determinant of matrix. L1 is determinant of matrix consisting of component x y 1, x2 y2 1, x3 y3 1 divided by determinant of matrix consisting of x1 y1 1, x2 y2 1 and x3 y3 1. If you verify that these determinants are nothing but twice the area opposite to node 1. One of the determinants is twice area opposite to node 1. And the other determinant is twice total area of triangle. L1 is ratio of area opposite to node 1 divided by or area opposite to node 1 to total area ratio of area opposite to node 1 to total area. And that can be calculated using the determinant of these matrices which we can obtain from the information about the nodal coordinates of the particular triangular element, so this is how L1 can be calculated. Please note that this is obtained from the previous equation by inverting the equation or the solution of the previous equation based on that we got this L1.

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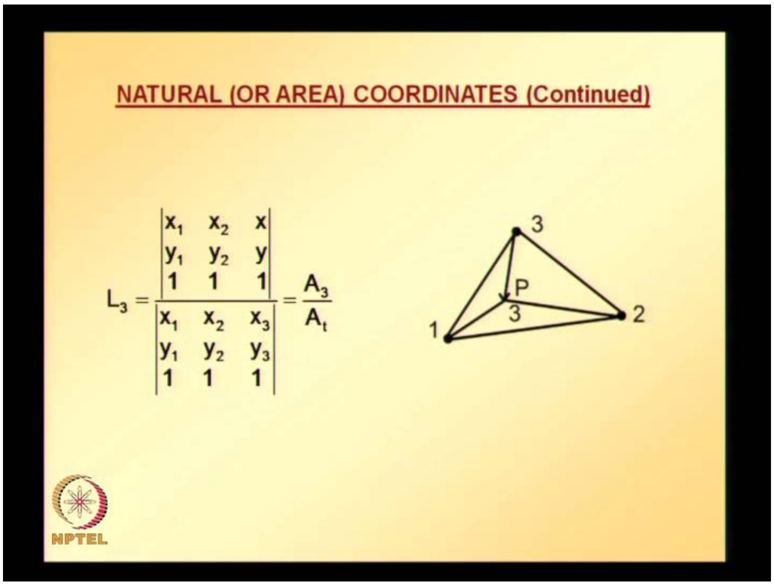

Similarly, we can find L2 and L3. It is easy to remember these formulas for L1, L2, L3. If you see the denominator, you have determinant of matrix having components x1 y1 1, x2 y2 1 and x3 y3 1. If it is you are trying to calculate L1, you copy whatever is there in the denominator to the numerator, except replace the first column of the matrix with x, y, 1. Similarly, when you are calculating L2 replace second column of the matrix in the numerator with x, y, 1. Similarly, when you are calculating L3 after copying whatever is there in denominator to the numerator replace whatever is there in third column of the numerator replace that with x y 1 that is how you can easily remember these formulas for L1, L2, L3.

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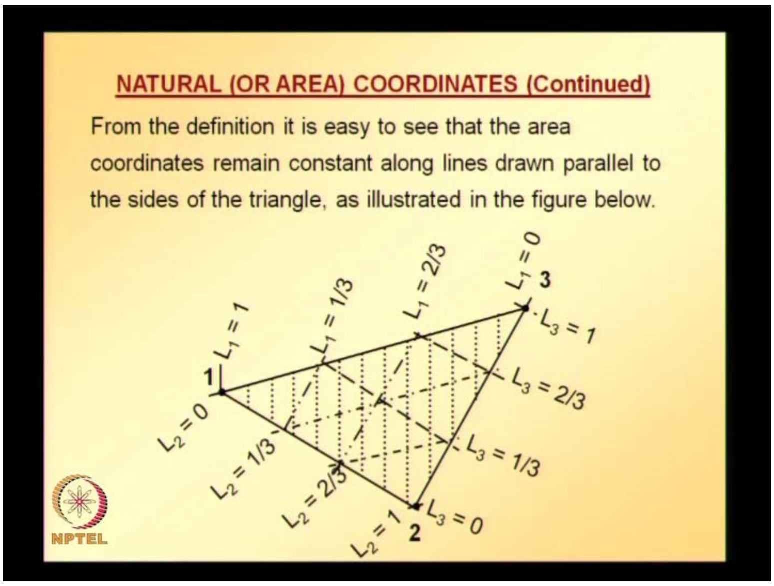

Also, from the definition it is easy to see that area coordinates remain constant along lines drawn parallel to sides of triangle, it is illustrated if you have a triangle. And if you try to draw lines parallel to each of the sides along those lines, the area coordinates remain constant as indicated in this figure. This can be easily verified using the formulas for L1, L2, L3. This is how one can find area coordinates of any point. The procedure for deriving shape functions for triangular elements is based on this concept of area coordinates or natural coordinates. Shape function formula that is that we are going to look in a while requires each node be represented by three subscripts, three subscripts related to L1, L2, L3 values at each. At each we need to find L1, L2, L3; and we need to use those values of L1, L2, L3 for deriving shape functions using the formula that we are going to look now.

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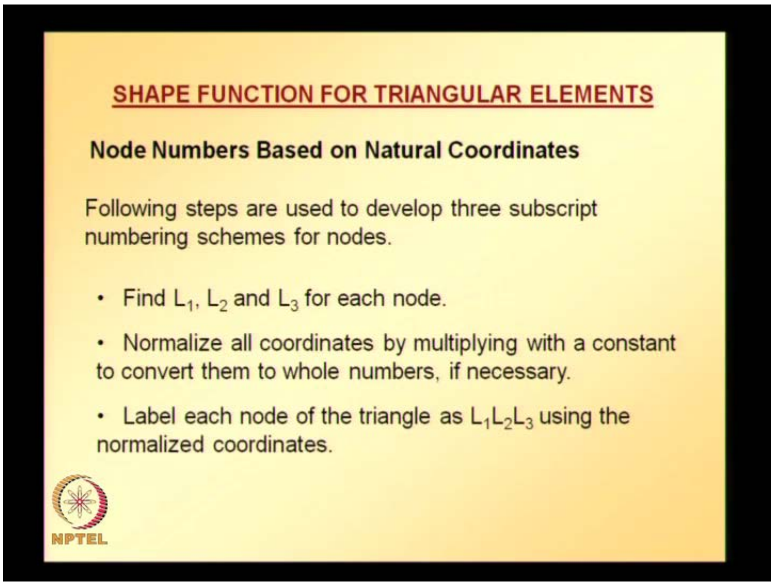

Let us look at the procedure, shape functions for triangular element as I mentioned; this procedure is applicable for both higher order triangular elements and the linear triangular element that we already looked at. So, this is how procedure goes node numbers based on natural coordinates following steps are used to develop three subscript numbering schemes for nodes. We look at some steps, and then we make them clear by looking at some example, and then come back to the formula that is how we will proceed.

First step is, find L1, L2 and L3 for each node given a triangular element; normalize all coordinates by multiplying with a constant to convert them to whole numbers, if necessary. This step of normalization is only required, if necessary. Normalize all coordinates by multiplying with a constant to convert them into whole numbers, if necessary. And this point will become more clearer; when we look at an example. Label each node of triangle as L1, L2, L3 using the normalize coordinates. To understand these first three steps, let us take two examples. And try to understand these three steps before we proceed further with this procedure of deriving shape functions for triangular elements. Let us try to understand, how to find L1, L2, L3 for each node, and also or if necessary how to normalize this L1, L2, L3; and also how to label each of the node as L1, L2, L3.

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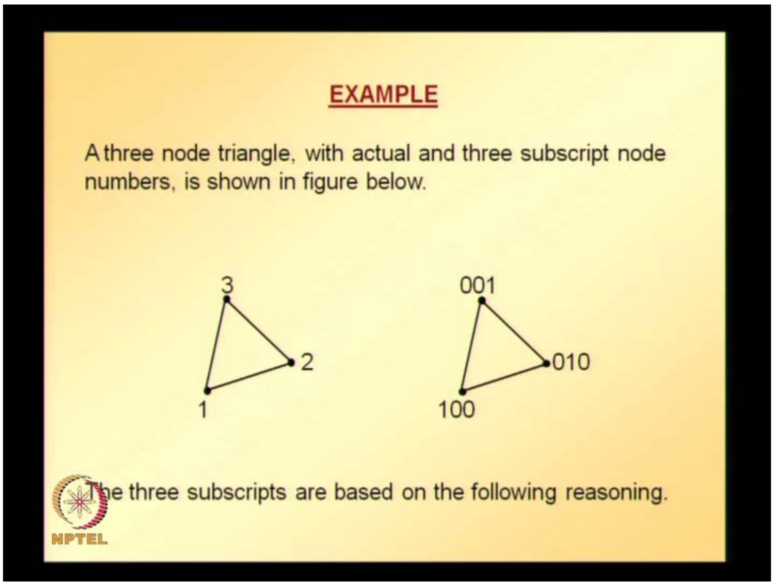

A three node triangle with actual and three subscript node number is shown in figure below. The figure on the left hand side is numbered as 1, 2, 3; imagine that to be in x, y coordinate system, and the figure on the right hand side the nodes are numbered using 3 subscript numbering scheme. So, how this three subscript numbering scheme is obtained? Let us try to understand.

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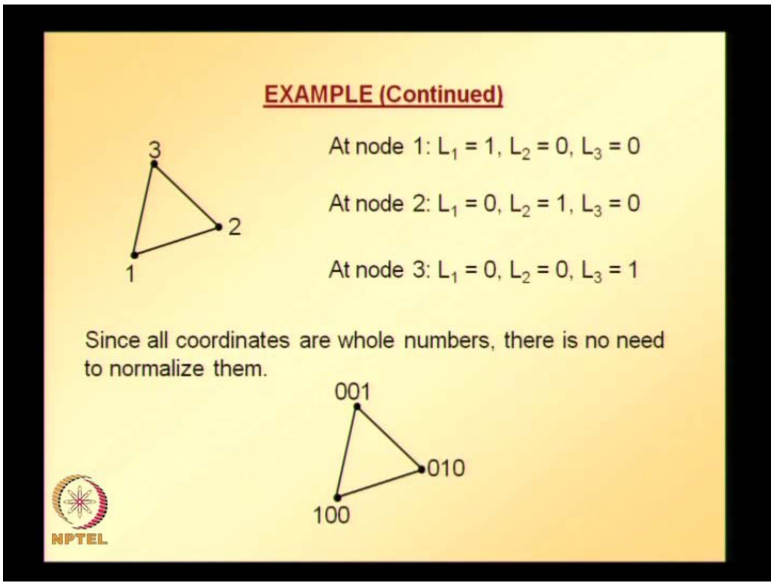

The three subscripts are based on the following reasoning. This triangle that is given having nodes 1, 2, 3. And I already explained at node 1 L1 is equal to 1, L2 is equal to 0, L3 is equal to 0; that is how node 1 is numbered as 100 in 3 subscripts numbering scheme. Similarly, at node 2, L1 is 0, L2 is 1, L3 is 0. So, node number under three subscript numbering schemes for node 2 is 010. And similarly, at node 3 L1, L2, L3 values are these, so node number for node 3 under three subscript numbering schemes is going to be 001. And here, we can see for this particular three node element all the area coordinates that is L1, L2, L3 are whole numbers, there is no need to normalize, we can directly write L1, L2, L3 side by side to get this three node or three subscript numbering for each of the nodes of the given triangle. So, this is how we get these three subscript numbering schemes for this three node triangular element.

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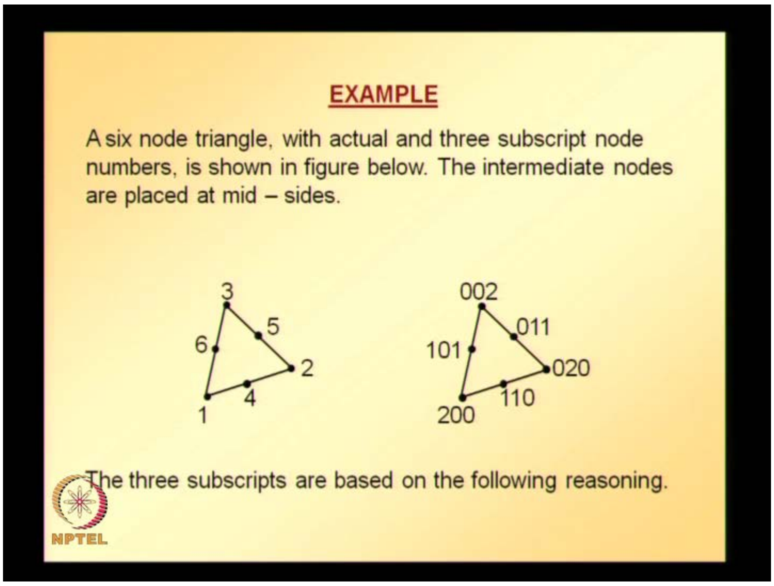

Now let us see higher order element. A six node triangle with actual and three subscript numbers and the element is shown here. The intermediate nodes are placed at mid sides that information is required, because unless we know where these intermediate nodes are placed. It is difficult for us to find L1, L2, L3 for that particular node that information is given. Intermediate nodes are placed at mid sides. The actual six node element triangle element is shown. And also for that six node triangle, three subscript numbering schemes is also shown on the right hand side. And this three subscripts are based on the following reasoning.

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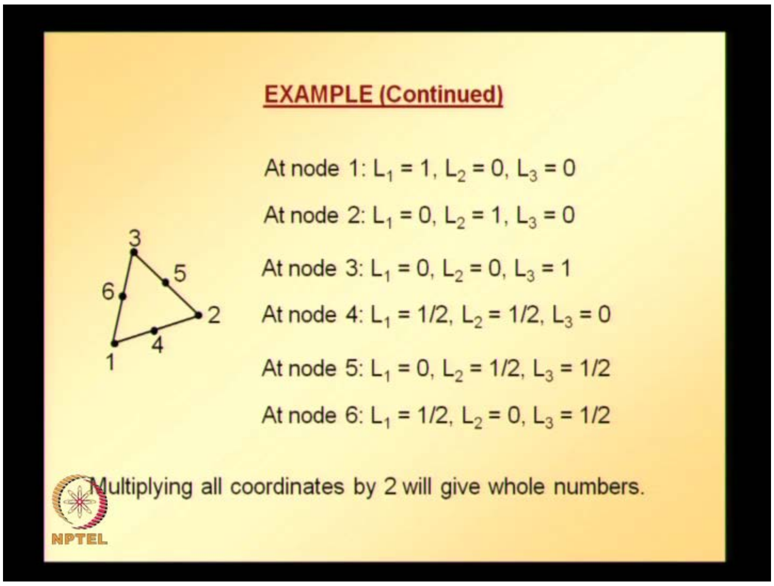

At node 1 is similar to three node triangle L1 is going to be 1, L2 is 0, L3 is 0. At node 2 same thing as for three node triangle. And same again as for three node triangle. Now comes the new thing that is at node 4, imagine node 4 joining or joined with each of the vertices already node 4 is alongside 1, 2 so it is already connected with 1 and 2. And now connect by an imaginary line node 4 to node 3. And it can be easily checked that when you do that kind of joining of node 3 to node 4 that imaginary line is going to bisect this triangle into two equal parts. And in that case, area opposite to node 1 is going to be half the area. And area opposite to node 2 is going to be half the area of total area of the triangle. So, L1 is going to be half, L2 is going to be half. And L3 area opposite to node 3 is going to be 0. So, L3 is going to be zero. That is how L1 is going to be half, L2 is going to be half and L3 is going to be 0 for node 4.

Similarly at node 5, node 5 is alongside 2, 3. So, imagine a line connecting nodes 1 and 5 that line is going to bisect again triangle into two equal parts. And area opposite to node 1 is going to be 0. Area opposite to node 2 is going to be half the area. And area opposite to node 3 is going to be half the area. In that case L1 is going to be 0, L2 is going to be half and L3 is going to be half. Similarly for node six, I hope this is clear. Now you can easily verify that L1, L2, L3 for all these written for all these nodes they are not whole numbers, because we have half for some of the quantities. So, what we need to do is, if require, we need to normalize that is second step. Now it is required, so multiplying all coordinates by 2 gives us whole numbers multiply for node one you take multiply L1 with 2, L2 with 2 and L3 with 2. We get L1 is equal to 2, L2 is equal to 0 and L3 is equal to 0. Similarly, if we take node 4, L1 is half multiplying that with 2. We get L1 is equal to 0, L2 is equal to 1, and L3 is equal to 0.

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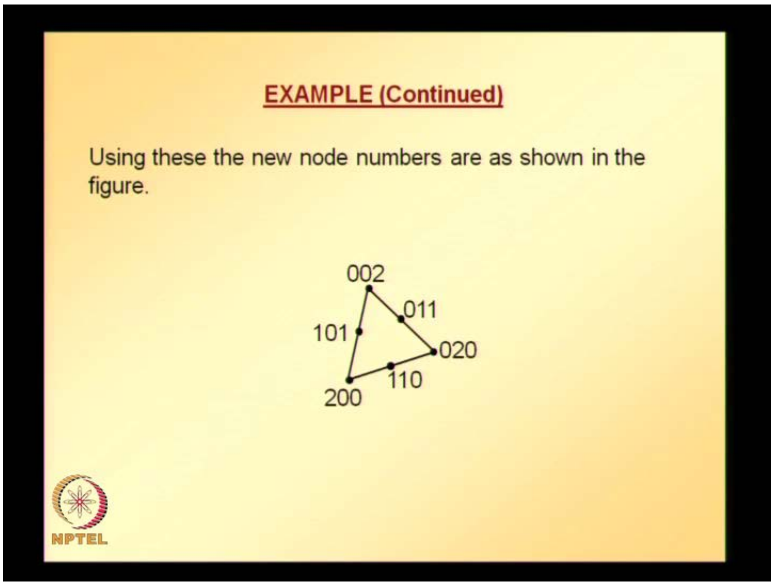

Using these new node numbers, we can write three subscript numbering schemes. L1, L2, L3 are normalized with 2 to convert into whole number. And they are written side by side for each of the nodes L1, L2, L3 that is how we obtain this three subscripts numbering scheme for six node triangle this is how you can obtain this three subscript numbering scheme for any triangle any nodded triangle.

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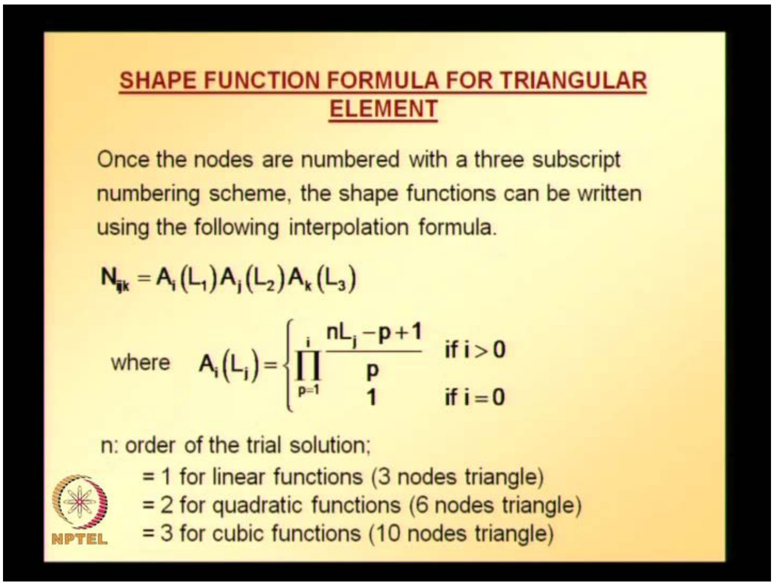

Next step, once the nodes are numbered with three subscript numbering scheme, shape functions can be written using the following interpolation formula. Now each of the nodes 1, 2 and 3 the numbering is replaced with three subscript numbering scheme. Each node is referred to with respect to three subscript numbering scheme corresponding shape functions can be obtained using this formula N i j k is equal to A i L1, A j L2, and A k L3. A i, L j is defined like this P goes from 1 to i; i depends on the value of subscript for A i and N is the order of trial solution. It depends on what nodded element you are looking for. If it is three node triangular elements, it is linear. It involves linear functions. N is equal to 1. If it is six node triangular elements, it involves quadratic functions, so N is equal to 2. And if it is ten node triangular elements, it consists of cubic function. So, N is equal to 3.

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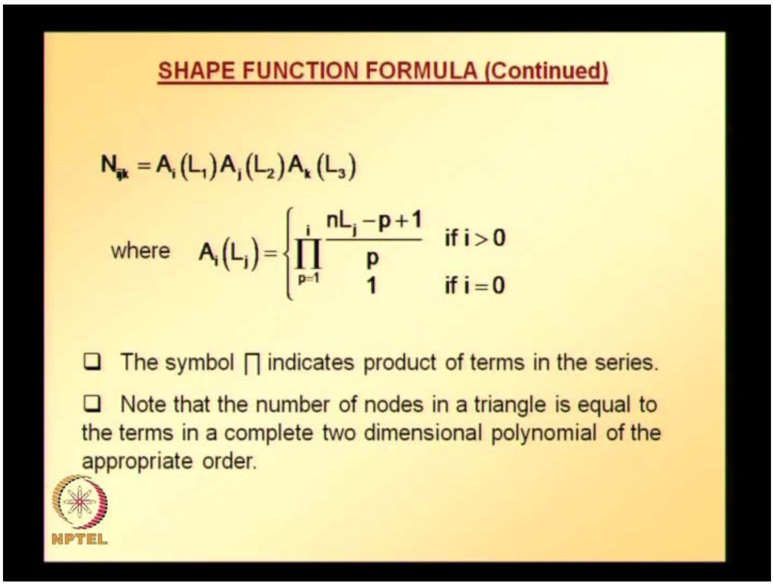

Here in this formula, we have the symbol pi, which indicates product of terms in series. So, first term multiplied by second term multiplied by third term like that. And note the number of nodes in a triangle is equal to the terms in a complete two dimensional polynomial of appropriate order. Suppose one is looking for triangular element of certain order that is linear order in the sense, we are referring to whether it is linear element or quadratic element or cubic element. If one is looking at a triangle element of certain order, then then one has to include all the terms in a complete two-dimensional polynomial of that particular order.

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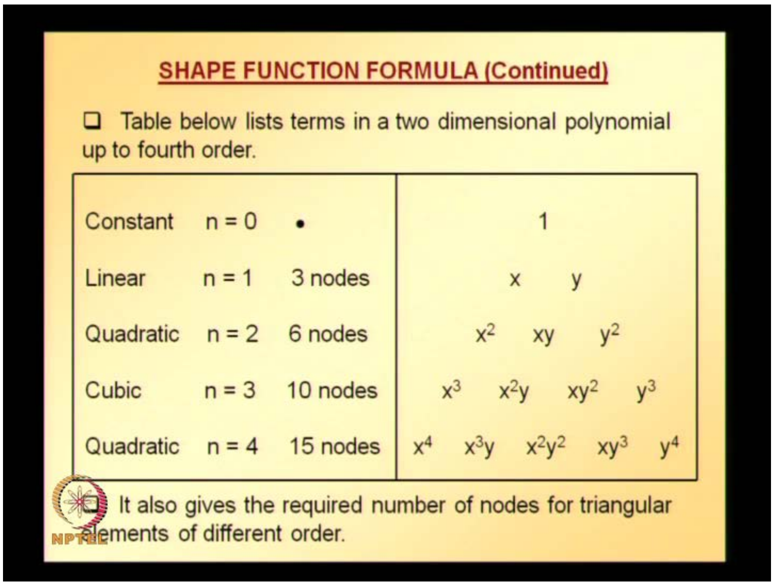

Let us see that these terms are coming from the table as given here. Table below list the terms in two-dimensional in a two-dimensional polynomial up to fourth order, n is equal to 0 corresponds to constant. It consists of only 1. A linear element consists of functions in the trial function of a linear element will have 1, x y. And it consist of three nodes. And in a quadratic trial function we have terms 1 x y, x square x y, y square. So, we need to take n is equal to 2. And it corresponds to six node triangle. Similarly, if one selects all the terms up to cubic order then ten terms in total trial solution will have. The triangle element is of order three, it is going to be cubic n is equal to 3. And it should consists of same number of terms as the number of nodes in a triangle should be equal to number of terms. So, it should consist of ten nodes.

Similarly, a quadratic triangle element consists of fifteen nodes and it consists of fifteen terms which are all the terms shown in the table which can be easily verified which counts to fifteen. And the order of element is going to be quadratic sorry quartic. And n is equal to 4. It also gives the required number of nodes for triangular element of different order. We can extend this table and we can easily get what are the number of nodes required for triangular element of different order.

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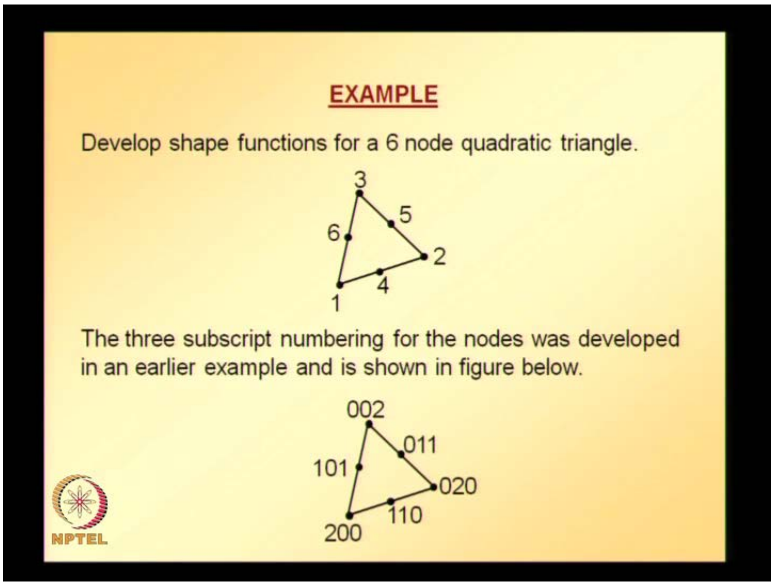

Now, let us try to understand, how we can apply this formula for deriving shape functions for six node elements, for which we already found what are these L1, L2, L3. And we already looked at three subscript numbering scheme. So, we are ready to use this formula, because we already have three subscript numbering schemes for each of the nodes. The three subscript numberings for nodes was developed in an earlier example and is shown in figure above. Now, we are ready to use the formula N 1 shape function corresponding to node one is equal to N 200 and before that, we are dealing with six node element.

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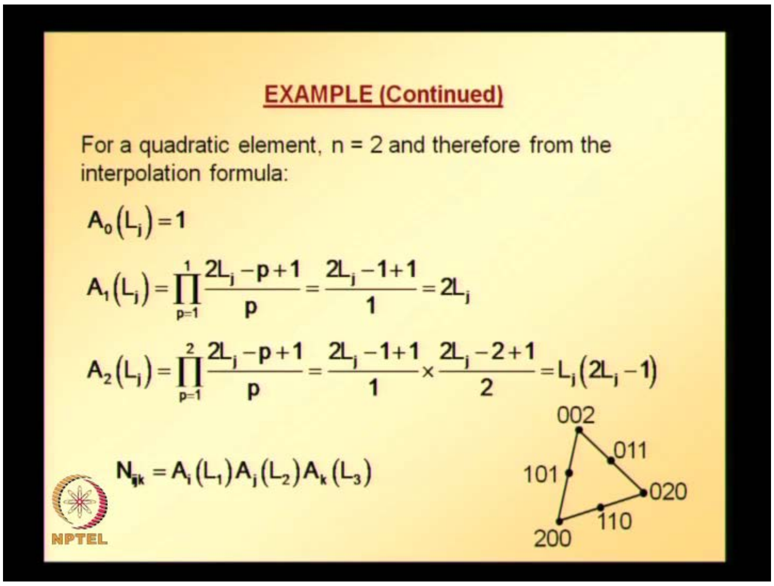

So for a quadratic element quadratic element n is equal to 2 and the formula for all A s, A i here substituting into 1 substituting in the three subscript numbering corresponding to node one that is 200 into the interpolation formula that we have seen in the earlier slides, we get shape function corresponding to node 1. Before that we require to calculate what are these A naught L j, A 1 L j, A 2 L j, because these will be required, when we are actually using the interpolation formula. A naught L j using the formula for A i L j can be verified as is equal to 1. A1 L j again here I take value 1. So, the counter over is pi goes from P is equal to 121, which gives us only term. So, A1 L j turns out to be 2 L j. And also it can be verified that A2 L j, i takes values 1 and 2. Counter goes from 1 to 2, so two terms we need to take product of those finally it results in L j times two L j minus 1. Depending on J value, we can use this A naught L j, A1 L j and A2 L j. For a particular node with three subscript numbering schemes; this is the formula. So now in the figure on the right hand side all the three subscript numbering schemes for all the nodes are shown.

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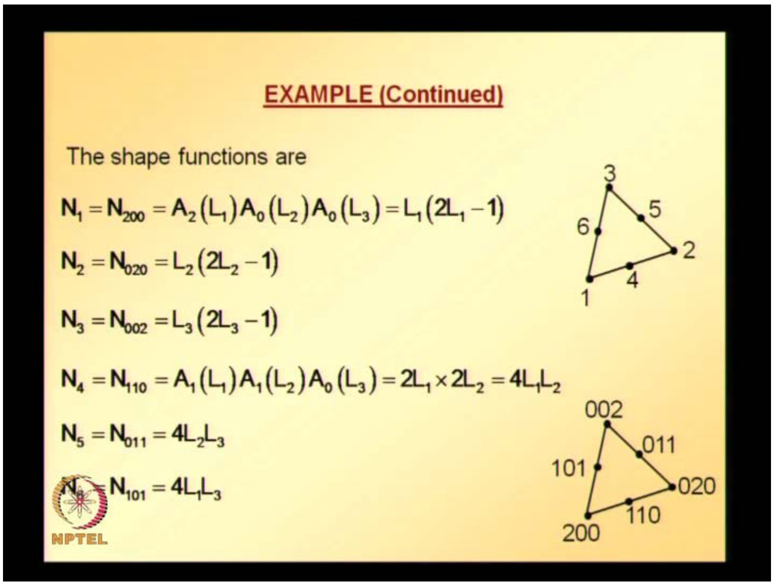

Shape functions, for node 1 is same as in three subscript numbering scheme, node 1 corresponds to 200. So, N 200 that gives us i is equal to 2, j is equal 0, k is equal to 0 substituting into the interpolation formula we get this. After that we need to substitute what is this value of A2 L1, A naught L2, A naught L3 by substituting all those. And simplifying N 1 is equal to L1 times two L1 minus 1. Similarly, shape function for node 2, node 3, node 4, node 5 and node 6. With appropriate change of variables, all integrations and differentiations necessary for developing finite element equations can be carried out in the natural coordinates. If necessary, the shape functions can also be expressed in terms of x, y coordinates, which we will see in an example, so this is how shape functions can be calculated.

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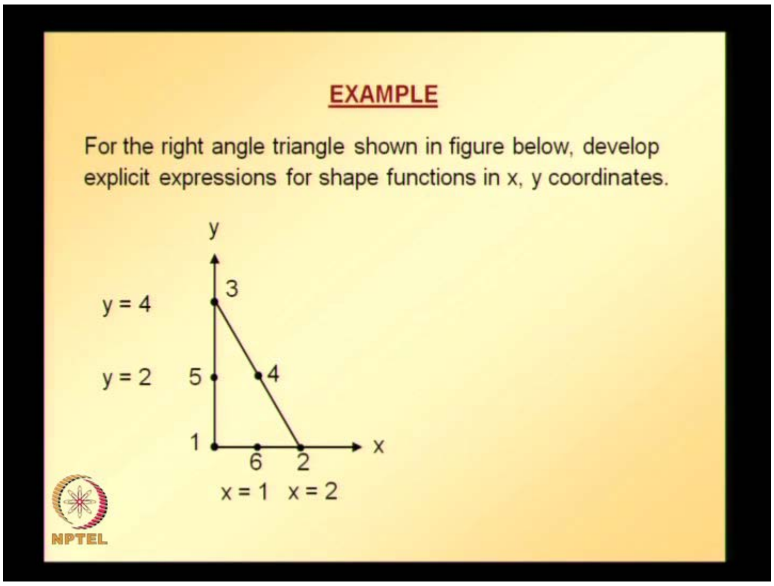

Now let us take an example. For right triangle shown in figure above, develop explicit expressions for shape functions in x, y coordinates. All the coordinate information is given in the figure. We can easily calculate what are L1, L2, L3, which are required for writing the natural coordinates.

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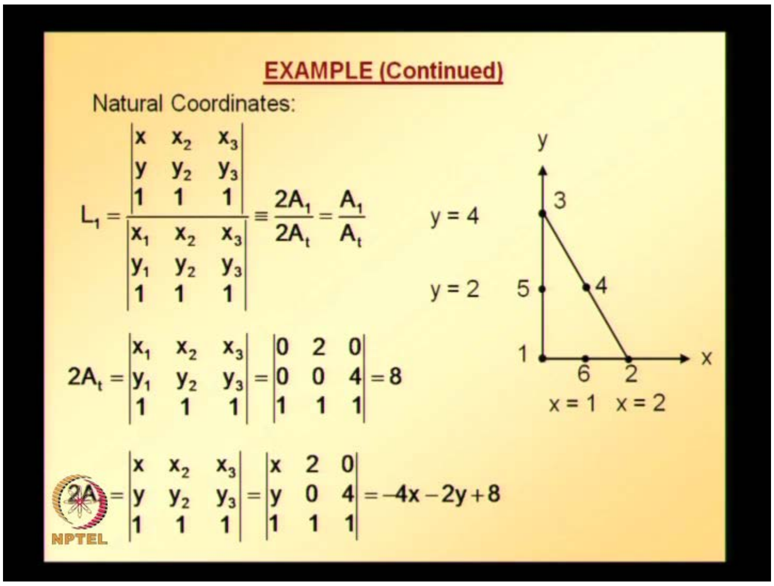

Natural coordinates of all the nodes, and also finally to get three subscript numbering schemes, L1 is given by this formula. And to find L1, we need to calculate what is A t and A1, they are also shown and finally we can calculate L1.

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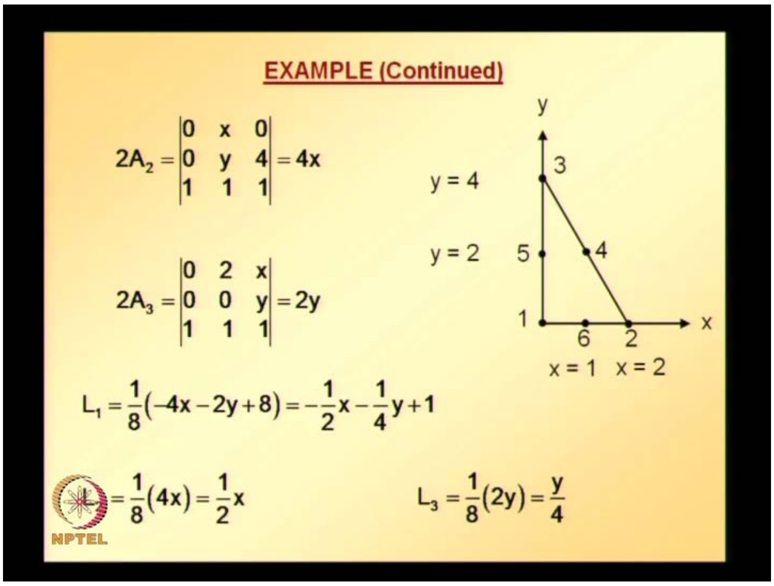

To calculate L2 and L3, we need to know what are A2 and A 3 those are also shown. And using that A t, we can calculate L2 and L3. L1, L2, L3 for this particular right triangle with the coordinates that are given by this L1, L2, L3.

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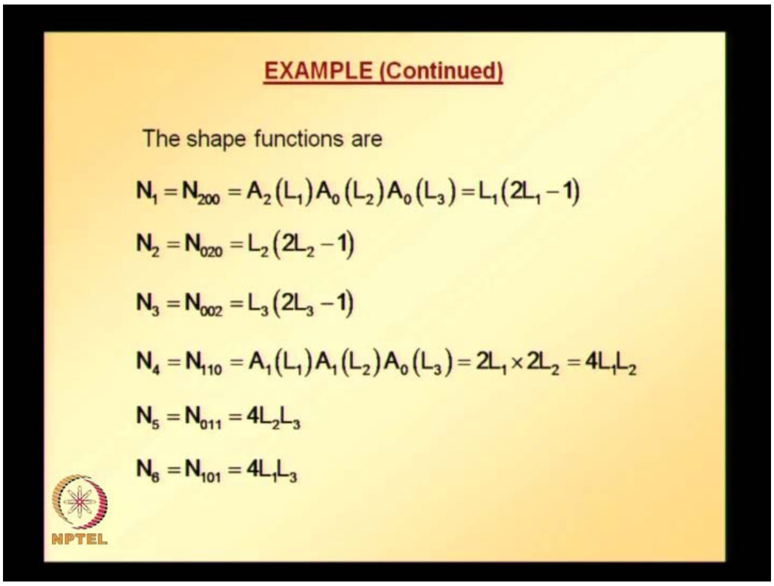

In the earlier example, we have already derived shape function for all the six nodes in terms of L1, L2, L3.And for this right triangle element we got L1, L2, L3. Now, you need to substitute those values of L1, L2, L3 into these. And then we can get the shape functions for all the six nodes in terms of x, y. So, this is how one can derive shape function.

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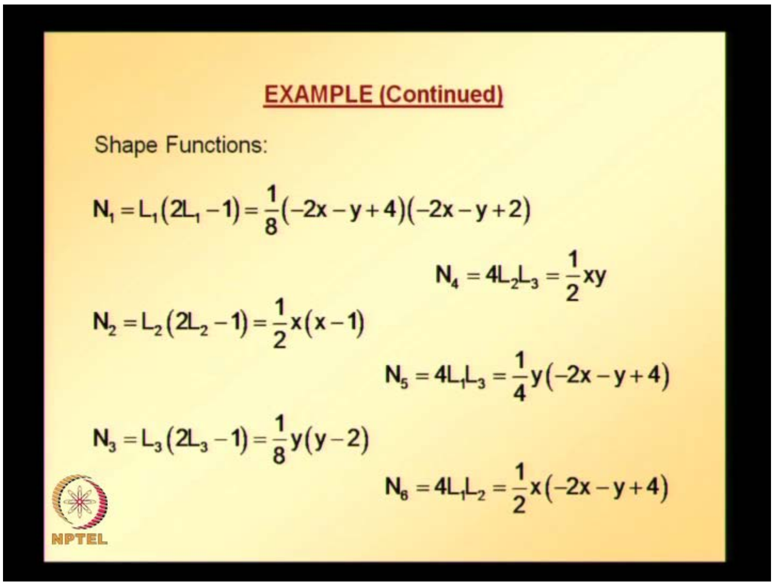

For higher order triangular element and not necessary in x y coordinate system, but we can also derive in parent element coordinate system that is in s and t.

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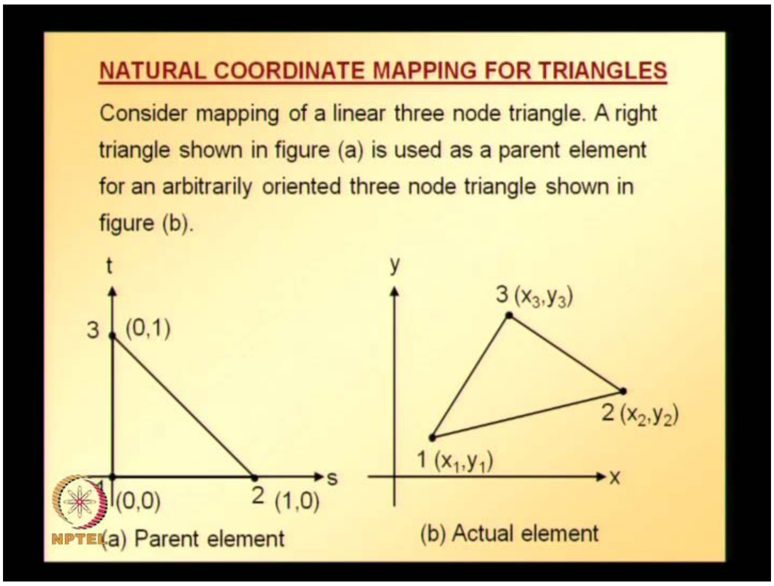

Now, let us look at mapping for triangles. Consider mapping of a linear three node triangle. A right triangle is shown in the figure on the left hand side. And that is used as a parent element for arbitrarily oriented three node triangle element, which is shown on the right hand side. So, the left hand side shows the parent element and on the right hand side shows the actual element. Any element we are going to map on to an element which looks like what is shown as parent element in which the nodal coordinates are 1 2 and 3. And they are located at 00 node 1 and node 2 is located at 10 and node 3 is located at 01. To get the isoparametric mapping if you recall even the way we did for quadrilateral element, we required to find shape functions in the parent element coordinate system. And use the actual element x, y coordinates to get the relation between x and s and t and y and s and t from where we can find all the derivatives of x with respect s and t, y with respect s and t and finally, we can get jacobian and determinant of jacobian.

Using the formulas that we have seen till now in today's class to get the shape functions for this parent element in s and t coordinate system. What we need to do is? we need to find L1, L2, L3 for each of these nodes. And then for three node element L1, L2, L3 are same as N 1, N 2, and N 3 that can be easily be verified. So, we need to find L1, L2, L3 for the parent three node element.

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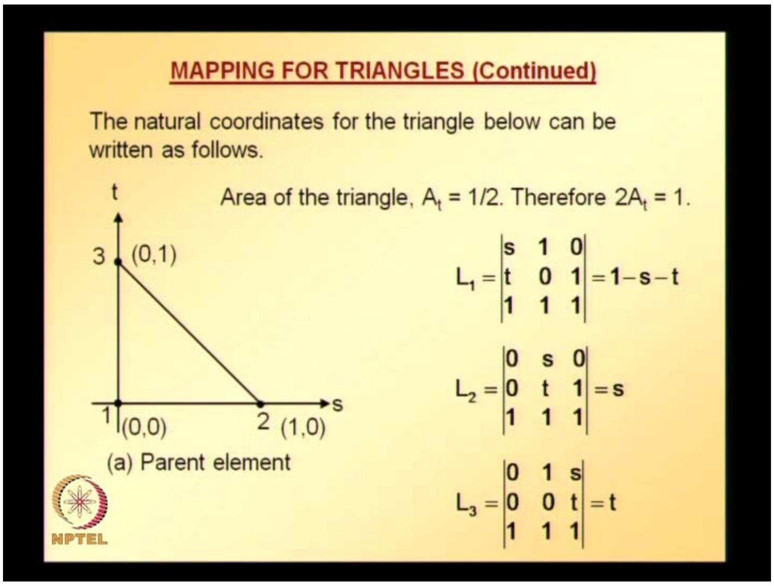

Natural coordinates for the triangle above can be written as follows. For this triangle, one can easily calculate area which total area is going to be half. And therefore 2 A t is going to be 1. And L t as I mentioned replace, when you are trying to find, what is L t replace the first column of the numerator with instead of x, y since we are dealing with s and t replace them with s and t. And the second column and third column consists of nodal coordinates of node 2 and 3.And since total area is equal to 1. So, denominator is going to be 1. So, L1 is given by this L2 is given by this and L3 is given by this.

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We got L1, L2, L3 for these nodes. Using the three subscript numbering scheme, the shape functions for this triangle can be readily written. And you can do that as a part of homework. And it can be easily verified that L1 is same as N1, L2 is same as N2, and L3 is same as N3.

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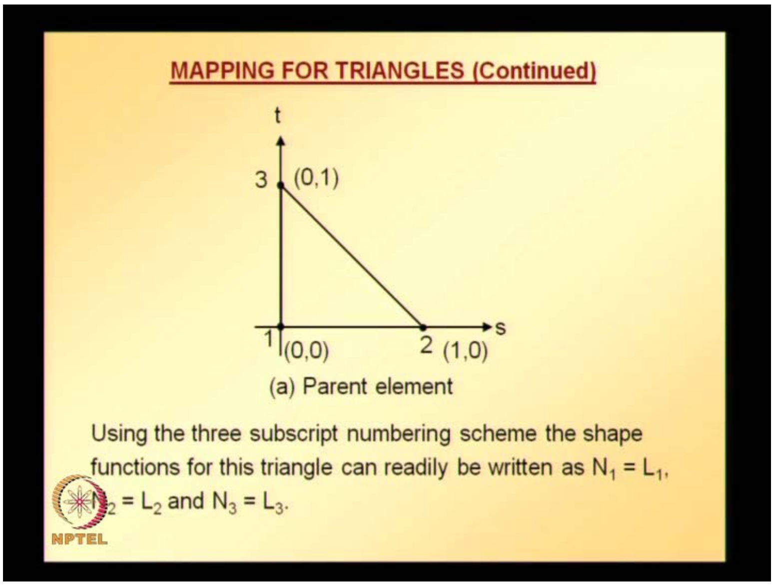

We got shape functions N1, N2, and N3 of the parent element. And we also know the coordinates of all nodes of the actual element. We can easily write the mapping.

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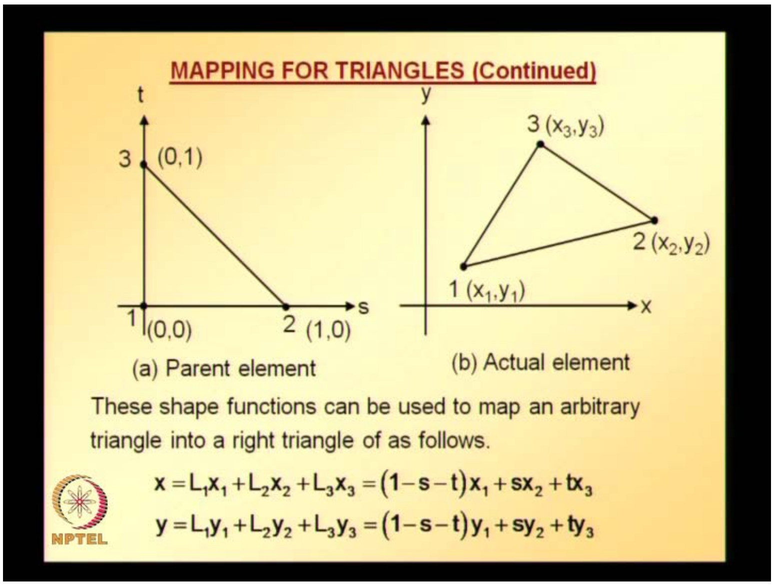

The shape functions can be used to map an arbitrary triangle into right triangle as follows. This expression looks similar to what we have earlier for quadrilateral element, except that here L1, L2, L3 are used, because L1, L2, L3 are same as N1, N2, and N3 for only for three node triangle element. x is substituting L1 in terms of s and t, L2 in terms of s and t and L3 in terms of s and t. We get relation between x and s and t and y as a function of s and t.

Now we can easily verify from these two equations when s is equal to 0 and t is equal to 0. Substitute s is equal to 0, t is equal to 0 that corresponds to x is equal to x1 and y is equal to y1. Similarly, s is equal to 1 and t is equal to 0 gives us x is equal to x2, y is equal to y2. And s is equal to 0, t is equal to 1 gives us x is equal to x3, y is equal to y3. Thus an arbitrary triangle in the x, y coordinate system is map to a right triangle in the eccentric coordinate system using the terminology used for isoparametric elements the right triangle is parent element for arbitrary triangular element.

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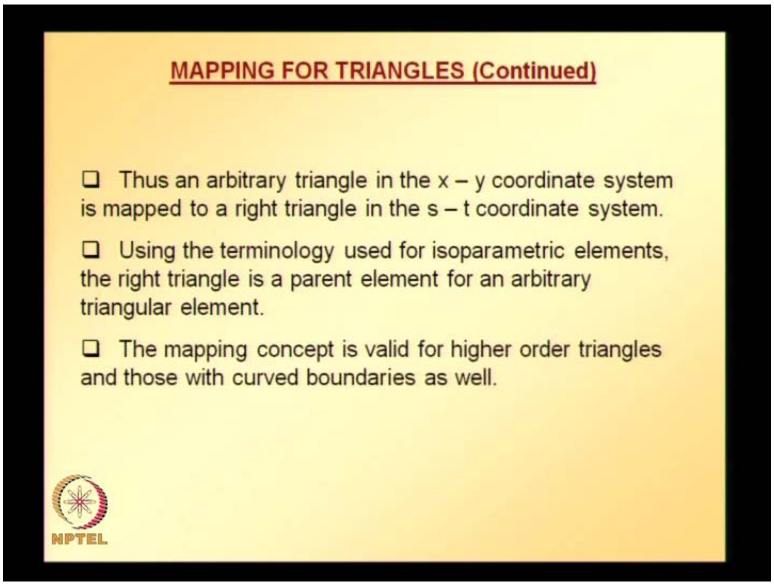

This mapping concept is as well valid for higher order triangle elements and those with curved boundaries.

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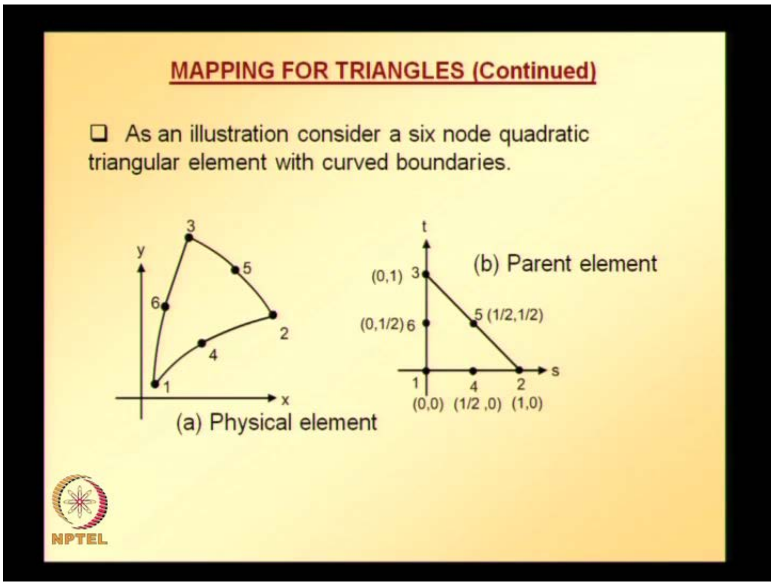

Let us take quickly as an illustration, consider a six node quadrilateral sorry quadratic triangle element with curved boundaries. Physical element is shown physical element is same as actual element with curved boundaries is shown on the left hand side. And parent element is shown on the right hand side. And all the coordinates are indicated for the parent element; the intermediate nodes are located at the midpoint of each of the sides. With the information that is given for parent element, coordinates we can easily derive what are the shape functions by calculating L1, L2, L3 for each of the nodes.

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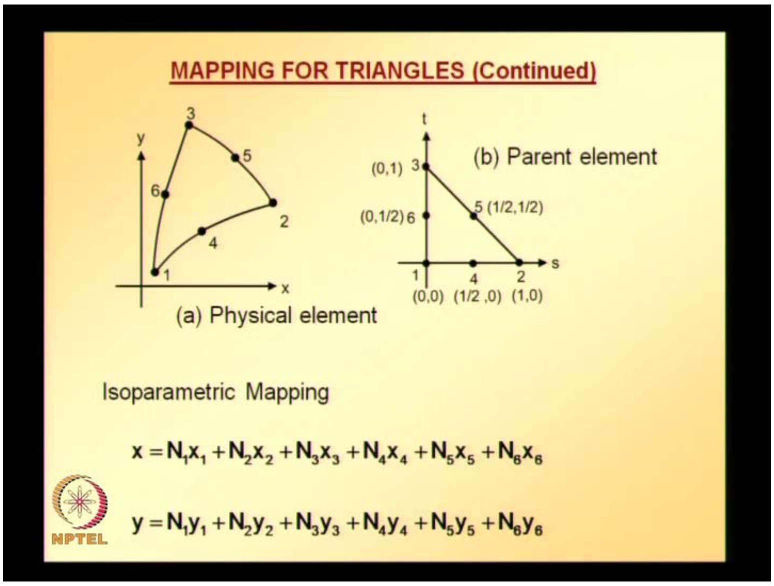

Also L1, L2, L3 for each of the nodes we can easily find what is isoparametric mapping for this particular element, which is going to be x is equal to N1 x1 plus N2 x2 plus N3 x3 plus N4 x 4 plus N5 x5 plus N6 x6. Similarly, y isoparametric mapping for y, but to use these two expressions, we require N1 to N6 of parent element.

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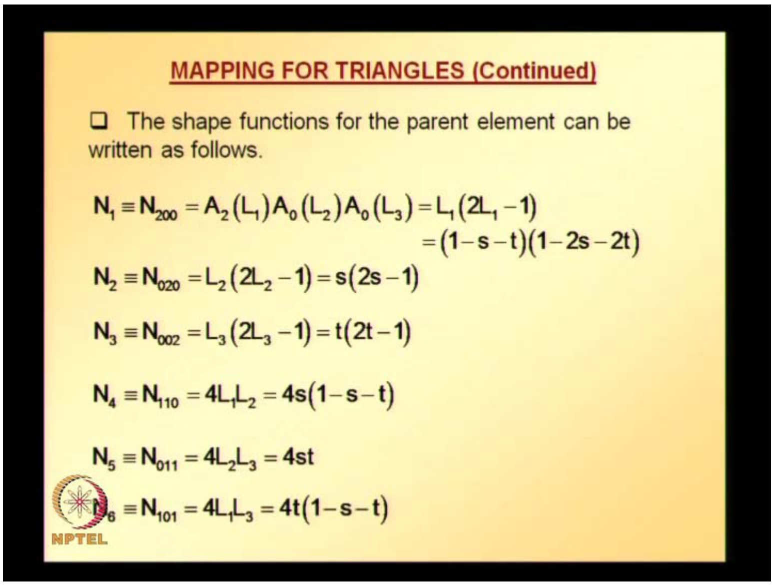

For six node element, we have already seen earlier in today's class. The shape function expressions for N1 to N6 in terms of L1, L2, L3 and also L1, L2, L3 we already have. Using that information we can write shape functions for all the nodes N1 to N6.

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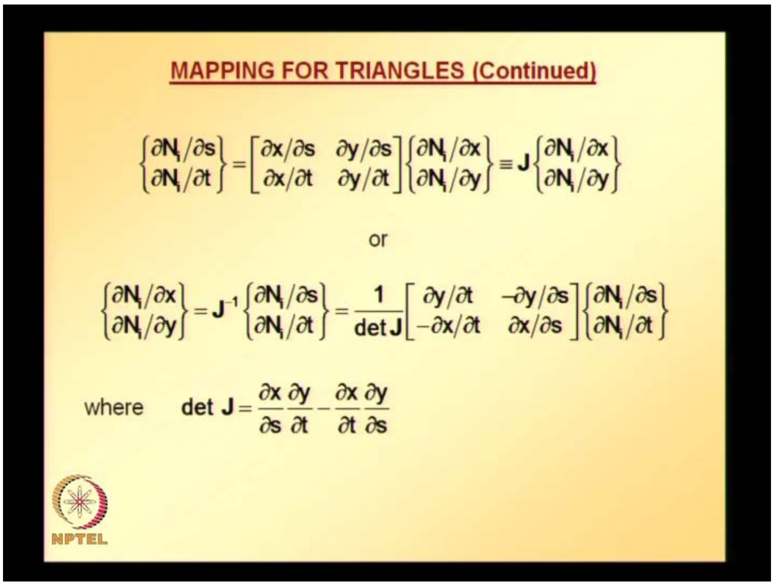

Once we have this shape function expressions, we can write isoparametric mapping and also we can find derivative of shape functions to finally calculate jacobian and determinant of jacobian using these relations. We will see applications of these equations in the next class, again to check the validity of mapping for triangular elements.

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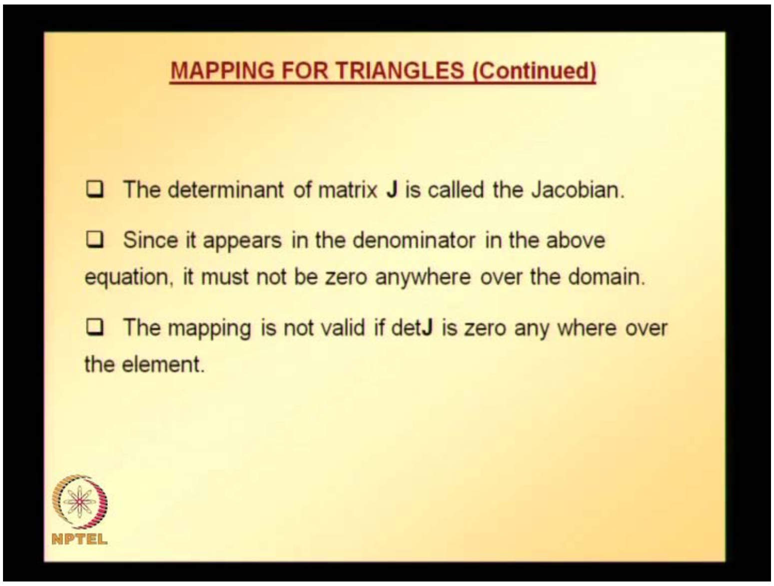

Similar to quadratic elements, the determinant of matrix J is called jacobian. Since, it appears in the denominator in the above equation, it must not be 0 anywhere over the domain. Mapping is not valid if determinant of J is 0 anywhere over the element. So all these points are similar to quadrilateral element that we already looked at for four nodes and eight node quadrilateral elements validity check we already perform. So, similar points we already took care for those elements, same points are valid here. With this, we will continue in the next class.