Finite Element Analysis
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Lecture No. # 27

In the last class, we have seen element equations development for two-dimensional general boundary value problem using 4 node elements. We started with governing differential equation applied Galerkin criteria, and then applied Green's theorem, and then later we substituted approximation of trial solution and derivative of trial solution in terms of finite element shape functions, and we obtained complete element equations for two-dimensional general boundary value problem for 4 node quadrilateral element.

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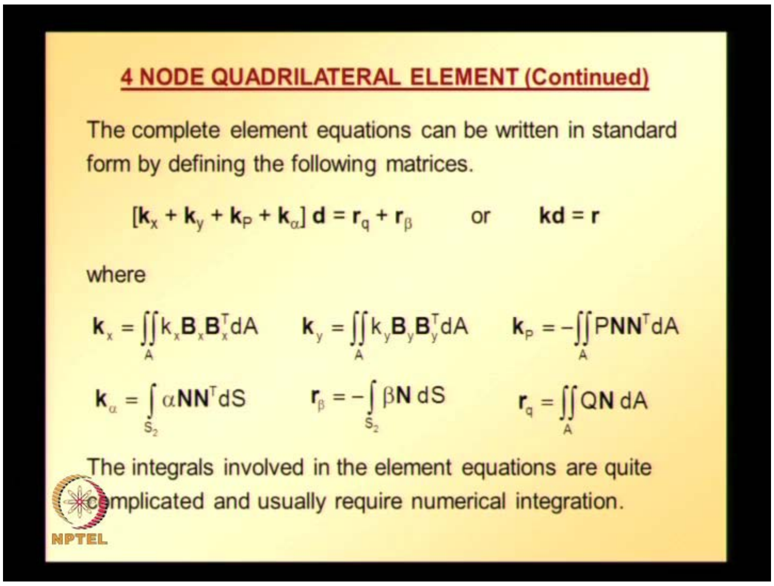

I just briefly summarize what we have done in the last class. The complete element equations can be written in standard form by defining following matrices in which, each of these quantities k x, k y, k p, k alpha, r q, r beta are defined here. And this equation can also be compactly written as k d equal to r, where contribution to k comes from k x, k y, k p, k alpha and contribution to r comes from r q and r beta. If you see each of these components k x, k y, k p, k alpha, r beta, and r q, some of these are area integrals, and some of these are boundary integrals.

Let us discuss in today's class, how to evaluate each of these integrals. First let us look at evaluation of area integrals, and later let us look at evaluation of boundary integrals for given a 4 node quadrilateral element. And these integrals can be quite complicated and usually require numerical integration, because once we start using a bilinear element or quadratic element, these integrals are no longer going to be constant unlike in three node triangular element which is linear, so we need to adopt numerical integration. So, all those details will be illustrated in the numerical examples that we are going to see in today's class.

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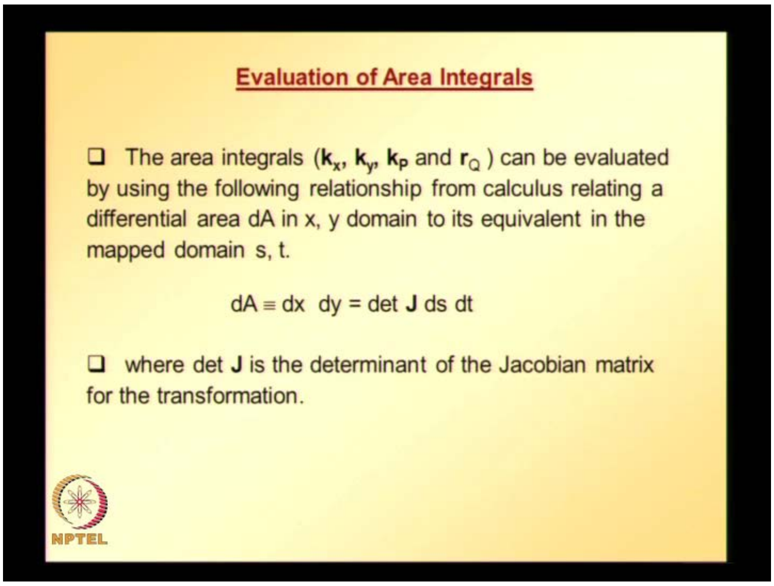

Let us start with discussing how to evaluate area integrals. Area integrals k x, k y, k p, r q can be evaluated by using following relationship from calculus relating differential area in x y domain to it's equivalent in the mapped domain s and t, so we are working for the actual element we are working in x y domain and that actual element we are going to map it on to a parent element which is in s and t domain. We need to know what is the relationship between differential area in x y domain to it is equivalent or differential area in mapped domain s and t, so this is how it is related differential area in x y domain is related to its equivalent in mapped domain s and t via this relation that is d x times d y is equal to determinant of J times ds dt where determinant of J is determinant of jacobian matrix that we get from partial derivatives of x with respect to s, x with respect to d, y with respect to s, and y with respect to t.

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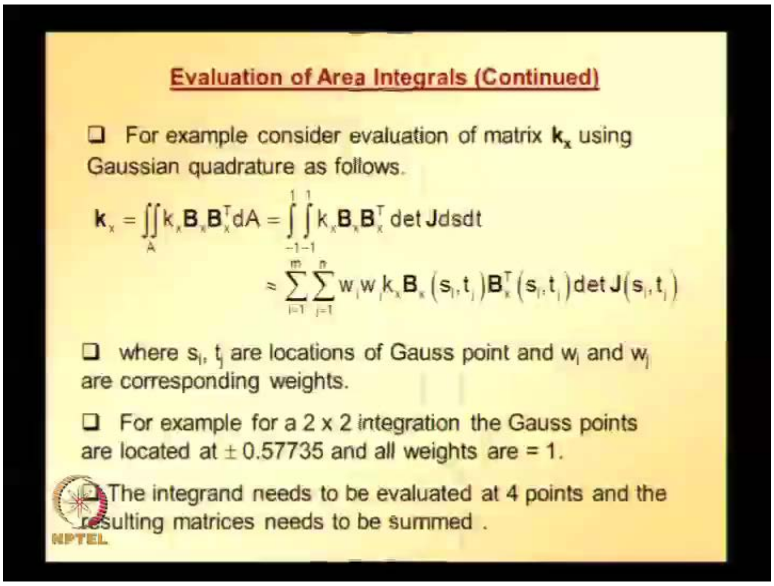

Once we have this relation of differential area from x y domain to s and t, it is equivalent in s and t domain we can evaluate the matrix k x. For example, consider evaluation of matrix k using Gaussian quadrature, so this is how k x matrix is defined integral over area k x times B x, B x transpose. We need to evaluate B x, B x transpose times k x over this area this limits of integration can be changed to minus 1 to 1 in the parent domain or using the relation that d A is equal to determinant of jacobian times dsdt. We can change the limits of integration as given in the question which can be further approximated by taking certain number of integration points along s direction and t direction and the formula is written for the case, where m number of points are taken along s direction and n number of points are taken along t direction.

If you see the formula s and t, s i, t j are locations of gauss point W i, W j are corresponding weights. For example, for 2 by 2 integration gauss points are located at plus or minus 0.57735 and weights that all plus or minus 0.57735 that means we need to use all kind of combinations to get the coordinate of all the four integration points and weights at each of these integration points is going to be 1, because weight in the s direction is one and weight in the t direction is one. If we decide to use 2 by 2 integration we need to evaluate or the integrant needs to be evaluated at four points and the resulting matrices needs to be summed up, and whatever this formula that is shown in that k x is not assumed to be function of spatial coordinates x y or s and t, if coefficient is not constant unlike what is shown here. First we need to express k x in terms of s and t before we go ahead with numerical integration. How to express k x as a function of s and t, for that, we need to use isoparametric mapping, which relates x y to s and t.

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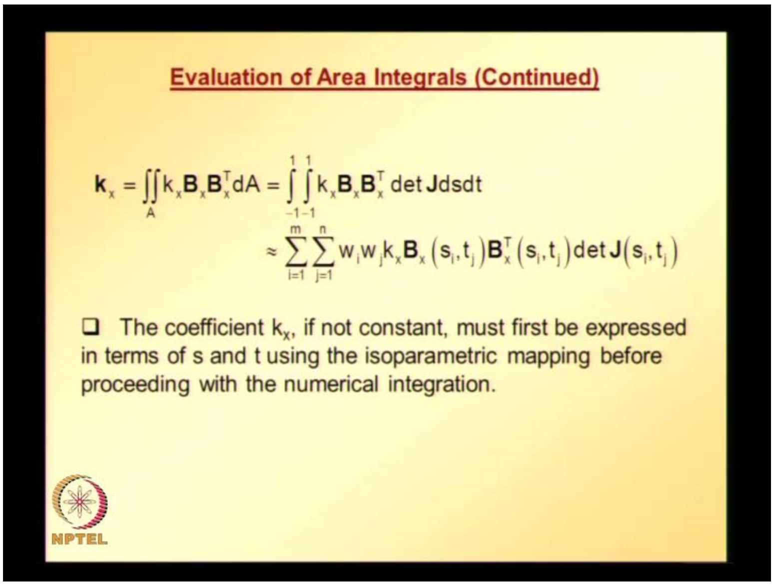

Let us take a problem and try to understand various steps in the evaluation of the area integrals. The coefficient k x, if not constant, it must be first expressed in terms of s and t using isoparametric mapping before proceeding with numerical integration.

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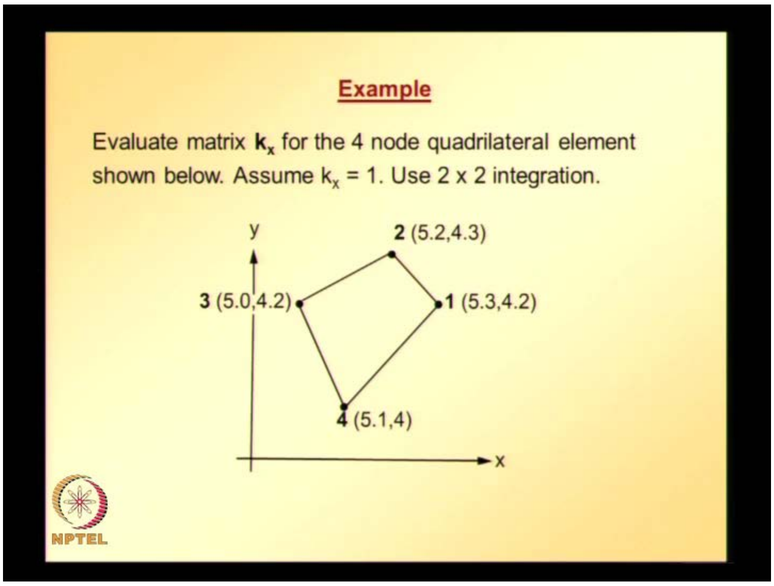

Now let us take an example evaluate matrix k x for the 4 node quadrilateral element as shown. Assume coefficient k x is equal to 1 use 2 by 2 integration coefficient k x is taken as a constant which is equal to 1 and the 4 node quadrilateral element is shown in the figure nodal coordinates of all nodes are indicated in the figure we can easily find what are the nodal coordinates and we can put all the x coordinates of all nodes in one vector and y coordinates of all nodes in another vector.

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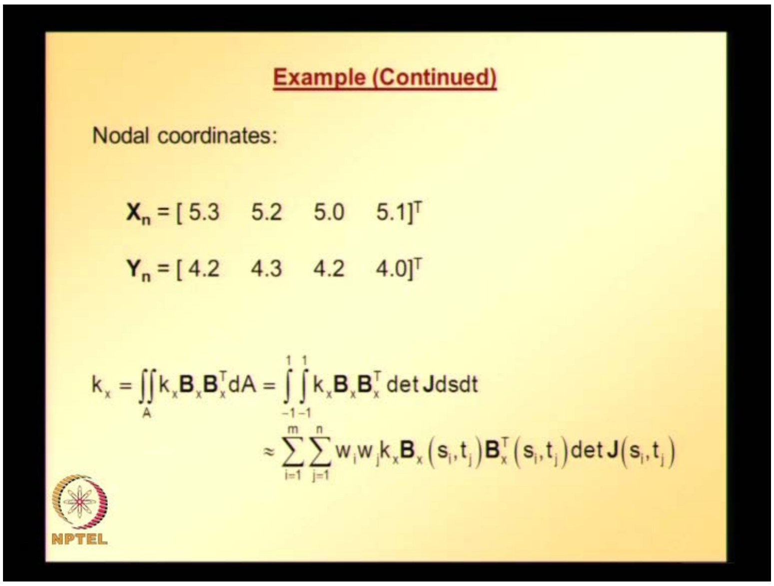

Nodal coordinates can be written in this manner and we need to evaluate this integral using 2 by 2 integration that means two points in the s direction and two points in the t direction, so we need to know what are the points.

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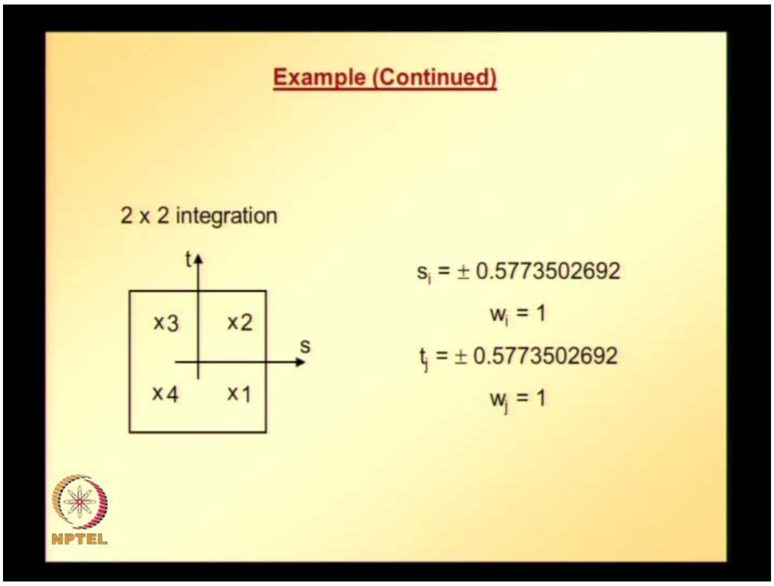

If you recall earlier, I gave this information 2 by 2 integration four points the locations of four points are as shown in the figure the corresponding coordinates can be obtained by all kinds of combinations of s i and t j which are equal plus or minus 0.57735 and weights at each of these integration points is obtained by weight in the s direction times weight in the t direction which is going to be equal to be 1 at all the four points. With this, we can easily figure out what are the locations of all the four integration points, now our job is to evaluate integrant using these weights and coordinates at each of the four integration points.

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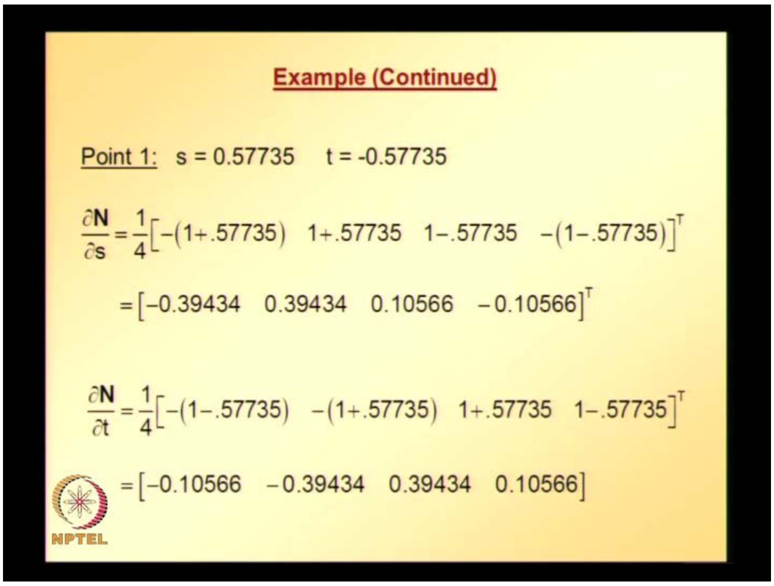

So now let us see the first point that we shows is s is equal to 0.57735 and t is equal to minus 0.57735. At this location we need to evaluate B x vector to get that we need to calculate what is partial derivative or derivative of shape function with respect to s, which is given by substituting all the values of s and t. Similarly, we can get derivative of shape function vector with respect to t and simplification of that gives us this. (Refer Slide Time: 11:00).

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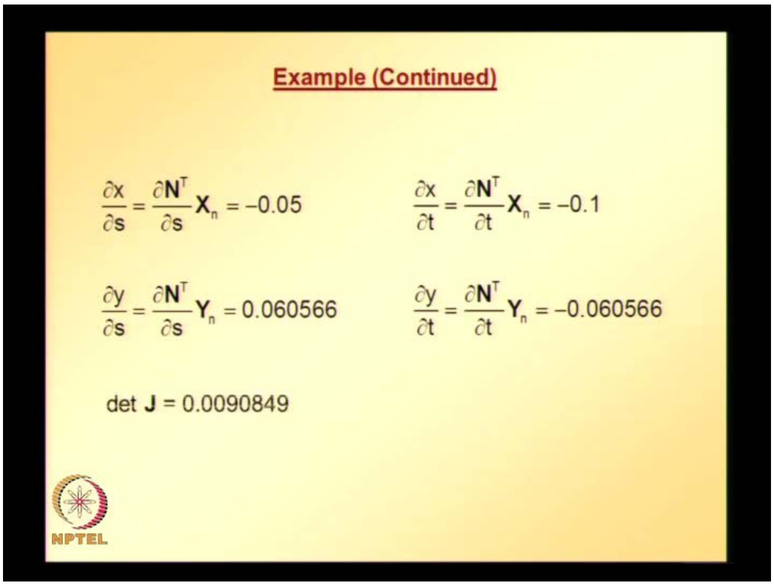

Once we have this two vectors we can easily calculate what is partial derivative of x with respect to s which is given by the formula and by substituting both the vectors and calculating the value it turns out to be that is minus 0.05. Similarly, other quantities partial derivative of x with respect to t, partial derivative of y with respect to s, and partial derivative of y with respect to t can be calculated, once we have all these quantities we can easily calculate what is determinant of jacobian and once we have determinant of jacobian, we can easily calculate what is B matrix B x vector.

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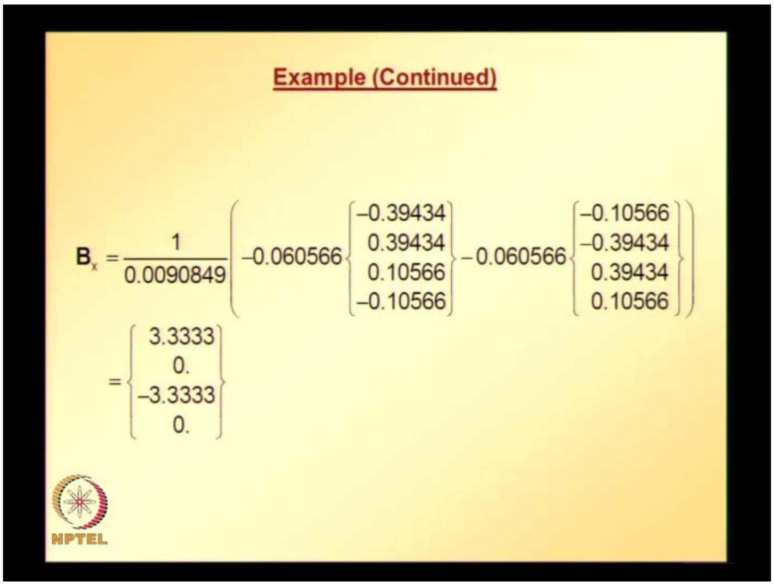

So, once we have this B x vector, we are ready to evaluate integrant value at the integration point that I have chosen that is s is equal to 0.57735 and t is equal to minus 0.57735.

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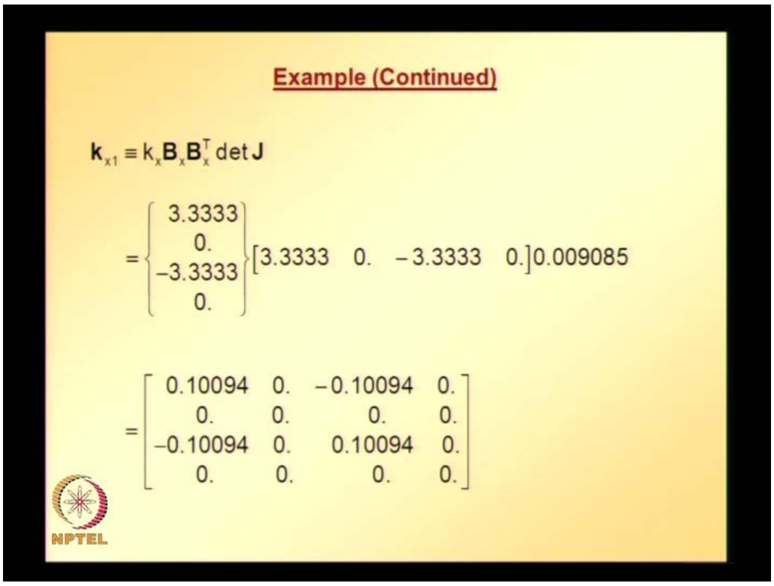

Substituting B x vector k x is constant which is equal to 1 and determinant of J we can evaluate what is contribution that comes from first integration point to the matrix k x. So carrying out vector multiplication we get this matrix, so this is contribution to k x matrix from first integration point that we have chosen.

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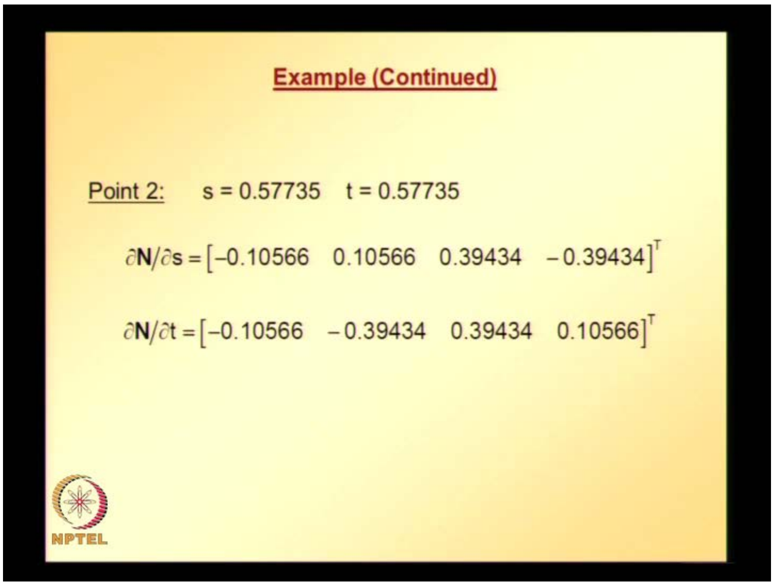

Similar procedure we need to follow for the other three integration points details are given. Let us go through those quickly this is a second integration point that is chosen and shape function vector derivative with respect to s, shape function vector derivative with respect to t.

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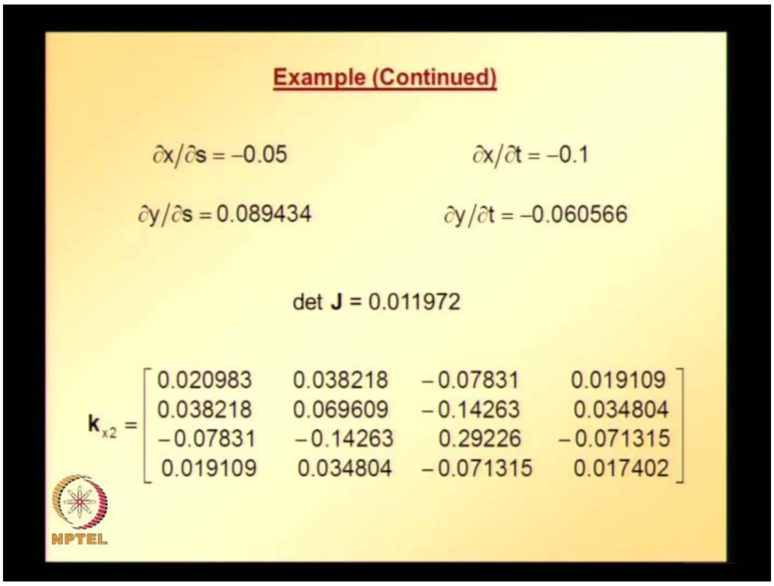

Once we have that information, we can easily calculate what is derivative of x with respect to s, derivative of x with respect to t, derivative of y with respect s, and derivative of y with respect to t, determinant of jacobian B x vector and then once we get B x vector we can easily calculate what is k x times B x, B x transpose times determinant of jacobian once we do all of that we get contribution to k x matrix from second integration point that we have chosen as this one.

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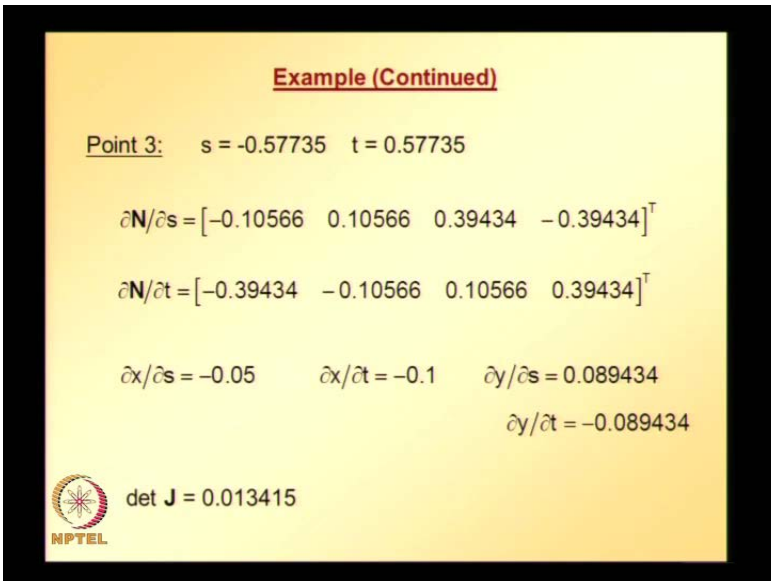

Similar thing we need to repeat for 0.3 and 0.4 details are given. 0.3 is chosen as s is equal to minus 0.57735 and t is equal to 0.57735 five derivative of shape function vector with respect to s, derivative of shape function vector with respect to t, derivative of x with respect to s, x with respect to t, y with respect to s, and y with respect to t determinant of J.

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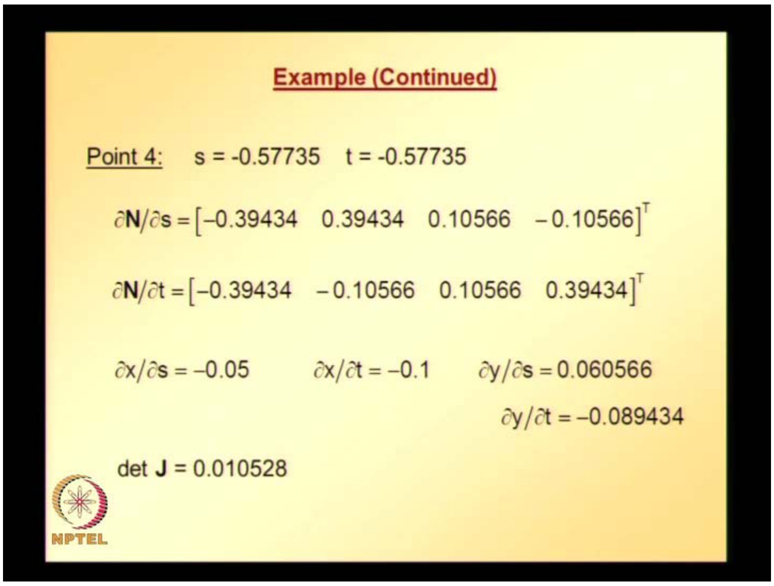

B x vector, we can calculate B x vector from that information and once we get B x vector we can calculate what is contribution from this third integration point to the matrix k x operation can be repeated for the fourth integration point. Derivative of shape function vector with respect to s with respect to t, derivative of s with respect to x, x with respect to s, x with respect to t, y with respect to s and y with respect to t, determinant of J.

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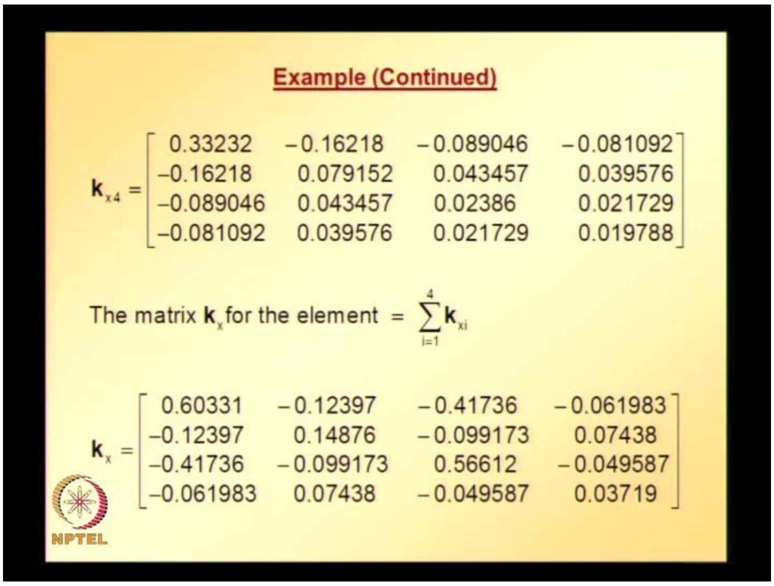

Then B x vector, once we get B x vector we can do k x times B x, B x transpose and once we do that k x times B x, B x transposed in determinant of J. Once we do that we get contribution from fourth integration point to the stiffness matrix as this and now we need to sum up all the contribution from four integration points and we get k x matrix as this, so this is how we can use Gaussian quadrature to evaluate this k x matrix, so this how we can evaluate area integrals.

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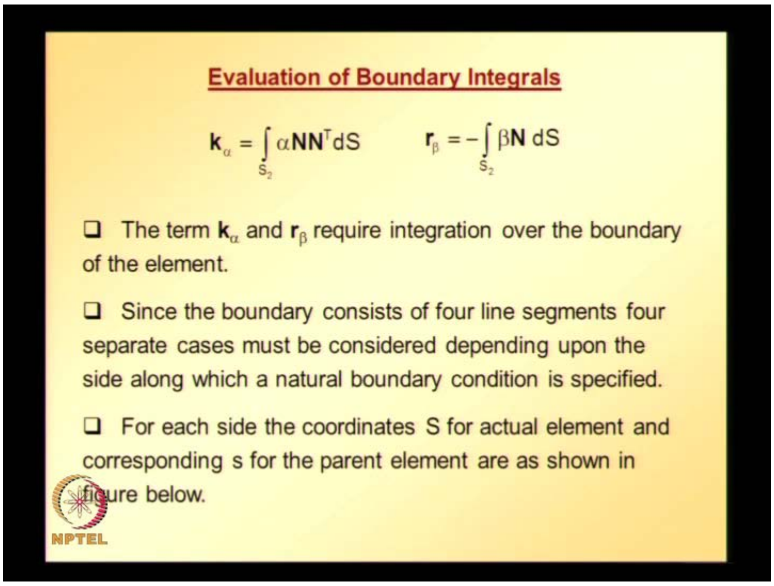

So now Let us discuss how to evaluate boundary integrals. Before doing that, please recall these are the two boundary integrals which we need to evaluate k alpha and r beta. The terms k alpha and r beta require integration along boundary of the element. Since we are dealing with 4 node quadrilateral element. Since the boundary consists of four line segments four separate cases must be considered depending upon the side along which natural boundary condition specified. Please note that we need to evaluate these boundary integrals only along this side on which natural condition is specified. For each side, the coordinates S for actual element and corresponding coordinates s for the parent element are shown. The coordinates for the actual element along each side is denoted with S.

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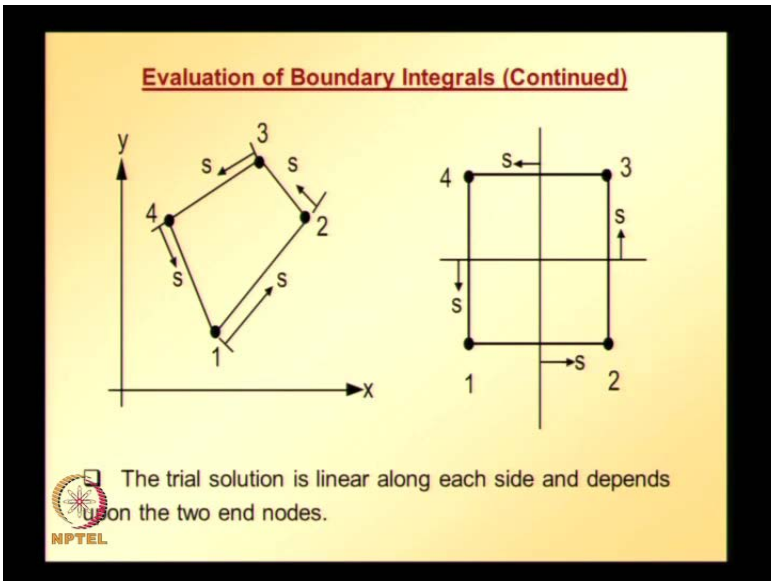

And for parent element it is denoted by s and it is shown in the figure. The actual element the figure on the left hand side is actual element, the figure on the right hand side is parent element. We can see S goes along each edge S goes from zero to length of that edge or length of that side whereas s along each of the sides goes from minus 1 to 1. And Shape function for each side of the parent element can be written by substituting constant coordinate into the shape function expressions you already know what are the shape functions of 4 node quadrilateral element in the parent element domain s and t. We already know the shape function expression in that shape functions for each side can be written by substituting constant coordinate value into the shape function expression. Since there is only one variable for each side, all shape functions are expressed in terms of s with positive directions of s for each side as shown in figure. You can also notice that the trial solution is going to be linear along each side and depends on only the two end values.

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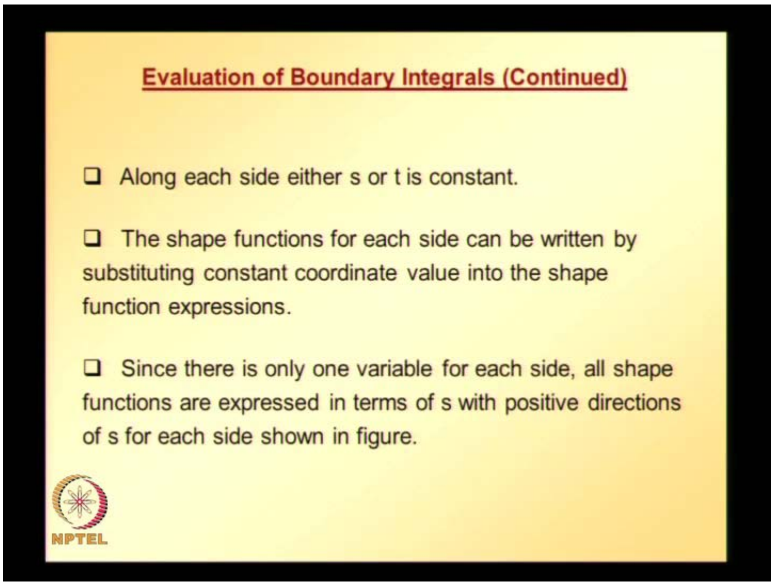

This is what I mentioned. Along each side either s or t is constant, shape functions for each side can be written by substituting out of these s or s and t whichever is constant by substituting that constant coordinate value into the shape function expressions. Since there is only one variable for each side either s or t all shape functions expressions are expressed in terms of s with positive directions of s for each side shown in figure.

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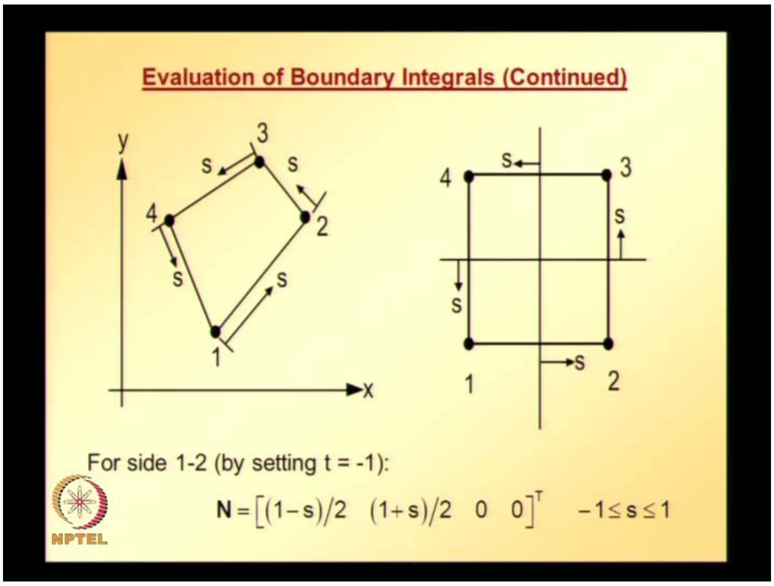

Now let us write the shape function vector along the each of the sides. For side 1-2, you can easily see from the parent element along side 1-2 t is going to be minus 1, so you know the shape function expression of all the 4 nodes for 4 node quadrilateral element in the parent domain. In those expression we substitute t is equal to minus 1 and put all the shape functions in a vector form this is how shape function vector is going to look for side 1-2 whereas goes from minus 1 to 1.

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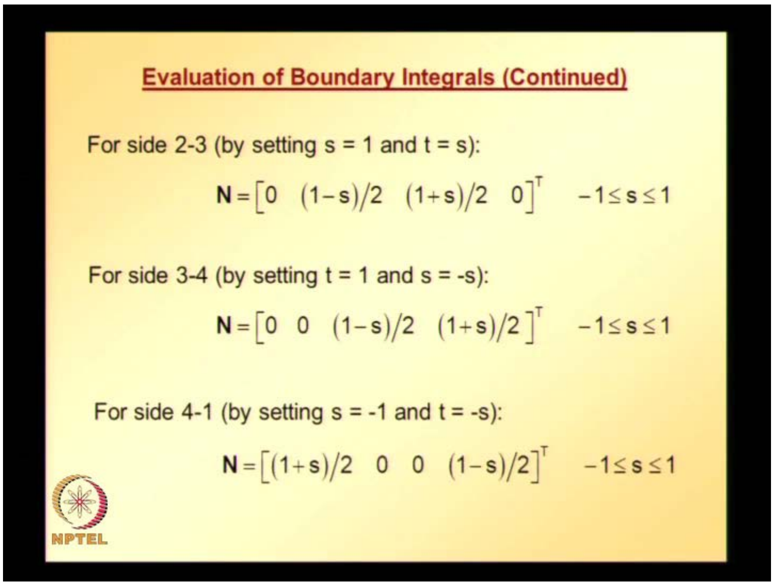

Similarly, for other sides, side 2-3, 3-4, and 4-1 we can write shape function vector by substituting the corresponding values of s and t.

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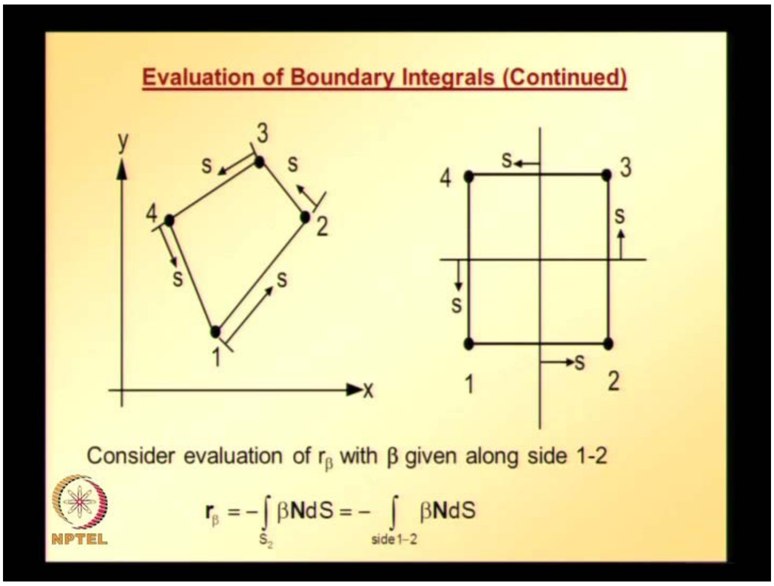

So we need to evaluate this consider evaluation of r beta with beta given alongside 1-2, so beta value assume that is given. We need to evaluate r beta along each of these boundaries, r beta is integral beta and D s. Here we will illustrate the procedure by taking side 1-2 that is why it is written side 1-2 there. So proceed to further we require what is the relationship between a differential element in the actual small differential element taken along each of the sides in the actual element domain and small differential element taken along each side in the parent element domain, so we need to find what is the relationship between d S and D s for that we require to find isoparametric mapping along each of these edges or sides.

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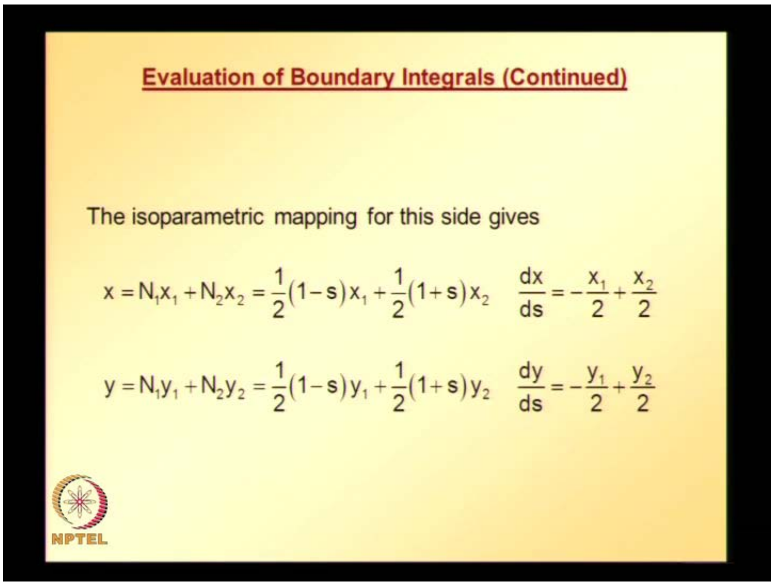

The isoparametric mapping for side 1-2 is x is equal to N 1, x 1 plus N 1, x 2 because N 3 and N 4 are going to be 0 alongside 1-2 and this gives us relation of X in terms of s. And from there, we can find what is derivative of X with respect to s. Similarly, y is equal to N 1, y 1 plus N 2, y 2 because N 3 and N 4 are 0 alongside 1-2. Once we simplify and take derivative that is derivative of y with respect to s, we get this (Refer Slide Time: 23:35). Finally, we are doing all this, please note that we are doing all this because we require to find what is relationship between a differential element in the actual domain relationship between differential element taken along a side in the actual element domain and what is it is relation with a differential element taken in each of the sides in the parent element.

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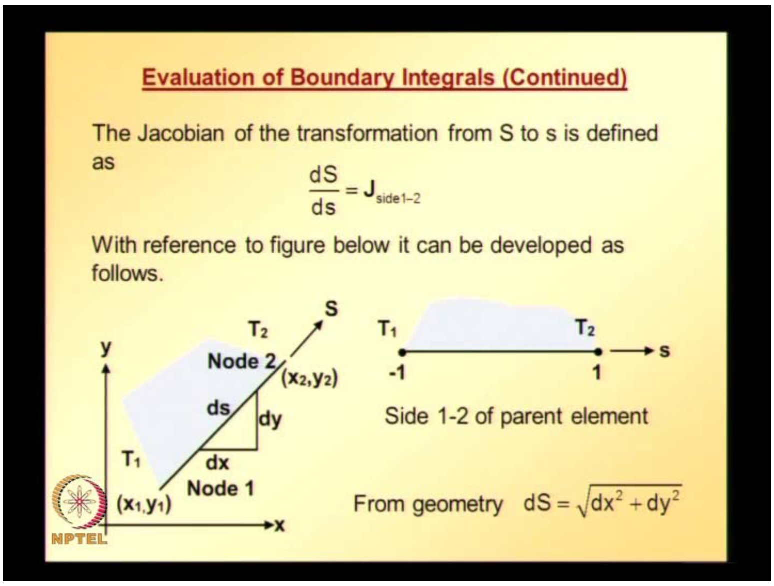

The jacobian of transformation from S to small s is defined as d S divided by d s that is what jacobian J side 1-2 and to get that let us closely look at the figure, which shows only side 1-2 in the actual element and also side 1-2 of the parent element. The actual element is shown in figure on the left hand side and the parent element side 1-2 is shown on the right hand side d S small differential element alongside 1-2 in the actual element is indicated there actually it should be d S in the figure it is printed as d s, please make that correction, d S can be resolved as d x and d y along x and y and it can be easily noted that the relation is t S is equal to square root of d x square plus d y square from pythagoras theorem. And now we just found what is d derivative of x with respect to s, derivative of y with respect to s, so now we have this relation that is d S is equal to square root of d x square plus d y square from the geometry of the actual element.

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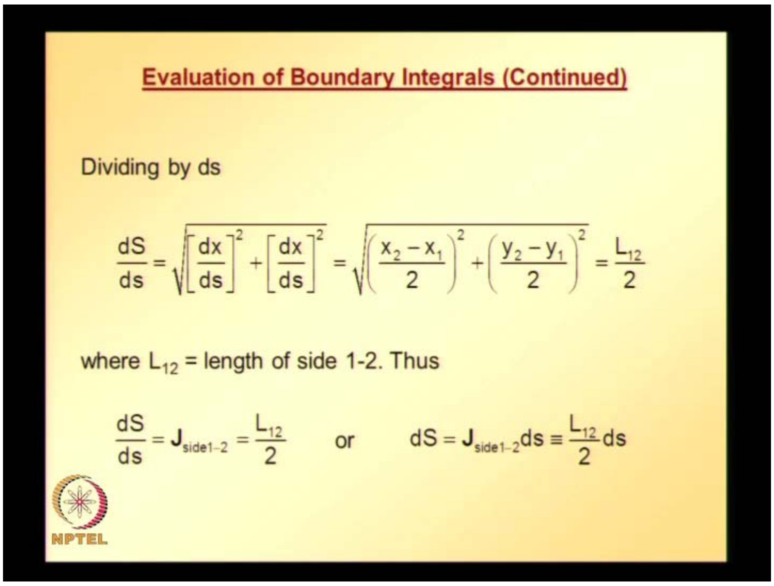

If we divide both sides of this equation with d s, we get this relation where we can substitute whatever we obtained from isoparametric mapping that is derivative of x with respect to s and derivative of y with respect to s. Here there is a mistake it should be d x over d s square plus d y over d s square, so instead of y, x is printed. This is the relation so L 1 to over 2 that is what is j for side 1-2. This L 12 length of side 1-2 can easily we obtained from the coordinates of the nodes and the relation is this one d S over d s is equal to L 12 over 2, which can be rearranged in this manner we can replace in the integral wherever we see d S that we can replace with L over 2 d s where s goes from minus 1 to 1, so the limits of integration also gets changed.

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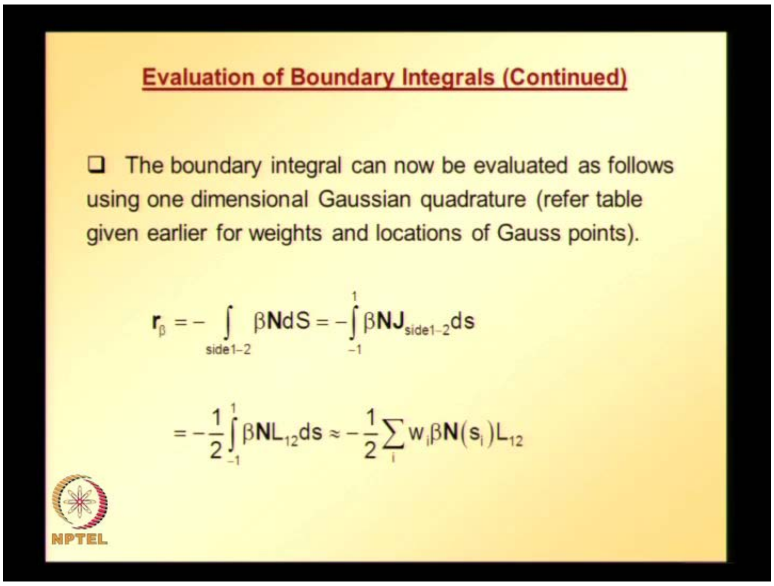

The boundary integral can now be evaluated as follows using one dimensional Gaussian quadrature and to get the points and weights we need to refer the tables that we already have from the earlier lectures. The previous integrals now take this form. Integral on side 1-2 is replaced with integral from minus 1 to 1 because d S is replaced with J times d s and now substituting the shape function vector alongside 1-2 we can evaluate this integral once beta is known. In this manner, using one dimensional Gaussian quadrature. Similarly, since k alpha is also a line integral or boundary integral we can evaluate k alpha also in a similar manner.

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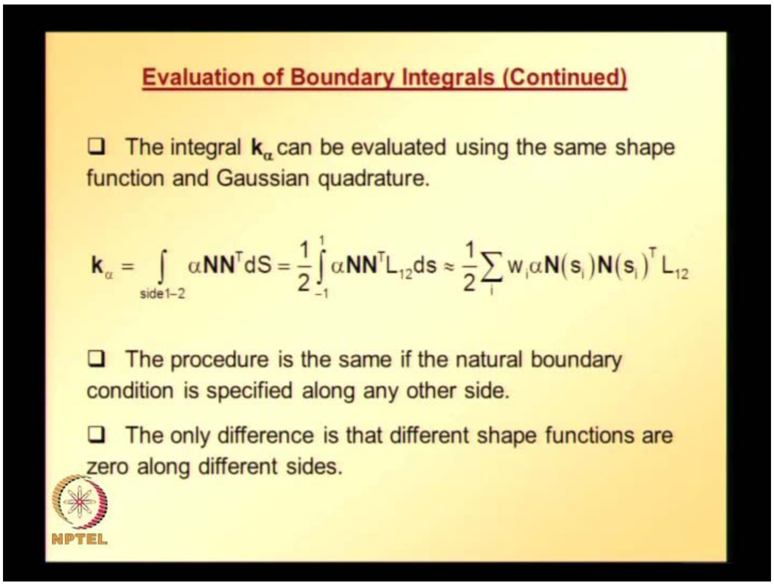

The integral k alpha can be evaluated using the same shape function and Gaussian quadrature. The procedure is same if natural boundary condition is specified along any other edges or sides the only difference is that different shape functions are zero along different sides, I think you agree with me on that that is a different shape functions are going to be zero along different edges, so we need to take care of that when we are evaluating these integrals and we need to be careful along which side we want to evaluate we need to substitute corresponding shape function vector. All these concepts will be made clear once we solve a numerical example. The procedure is same if natural boundary condition is specified along any other edges or sides with only difference shape functions are zero along different sides.

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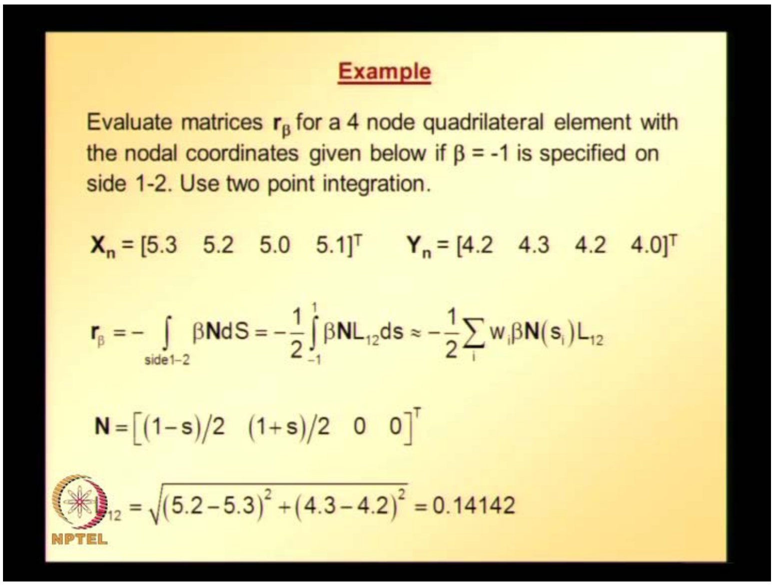

Now, let us take an example evaluate r beta for a 4 node quadrilateral element and beta is given as minus 1 and the coordinates of all nodes for this 4 node quadrilateral element are given and use two point integration. So this vectors of nodal coordinates are required for us because we require to find what is the isoparametric mapping and to find the relationship between d S and d s for that we require all this information. R beta is evaluated using this formula for which we require shape function vector alongside 1-2 as we already discussed shape function vector alongside 1-2 is given by this and length of side 1-2 is also required because jacobian is length of side 1-2 divided by 2. So we can get that or we can calculate L 12 from the information of the nodal coordinates using the formula.

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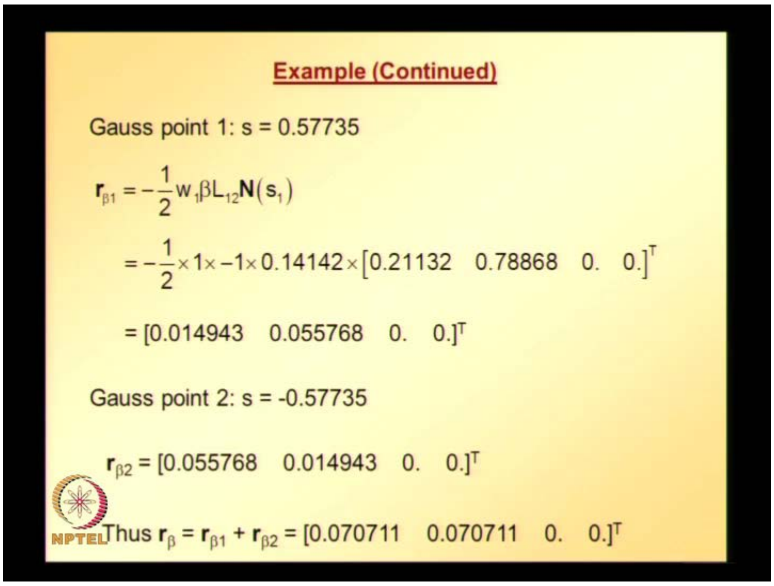

Now we need use two point integrations. This is the first point and weight at this point is equal to 1, so evaluate all the quantities that is integrant at this integration point by substituting s is equal to 0.57735 simplification of that gives us this. This is contribution from the first integration point to the vector r beta similar operation we need to repeat at the second integration point and we get the contribution from the second integration point to the vector or to the integral r beta and finally when we sum we get r beta vector, so this is how we can evaluate boundary integrals whether it is k alpha or r beta.

So let us look at similar kind of equations for 8 node serendipity element that is 8 node serendipity element for two dimensional boundary value problem. Before that why this 8 node serendipity element is required it is a higher order element and also curved boundaries can be model using this element.

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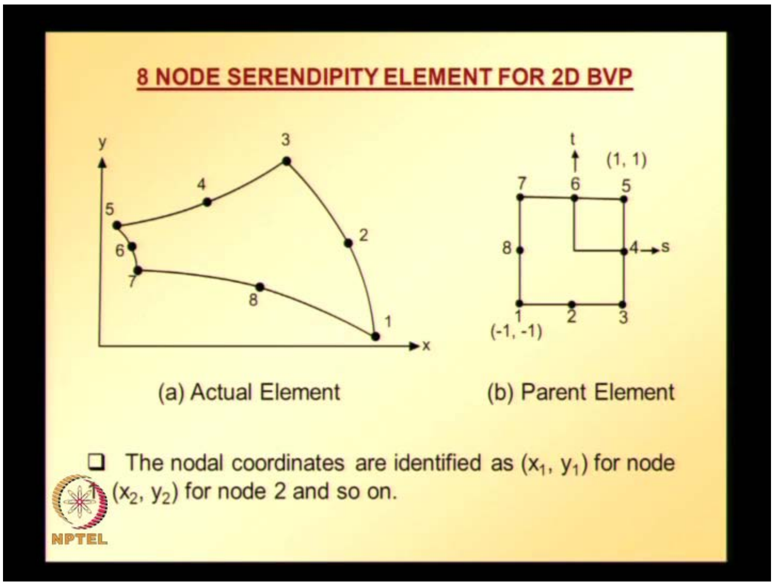

So, 8 node element actual element is shown and the parent element in s and t domain is also shown here, so consider an 8 node element shown here and also parent element is also shown on the right hand side as figure b. And let the nodal coordinates of actual element be x 1, y 1 of node 1, x 2, y 2 of node 2 and we can put all these nodal coordinates in two vectors all the x coordinates in one vector and all the y coordinates in another vector.

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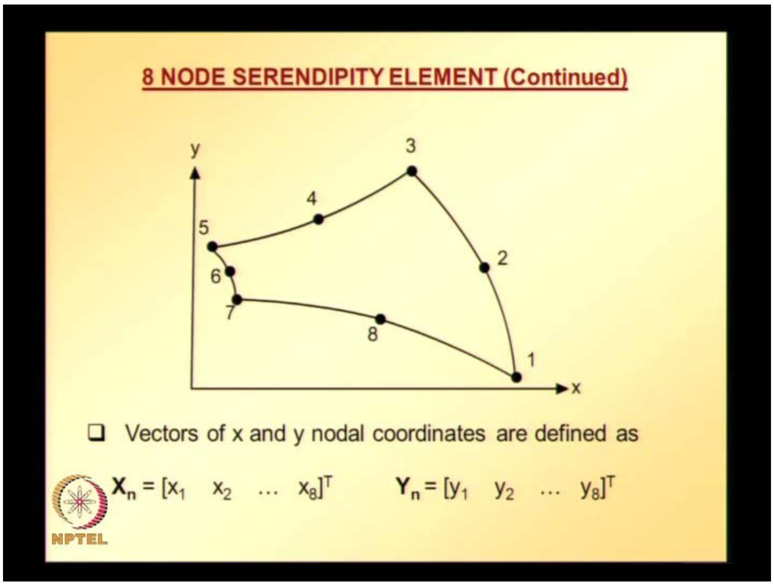

We can define two vectors X n and Y n in this manner because if it is advantages to put the X nodal coordinates and Y nodal coordinates in this manner because we can easily evaluate by vector operations what is partial derivative of x with respect to s, x with respect to t, y with respect to s, and y with respect to t calculate determinant of jacobian.

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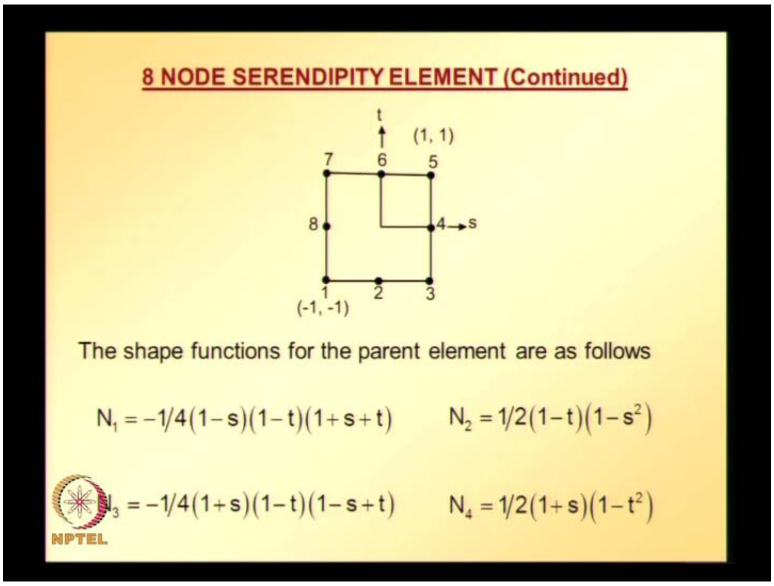

We also require shape functions of 8 node serendipity element in the parent domain. We already discussed how to get the serendipity element shape function earlier. The same shape function expressions are reproduced shape functions for parent element are as follows N 1 to N 4 are shown similarly, N 5 to N 8 are given.

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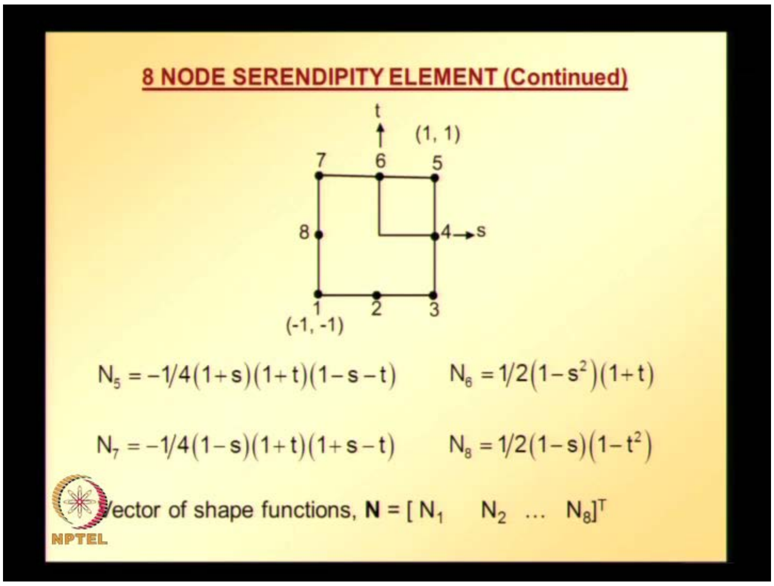

These shape function expressions are required for isoparametric mapping and all these shape functions, we can put them in a vector N.

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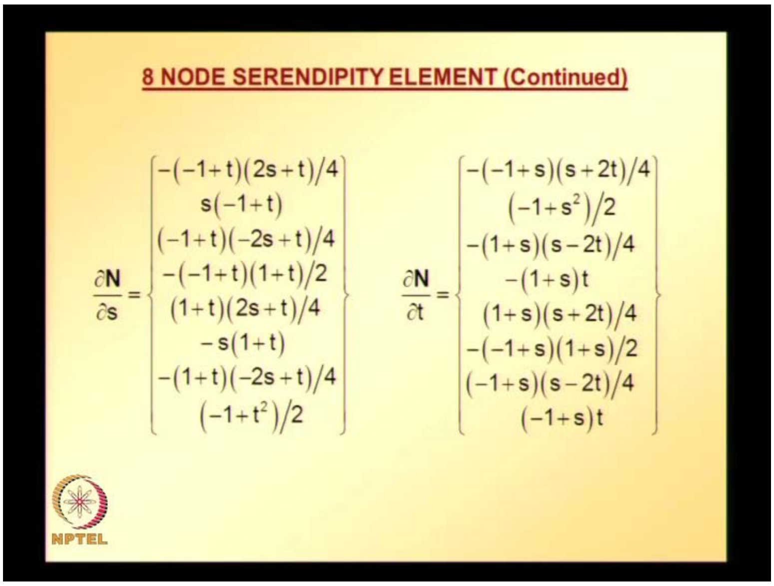

Since we have all the shape functions expressions we can easily calculate what is derivative of shape function vector with respect to s, derivative of shape function with respect to t, shape function vector with respect to t, because these two vectors are required for finding determinant of jacobian.

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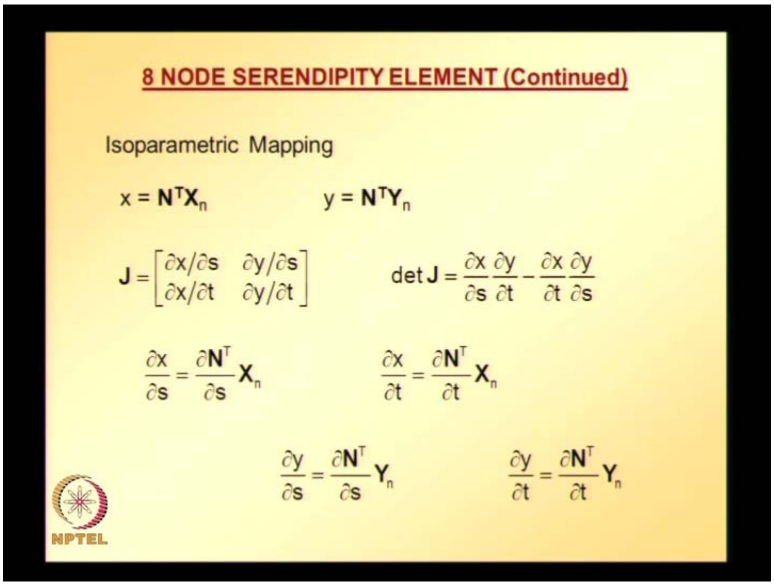

Now, let us see what is isoparametric mapping for 8 node serendipity element this is how isoparametric mapping can be performed and which is similar to 4 node quadrilateral element except that the shape function vector is now involves eight shape functions N 1 to N 8, X n involves X coordinates of all 8 nodes. Similarly y n involves y coordinates of all 8 nodes. So once we have this we can easily calculate what is J and determinant of J and from there we can calculate for getting J and determinant of J we require derivative of x with respect to s with respect to t, y with respect to s, y with respect to t that can be calculated using this expressions which are given.

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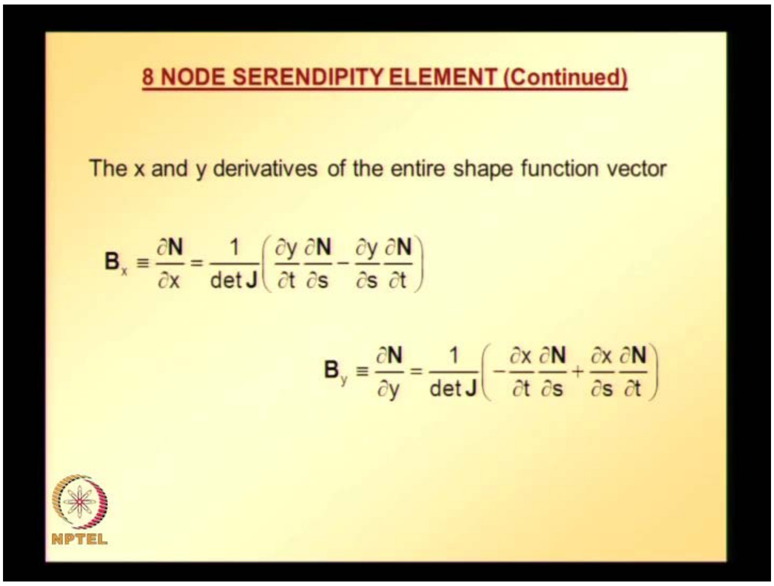

Also we require x y derivatives of entire shape function vector whatever we calculated those are shape function vector derivatives with respect to s and t what we also require with respect to x and y we can obtained that using chain rule or through this relation B x that is shape function vector derivative with respect to x similarly, shape function vector derivative with respect to y.

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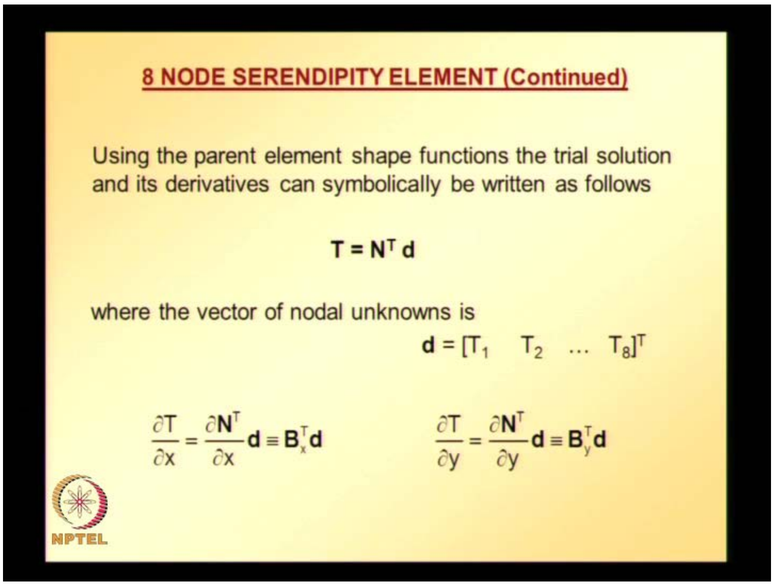

Using the parent element shape functions the trial solution and it is derivative can symbolically be written as follows. T is the field variable T value at any point is given by

N transpose d, d consists of all the nodal unknowns T 1 to T 8 and then we can also calculate derivative of T with respect to x with respect y through this formulas.

Once we have all this information, substituting the trial solution in the Galerkin criteria, which we already have when we are looking at 4 node quadrilateral element. We looked it in detail Galerkin criteria Greens theorem and all that stuff. Substituting trial solution and it's derivative into the Galerkin criteria and writing all eight equations together in a matrix form, we get earlier when we are dealing with4 node quadrilateral element we got four equations , now since we are dealing with 8 node serendipity we get eight equations.

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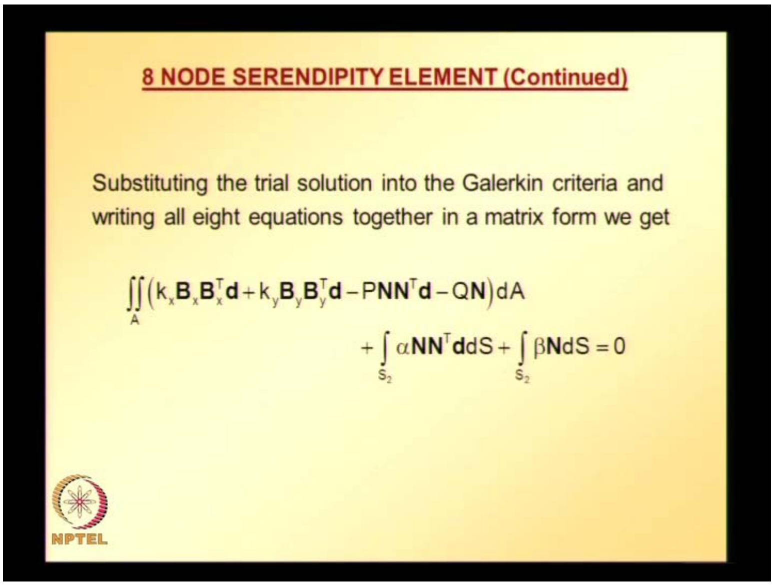

Writing all equation together in a matrix form we get this.

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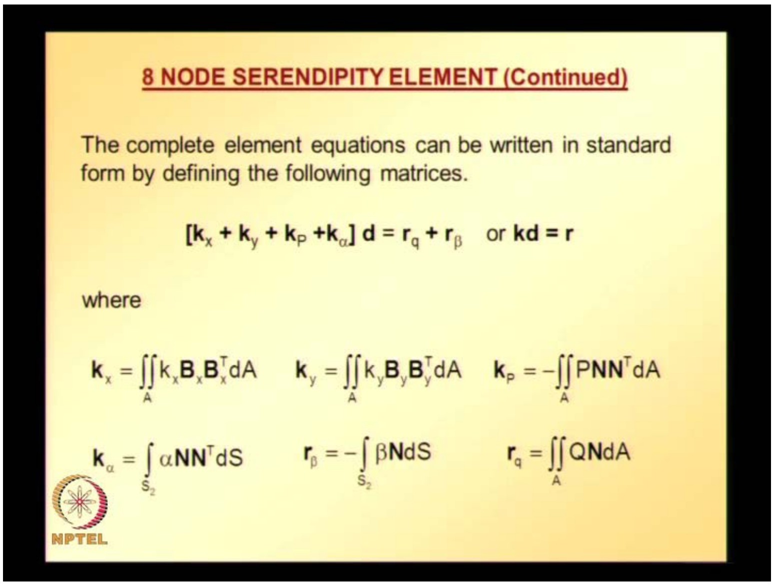

So complete element equations can be written in a standard form by defining the following matrices. This equation look similar to what we obtained for 4 node quadrilateral element, except that a dimension is more here all these matrices are going to be 8 by 8 and vectors are going to be 8 by 1 whereas, earlier we have 4 by 4 and vectors of dimension 4 by 1 and the details of each of this is similar to that of 4 node quadrilateral element, so this is how element equation looks for a general two dimensional boundary value problem with adopting 8 node serendipity elements.

Let us to discuss how to evaluate each of these area integrals and boundary integrals and also illustrate by looking at some examples (Refer Slide Time: 39:28).

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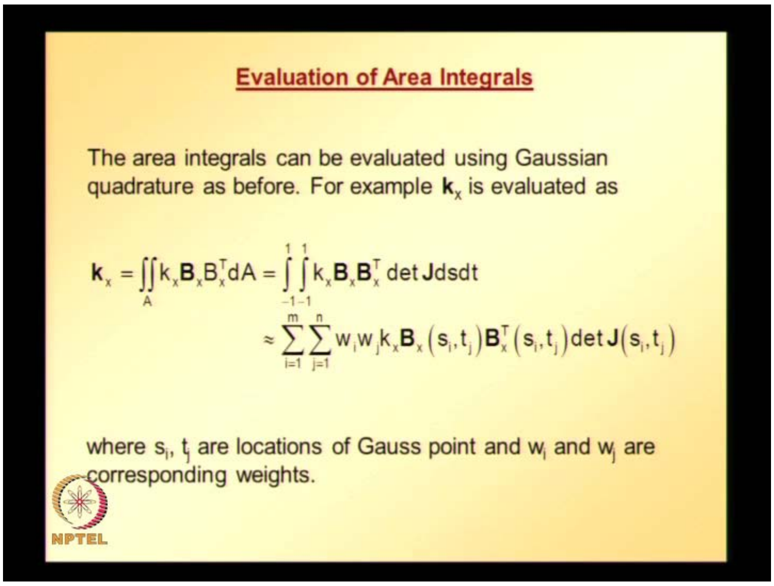

The evaluation of area integrals. The area integrals can be evaluated using Gaussian quadrature as we did for4 node quadrilateral element and k x matrix can be evaluated using this, (Refer Slide Time: 40:45) it is exactly same as what we did for 4 node quadrilateral element and s i, t j are the locations W i, W j are the corresponding weights and rest of the details are similar to that 4 node quadrilateral element.

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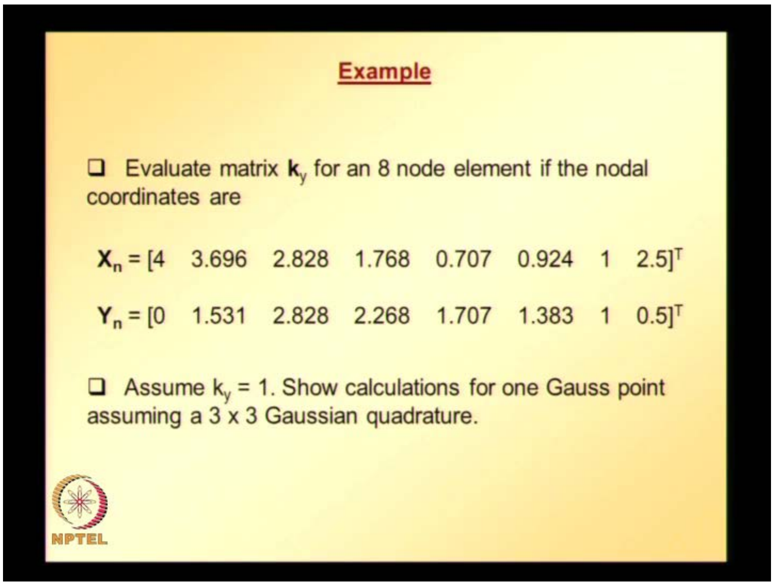

Now let us take an example. Evaluate matrix k y for an 8 nodded element if the nodal coordinates are as given here. An assume coefficient k y is equal to constant is a constant equal to 1 and show calculations for one gauss point and adopt 3 by 3 Gaussian quadrature. We need to know what are the weights and coordinates when we use 3 by 3 quadrature.

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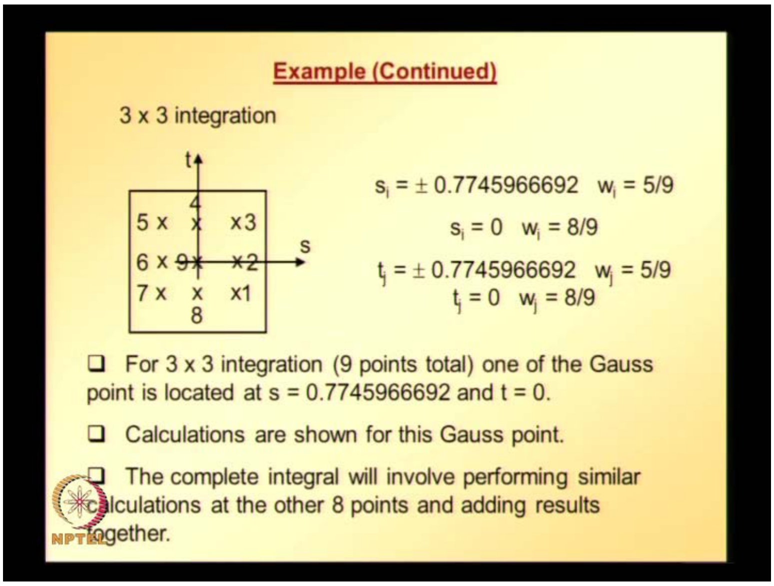

Already we discussed this earlier these are the coordinates and weights in the s direction and t direction. Using Cartesian product of these coordinates we get coordinates of all the nine integration points, please note that here we are going to get total nine integration points because we are using 3 by 3 integration and similar to 2 by 2 integration weight in the s direction times weight in the t direction if we find that gives us total weight at a particular point, but weight is not going to be constant is equal to 1 as in 2 by 2 integration it is going to be different for different points depending on the location of points it is going to be different.

For 3 by 3 integration nine points in total one of the gauss point is located at this and calculations details are going to be demonstrated or illustrated for this particular point. Calculations are shown for this point. Complete integral involve performing similar calculations at other eight points adding results together. Let us take the point and for this particular point weight in the s direction is going to be 5 by 9 and weight in the t direction is going to be 8 by 9 so, total weight is going to be 5 by 9 times 8 by 9.

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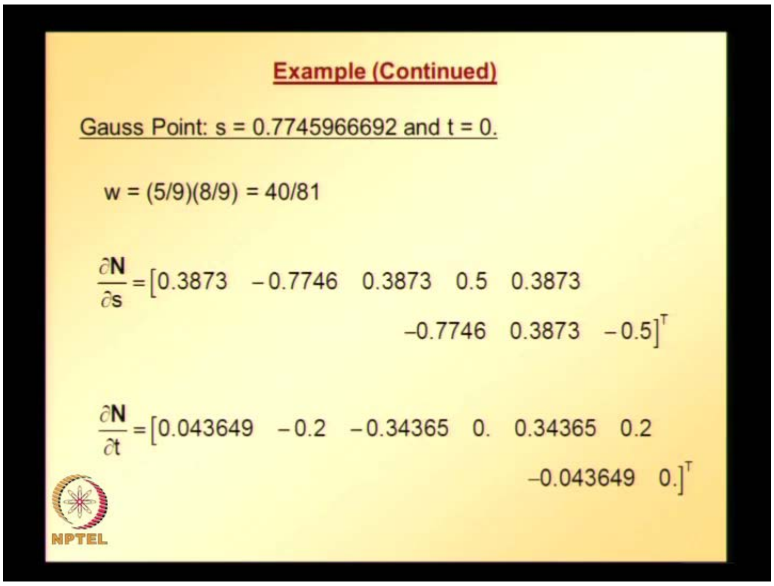

This is the point and this is what I mentioned about weight. Rest of the details are similar to that 4 node quadrilateral element, except that dimensions of vectors are going to be now 8 by 1 or 1 by 8 depending on how you put that vector. This is derivative of shape function with respect to s, shape function derivative with respect to t and once we have this information we can easily calculate what is derivative of x with respect to s, derivative of x with respect to t, y with respect to s, y with respect to t.

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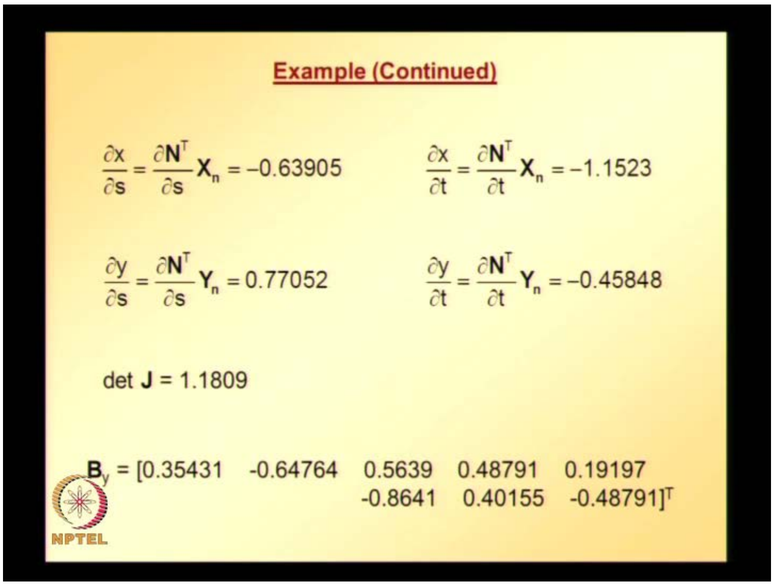

Then determinant of J, then b y vector all these follows mechanically once we get the information about point and weight. Once we get this vector multiply coefficient k y with B y, B y transpose determinant of jacobian times weight.

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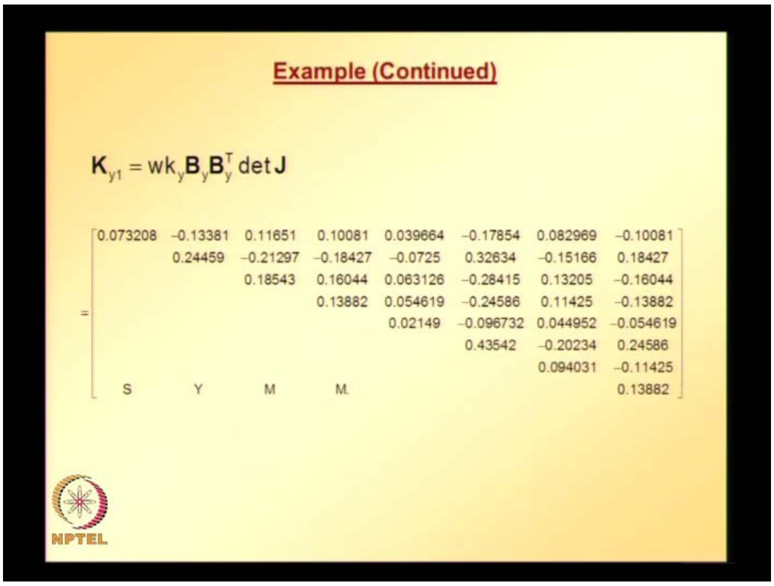

After doing vector multiplication we get this matrix this at one integration point similar operation we need to repeat at remaining eight integration points and sum them up to get the total value or the value of matrix k y from contribution from all the nine integration points, so this is how we can evaluate area integrals for 8 node serendipity elements.

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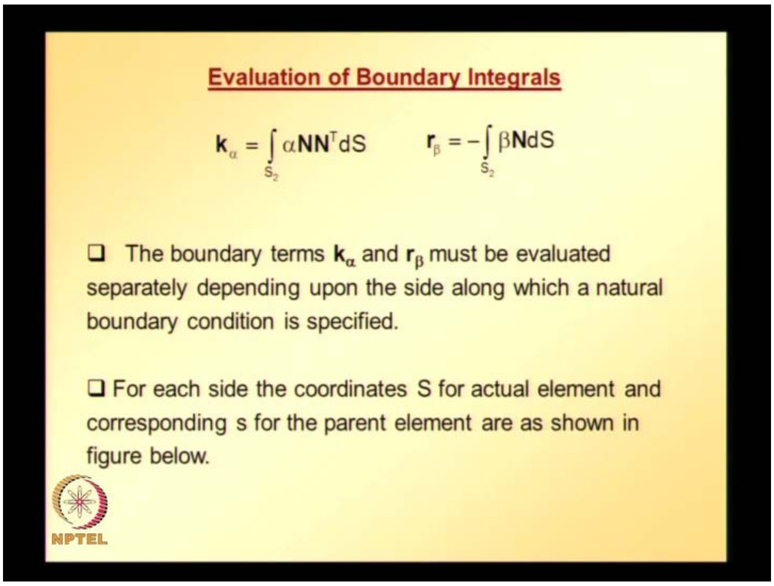

Now, let us discuss about boundary integrals. Once again these are the two boundary integrals if you look back the element equations, these are the two boundary integrals. Boundary terms k alpha r beta must be evaluated separately depending on the side along which natural boundary condition is specified this we already discussed when we are looking at 4 node quadrilateral element and also for each side, the coordinates S for actual element and the corresponding s for parent element.

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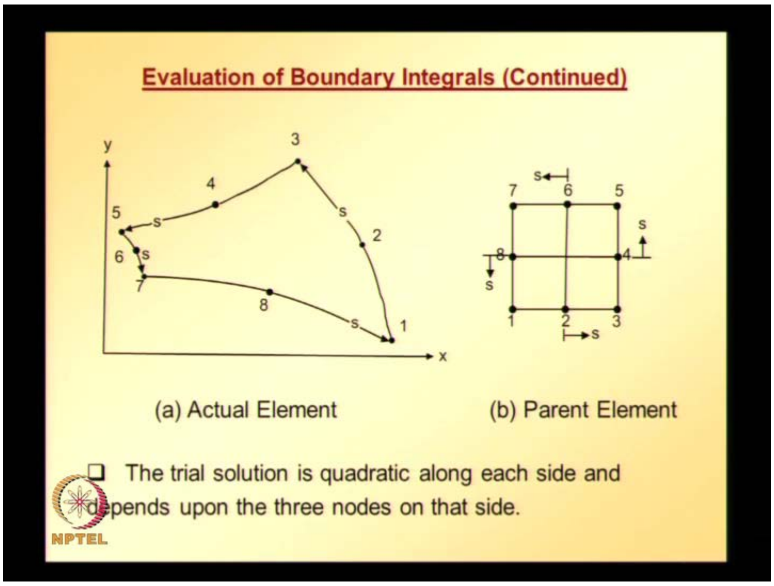

We need to find also the relation and they are shown actual element and parent element. If you see, there are three nodes along each side trial solution now it is going to be quadratic along each side and depends up on three nodes on that side. Let us take this parent element and for any of the edges let us say one side containing 1-2-3 nodes. The shape functions of rest of the nodes other than 1-2-3 shape functions of those nodes are going to be zero along the side 1-2-3. Similarly, alongside 3-4-5 except 3-4-5 other nodal shape functions are going to be zero along that edge 3-4-5 similarly for other edges.

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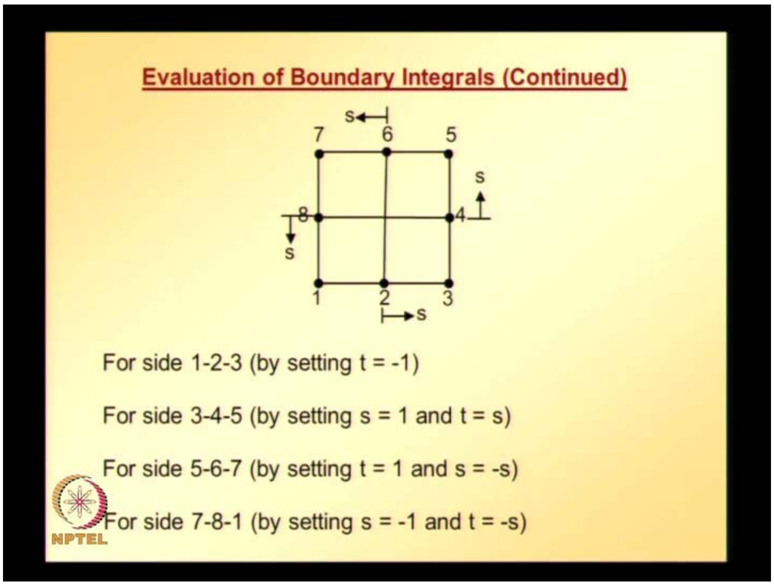

For side 1-2-3 we already have shape function expressions for 8 node serendipity element in the parent element domain in those expressions substituting t is equal to minus 1 for side 1-2-3. Similarly substituting s is equal to 1 t is equal to s, for side 3-4-5 and for sides 5-6-7 setting t is equal to 1, s is equal to minus 1 and for side 7-8-1 setting s is equal to minus 1, t is equal to minus s we get expressions or shape function vector along each of the four sides.

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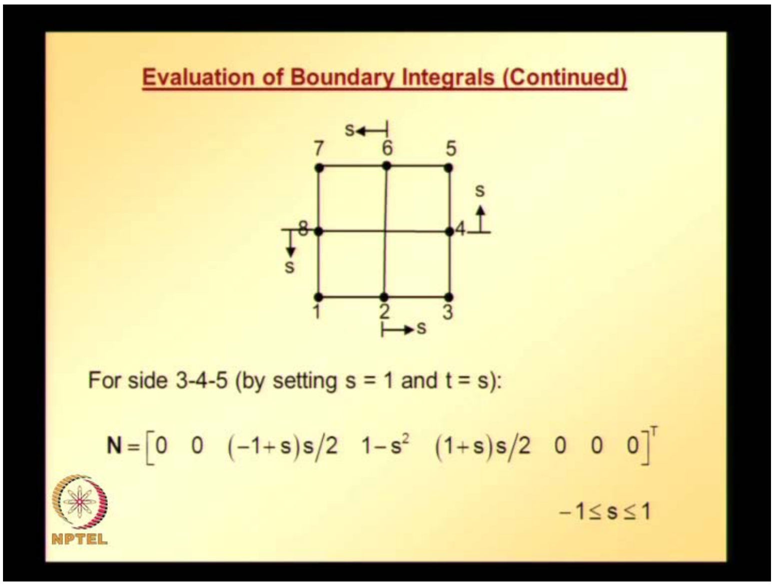

Those are given here where s goes from minus 1 to 1, This is the shape function vector alongside 1-2-3, similarly for side 3-4-5 side.

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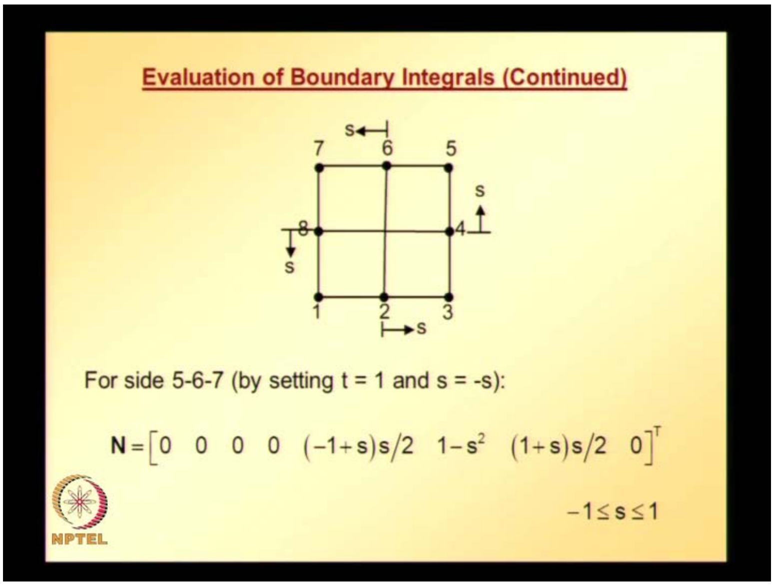

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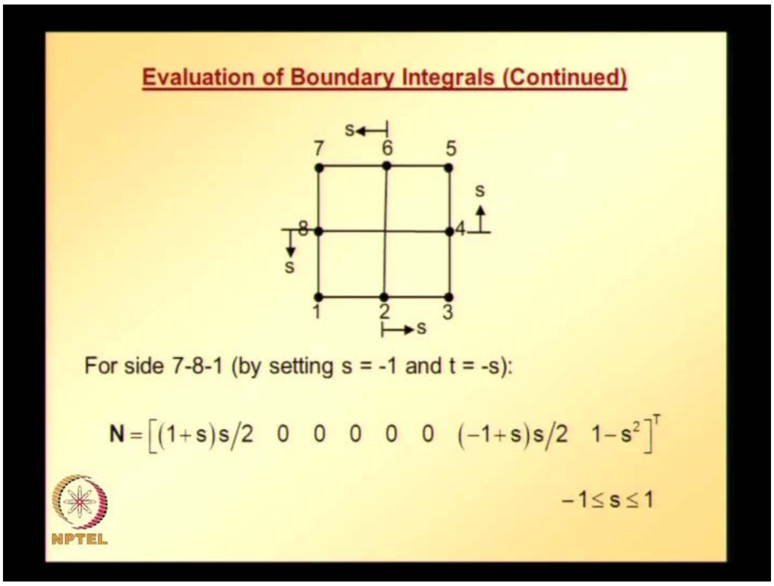

For side 5-6-7 and for side 7-8-1 we got shape function vector along each of these sides.

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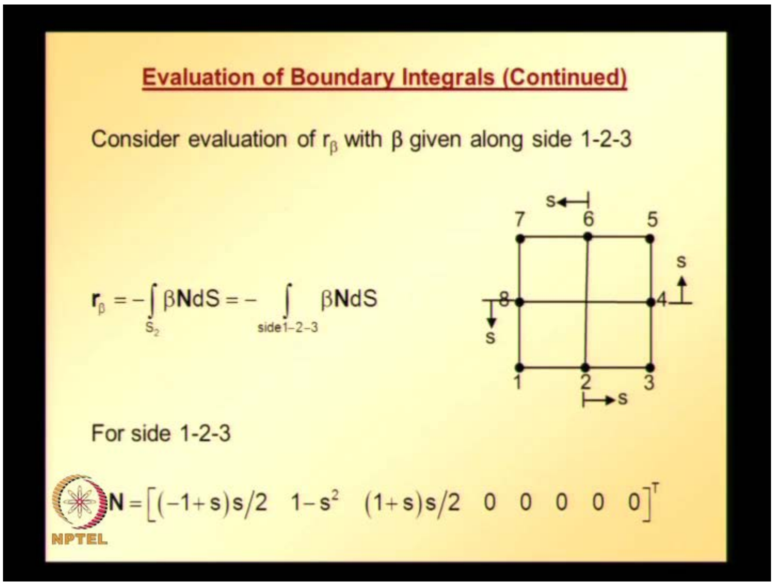

Now we can evaluate boundary integral r beta or k alpha. R beta with beta given alongside 1-2-3 is illustrated for that evaluation, we require shape function vector alongside 1-2-3 and also we require the information of how the differential element taken alongside 1-2-3 in the parent element is related to differential element along the same side in the actual element, so we require to find what is the relationship between d S and d s, for that we require to do isoparametric mapping along this edge.

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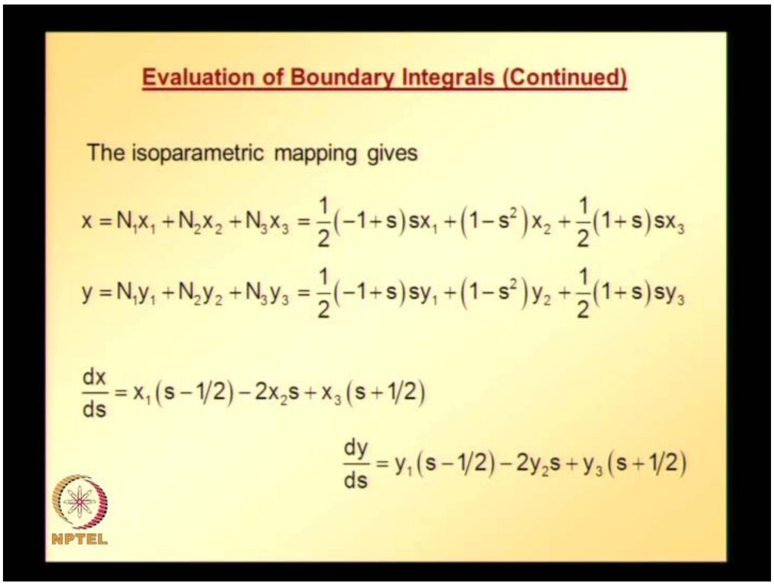

Isoparametric mapping gives us this x is equal to N 1, x 1 plus N 2, x 2 plus N 3, x 3 because shape functions of other nodes along the side 1-2-3 are going to be zero similar is the case for y. Once we have this relations x in terms of s, y in terms of s, we can take derivative of x with respect s and y with respect s similar to the way we did for 4 node quadrilateral element.

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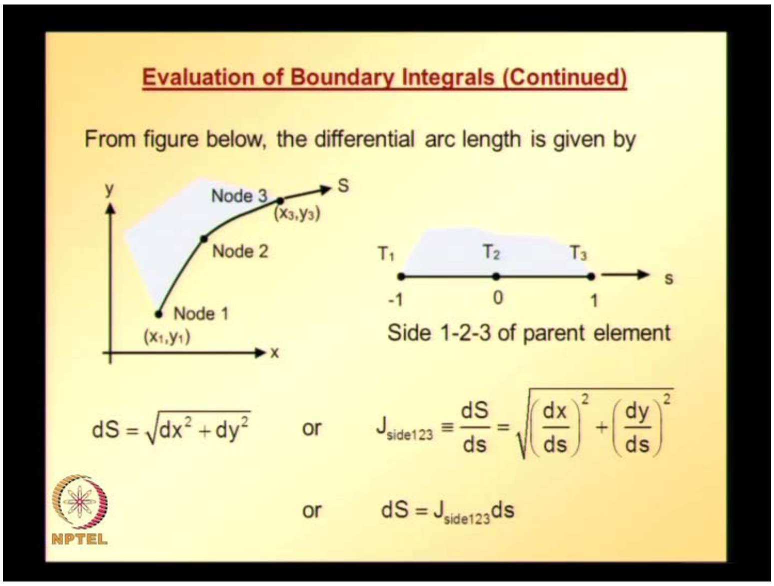

Once we get this two quantities and here side 1-2-3 for the actual element and parent element are shown. And similar to 4 node quadrilateral element d S is given by square root of d x square plus d y square when we divide that equation on both sides with d s , we get the equation or we get the relation for jacobian alongside 1-2-3.

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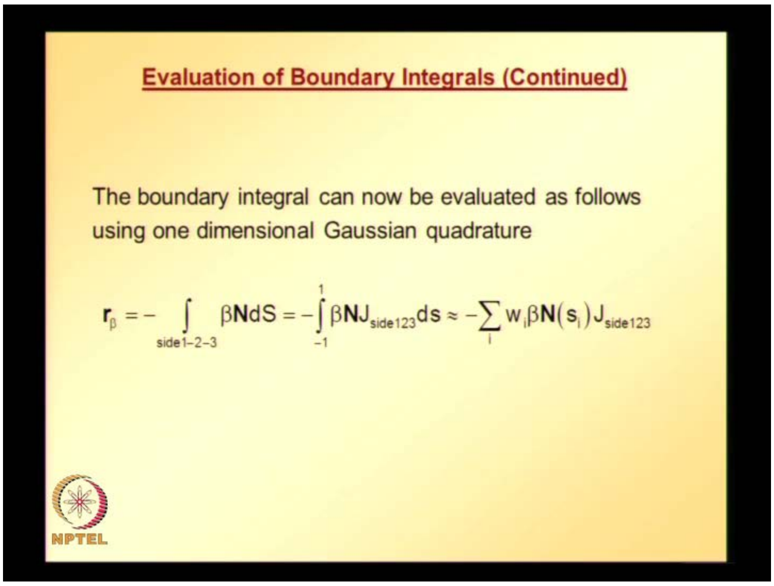

Once we have this, we can change the limits of integration and use one dimensional Gaussian quadrature this is for integral r beta.

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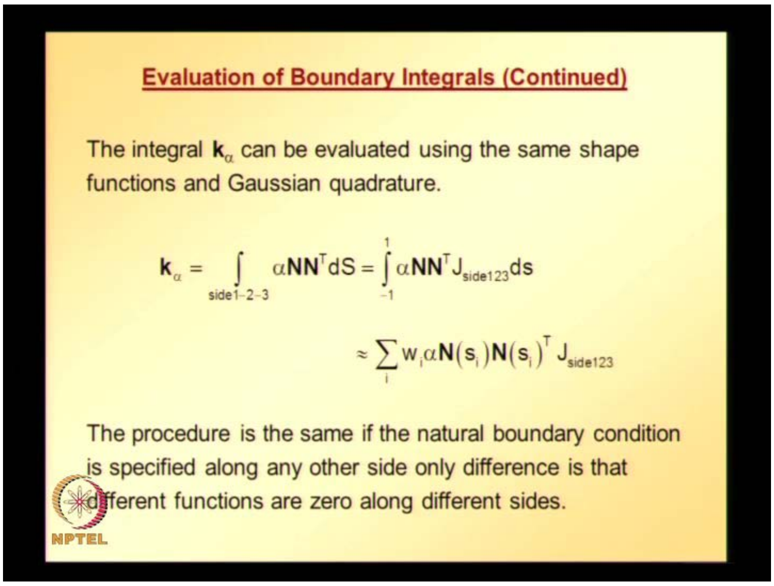

Similarly, we can evaluate k alpha using the same shape functions and using this formula. The procedure is same if natural boundary condition is specified along any other edge or side the only difference is that different shape functions are zero along different sides.

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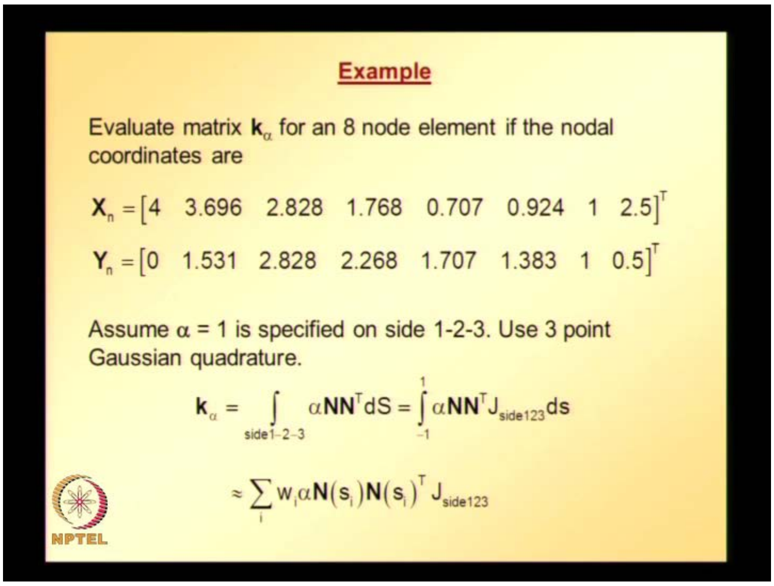

Now let us quickly take an example. Evaluate k alpha for 8 node element if nodal coordinates are x coordinates and y coordinates of all 8 nodes are given and assume alpha is equal to 1 alongside 1-2-3 and use three point gauss quadrature, so we require to evaluate this integral for that we require shape function vector alongside 1-2-3.

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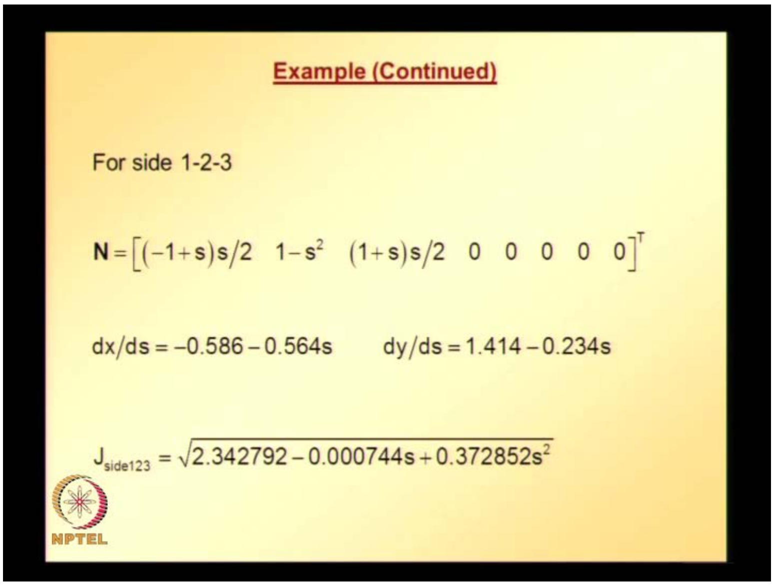

The shape function vector alongside 1-2-3 is given, and also derivative of x with respect to s with derivative of y with respect to s is given by this from, there we can find jacobian.

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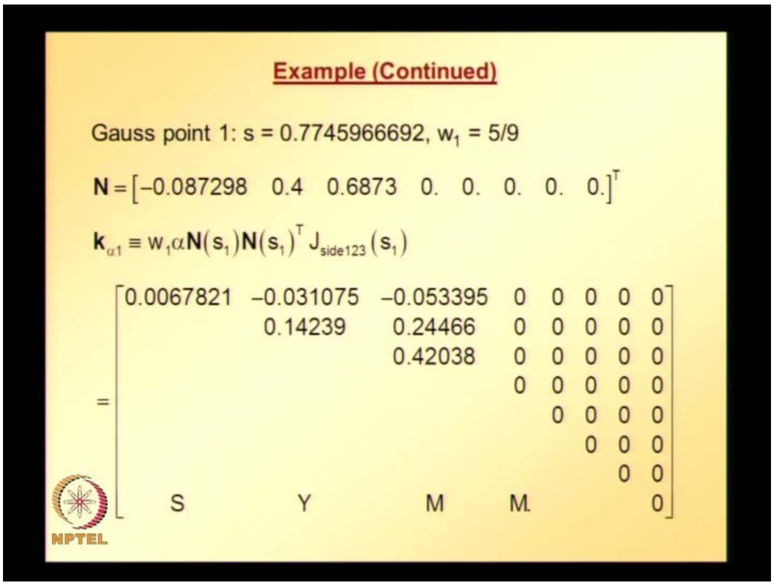

Now taking each of the integration point; this is first integration point details are given. The shape function vector and substituting shape function vector and jacobian alongside 1-2-3 at this integration point we can get the contribution of the first integration point to k alpha matrix.

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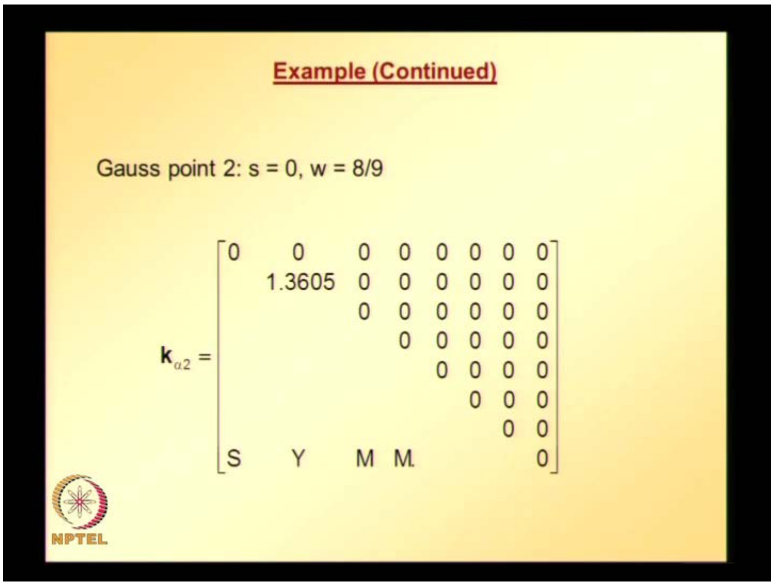

Similarly, second integration point contribution, by taking shape function vector at this point.

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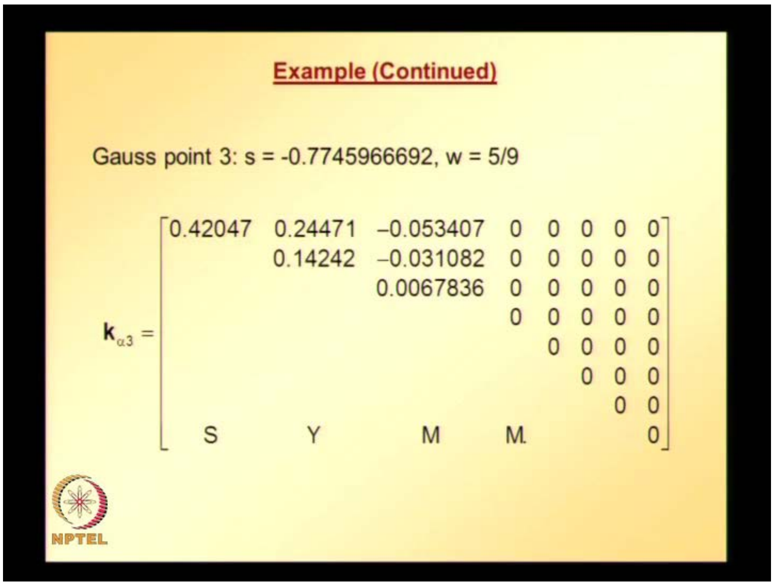

The third integration point contribution.

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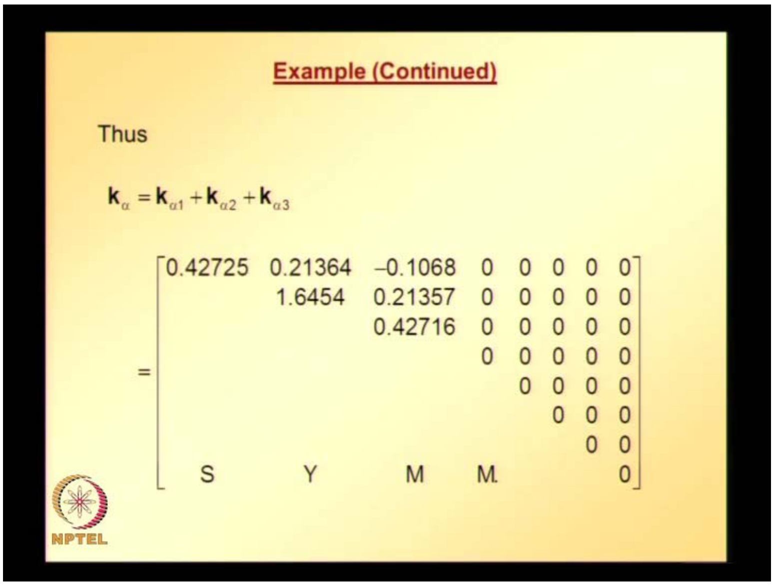

And summing all the contributions, we get k alpha matrix, so this is how we can evaluate area integrals and boundary integrals for both 4 node quadrilateral element and 8 node serendipity element.