Finite Element Analysis

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Lecture No. # 26

In the last class, we have seen the concept of isoparametric mapping, using which we can express the physical coordinates X Y in terms of parent element coordinates as s and t using shape functions of parent element and the spatial coordinates of the physical element. Basically isoparametric mapping is a relation, which relates physical coordinates of elements with the coordinates of parent element. After that we looked at how to calculate derivatives of shape functions with respect to the physical coordinates X Y, if the shape function of derivatives with respect to the parent element coordinates s and t are known; for that we require what is called Jacobian. So Jacobian matrix is the one which relates the derivative of shape functions in the physical element coordinates X Y with parent element coordinates s and t. To get the inverse relation that is if we know the derivatives of shape functions with respect to the parent element, how to get the shape function derivatives with respect to the physical coordinates X Y, we require determinant of Jacobian.

And in the last class, we have seen for a valid mapping, determinant of Jacobian should be greater than 0, because if determinant of Jacobian is equal to 0, then there is that relation that is we cannot calculate derivatives of shape function with respect to the physical coordinates, if you know shape function derivatives with respect to the parent element coordinates s and t. We took two types of elements four-noded quadrilateral element and eight-noded quadrilateral element, and we checked for isoparametric mapping validity by plotting determinant of Jacobian as a function of s and t, and also we plotted the other way of checking is we can also plot the physical coordinates X Y in terms of s and t, and see if it produces the shape of actual element or not. And if one plots determinant of J, determinant of J needs to be check whether it is a greater than 0 over the entire parent element domain. This is how we can check the validity of isoparametric mapping.

After going through four examples, we can make conclusions out of what we studied in the last class that is why we will be looking at proper modeling with isoparametric element in this lecture.

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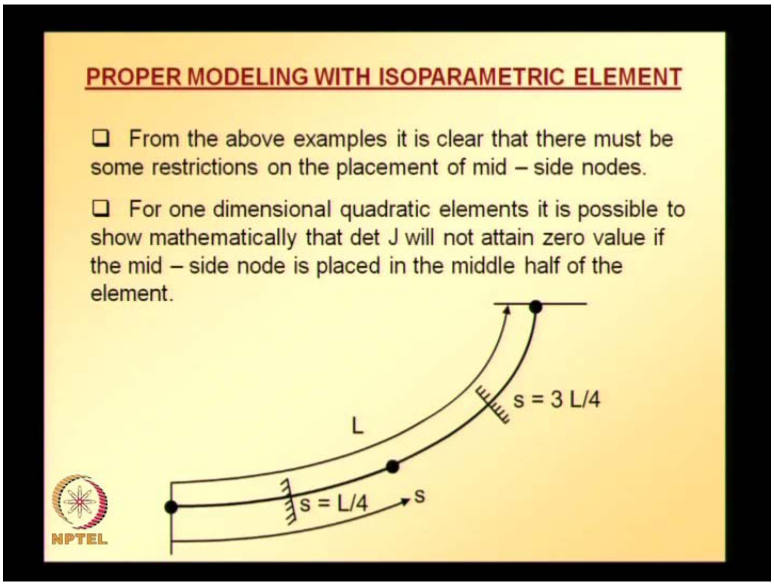

From the above four examples that we looked at in the last class, it is clear that there must be some restrictions on placement of mid-side nodes. For one-dimensional quadratic elements it is possible to show mathematically that determinant of J will not attain 0 if mid-side node is placed in the middle half of the element that is what basically we did when we looked at one dimensional quadratic element and what we derived or what the result that we got for one dimensional case is reproduced if we have an edge which is of length L then the excluding the two extreme nodes extreme end nodes the mid-side node the location of it should be between L over 4 and 3 L over 4 where s is measured from one of the extreme nodes and L is the total length of that side. This is the requirement if determinant of J is should not be equal to 0.

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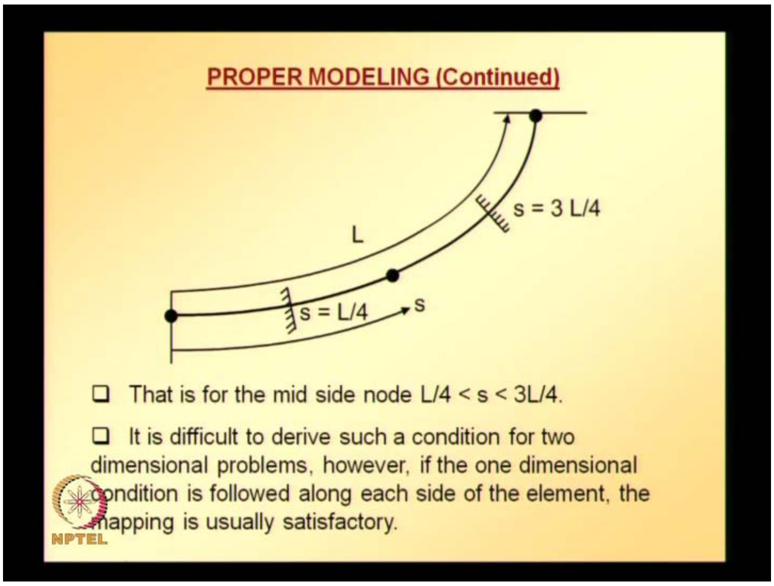

Same thing we can extend for two-dimensional case. The placement of mid side node should satisfy this condition s should be between L over 4 and 3 L over 4. It is difficult to derive such a condition for two-dimensional problems, however if one-dimensional condition is followed along each side of element mapping is usually satisfactory.

There is nothing special with respect to two-dimensional element whatever condition we derived for one dimensional element that we need to check for each of the sides of a two-dimensional element for mapping to be satisfactory.

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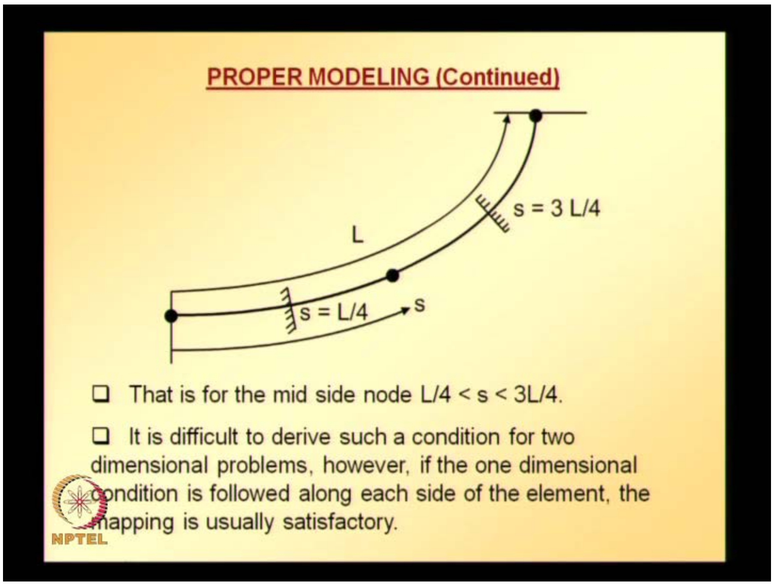

If you recall in yesterday's lecture we have taken the last example that we have taken is an eight-noded element where node two is placed very close to one of the nodes node three and there we have seen some problem with determinant of J being not greater than zero. So that happened because it violated this condition that the placement of mid-side node is not satisfying the condition that it should be between L over 4 and 3 L over 4 for that particular example.

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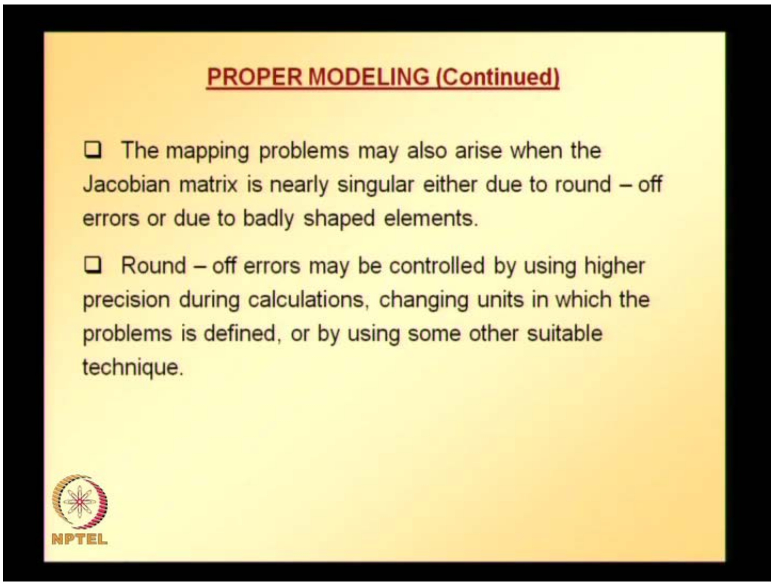

The mapping problems may also arise when Jacobian matrix is nearly singular due to round-off errors or due to badly shaped elements. These are the two reasons why Jacobian matrix maybe nearly singular that is almost singular. Round-off errors may be controlled by using higher precision during calculations and also changing units in which the problem is defined or using some other suitable technique, so this is how round-off errors can be controlled.

What about badly shaped elements?, we have looked at if you recall in the last class we have seen a four-noded element which is second example under validity checking of isoparametric mapping there we have taken an element intentionally which is very bad shaped element and there the mapping turned out to be bad. To avoid problems badly shaped elements, it is recommended that the inside angles in quadrilateral element be greater than 15 degrees and less than 165 degrees.

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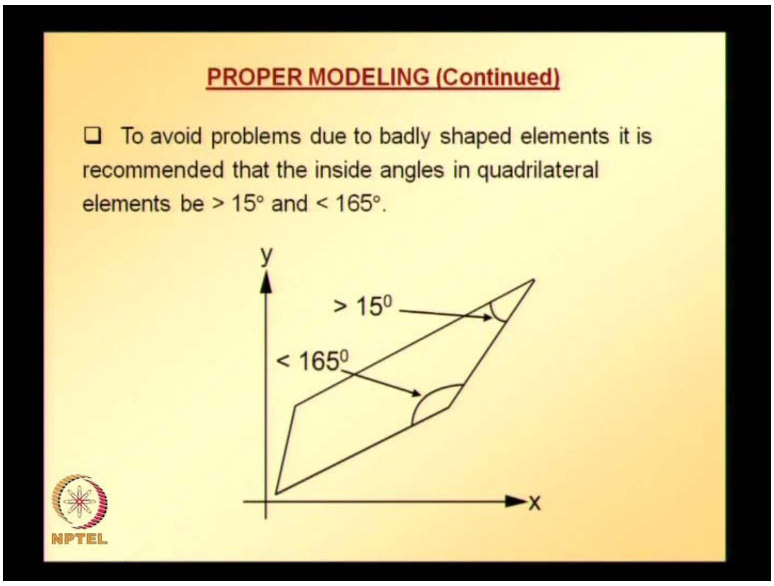

If we have violations of this inside angles, then you may have mapping problems similar to what we observed in the last class. So the two requirements are the mid side node should be positioned such a way that it is between L over 4 and 3 L over 4 of a particular side and the angle between or the inside angles in a quadrilateral element should be greater than 15 and less than 165 degrees.

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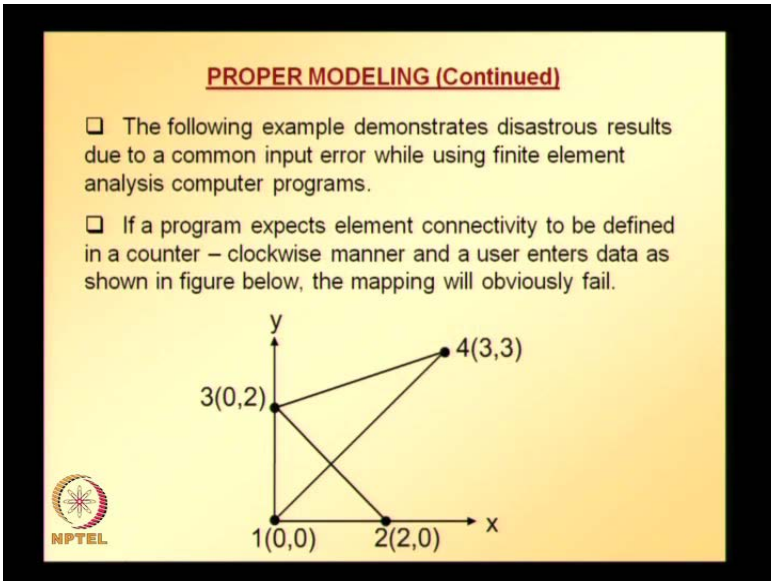

The following example demonstrates disastrous results due to a common input error while using finite element analysis computer programs. Sometimes the node numbering if we do not follow or if we violate for two dimensional cases either all the nodes should be numbered in clockwise direction or anticlockwise direction. Sometimes problem with results may arise and if a program expects element connectivity to be defined in a counter-clockwise manner and user enters a data as shown in figure below mapping will obviously fail.

The computer program is expecting the node numbering should be given in a particular direction that is counter-clockwise manner, but user in input data is not according to that requirement then mapping is going to fail to demonstrate that an example is given (Refer Slide Time: 09:17) actually it is a four-noded quadrilateral elements, but taking in counter-clockwise direction the node numbering should be 1 2 and after that 3 4, but if you see the node numbering is 1 2 and 4 and 3, if we take in the counter-clockwise direction. The computer program when it reads it takes the nodal connectivity as 1 2, 4 3 instead of 1 2, 3 4.

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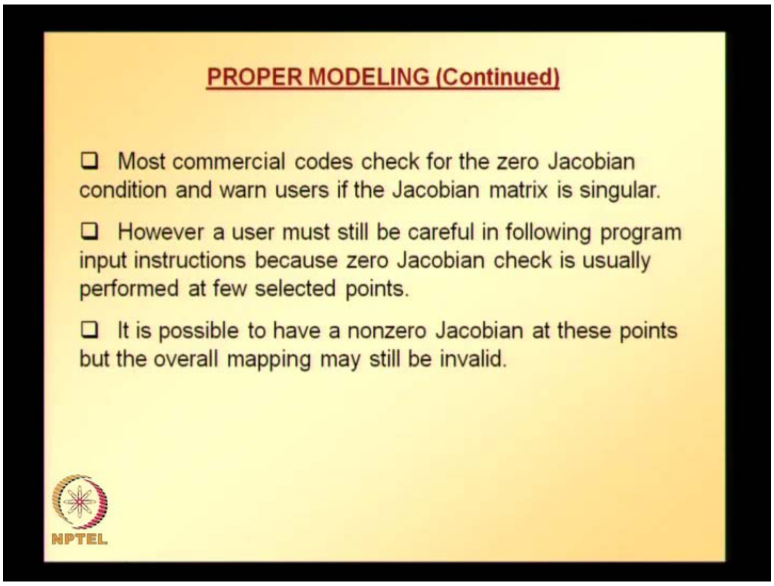

So, what happens most commercial codes checks for zero Jacobian condition and warn users if Jacobian matrix is singular, but this checking business that this commercial software is going to do is only at few points few selected points. However, a user must still be careful in following program input instructions because zero Jacobian is usually performed at few selected points, so what is going to happen, it is possible to have nonzero jacobian at those points where commercial software is checking, but overall mapping is still be invalid.

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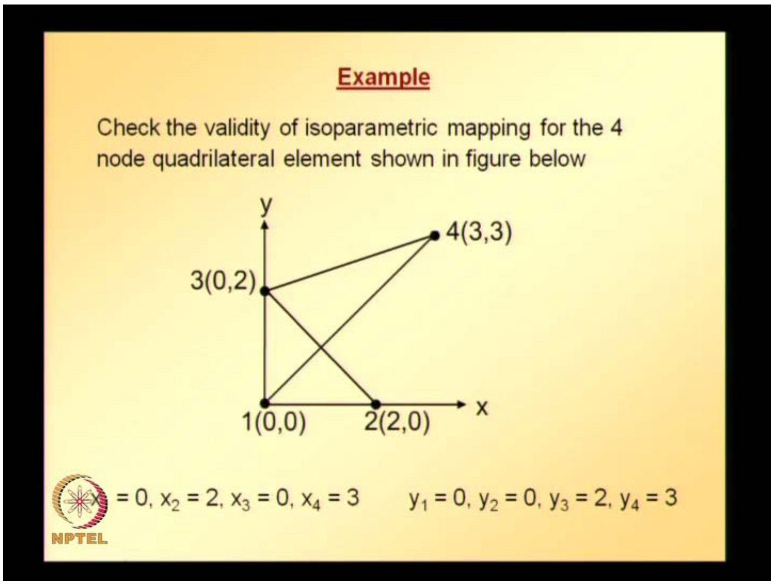

Now let us look at the example. What happens, if the commercial software or if a particular software expects the nodal information be given in a particular direction but if user inputs in a different way what is going to happen. Let us take that example nodes are numbered 1, 2, 3, 4 instead they should have been numbered as 1, 2, 4, 3 or the nodal connectivity should have been 1, 2, 4, 3 instead of 1, 2, 3, 4. So let us see what is going to happen and let us note down what are the physical coordinates X 1 to X 4 from the information that is given from the problem and y 1 to y 4. After this, this element is we are going to map it on to a 4 node parent element taking parent element shape functions we can write isoparametric mapping expressions that is X in terms of s and t, y in terms of s and t using parent element shape functions and physical element coordinates.

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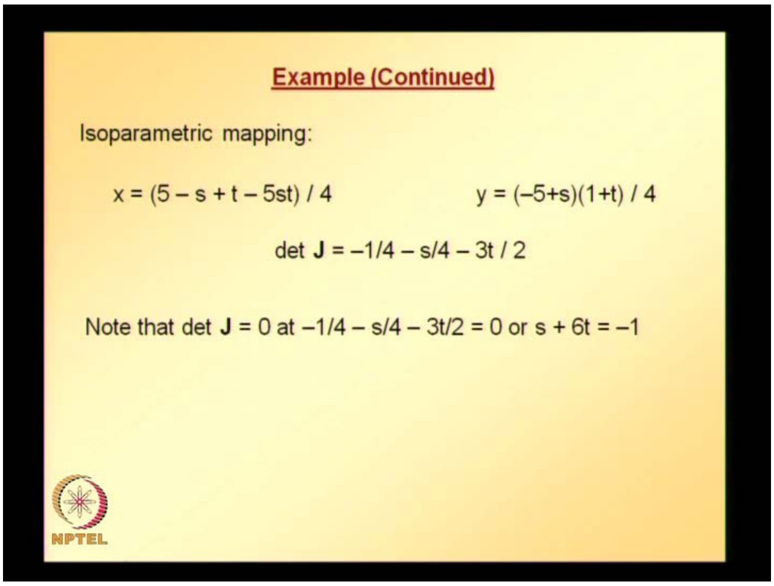

Isoparametric mapping, once we do that we get these relations between X and s and t, y and s and t using that waiting partial derivatives of X with respect s and t, y with respect s and t we can easily calculate determinant of J which is given. Let us check what line this determinant of J is going to be 0, equating determinant of J to 0, we get determinant of J to be 0 along the line which satisfies the condition s plus 60 is equal to minus 1. As I mentioned in the last class, if you plot this line or if you overlie this line that is s plus 60 is equal to minus 1 on the parent element domain s going from minus 1 to 1, T going from minus 1 to 1 if you overlie this line of that it can be clearly seen that this line is going to cut across the domain mapping is not good because this line is going to cut across the parent element domain.

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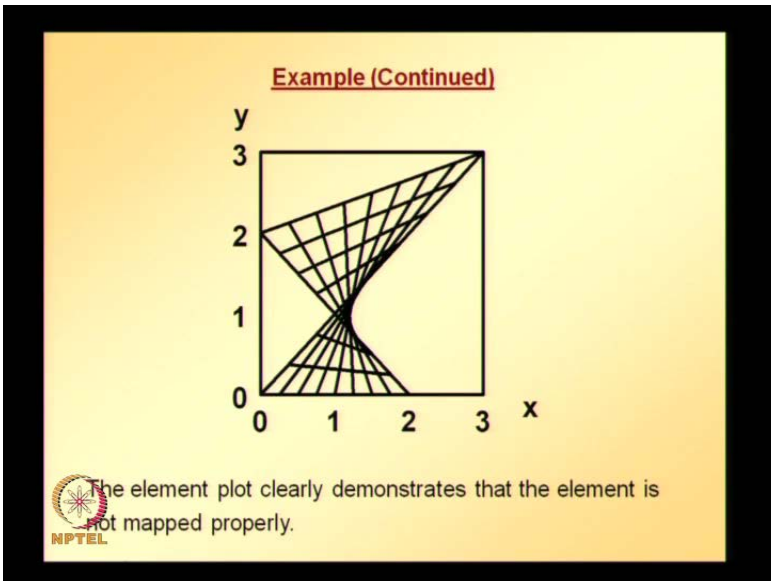

The other way of checking the same thing whether the mapping is valid or not is plotting x y as a function of s and t and it can be clearly seen there is a fold which indicates mapping is not good. The element plot clearly demonstrates that element is not mapped properly. If mapping was good, then we are not going to have this kind of fold and whatever the shape that we are going to get exactly looks like the physical or the actual element that is started out with. So placement of nodes and the internal angles of a quadrilateral these two requirements are very important for isoparametric mapping to be good. And whatever we have studied for two dimensional elements can easily be extended for three dimensional elements.

Now let us look at numerical integration for quadrilateral elements because the matrices and vectors that we are going to get for quadrilateral elements. The integrant for the integrals is not going to be a constant like in three node triangular element, which is linear, but they are going to be function of X and y or in turn function of s and t, so when we are conforming numerical integration or when we are trying to evaluate this integrals, we need to adopt numerical integration because we may not be able to find the closed form expressions of these so easily because as integrant becomes complicated.

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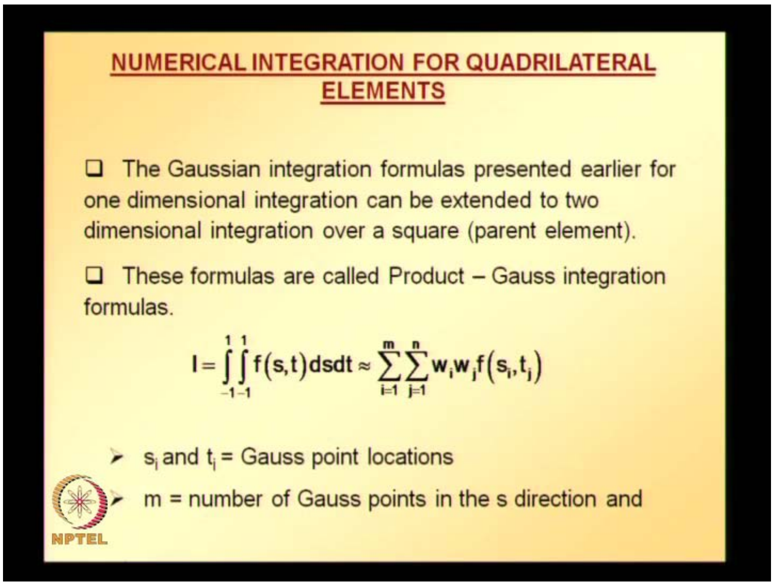

So now let us look at numerical integration for quadrilateral elements. The Gaussian integration formulas presented earlier for one dimensional integration can be extended to two dimensional integrations over a square. Basically, whatever quadrilateral element or rectangular element we have, we are going to map it on to a parent element where s goes from minus 1 to 1 and t goes from minus 1 to 1.

We need to understand how to do Gaussian integration over a square or a parent element with s going from minus 1 to 1 t going from minus 1 to 1. So the integration formulas presented for one dimensional integration now they will be extended for two dimensional integrations over this parent element. These formulas are also called Product-Gauss integration formulas. And basically this is how we will be evaluating in two dimensional case i is equal to minus 1 integral minus 1 to 1 double integral minus 1 to 1, minus 1 to 1 f as a function of s and t, dsdt.

So let us say suppose that is the function we need to evaluate that is numerically approximated as double summation i going from 1 to m, j going from 1 to and, weight in the s direction times weight in the t direction that is W i times W j types function evaluated at each of these integration points that is i going from 1 to m, j going from 1 to n that results in a grid. So at each point of this grid, we need to evaluate this function and multiply with corresponding weight which is going to be W i times W J and sum it up we are going to get approximate or approximation of the integral i.

(Refer Slide Time: 17:52) s i and t j are the gauss point locations m number of gauss points in the s direction.

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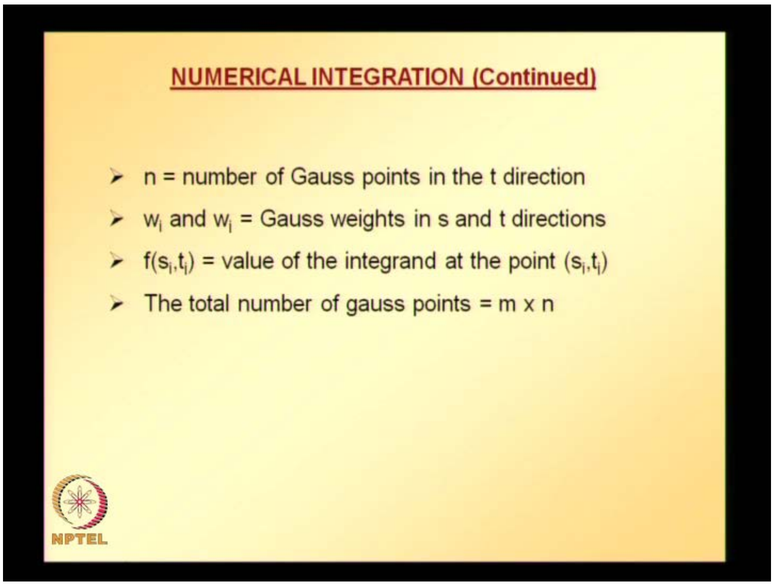

And n number of Gauss points in the t direction and W i, W j are the gauss weights in s and t directions, function s i, t j as the value of integrant at the point s i, t j and total number of gauss point there are m number of gauss points in the s direction and n number of gauss points in the t directions total number of gauss points is going to be m times n. The location of gauss points in each direction and corresponding weights are same as those which we already looked, when we are looking at one dimensional Gaussian integration.

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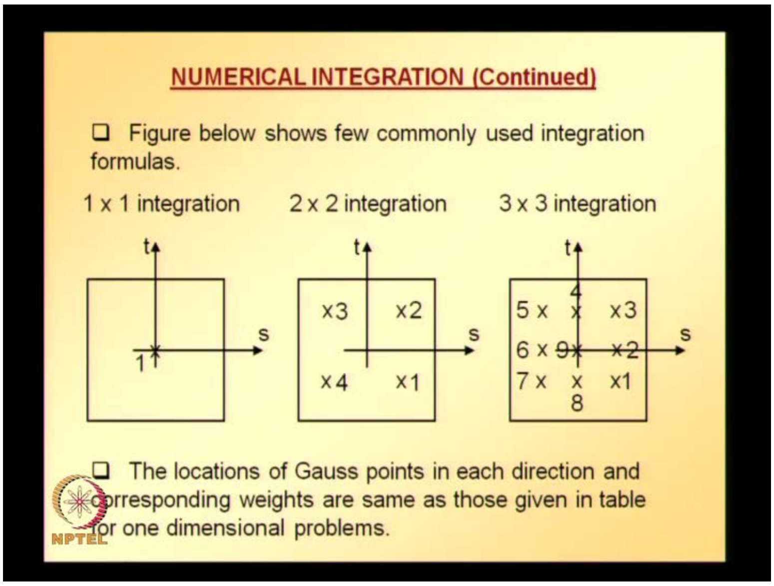

For completeness they are repeated here and before that we will see few commonly used integrations for two dimensional case, here 1 by 1 integration is shown that is along s direction one point is taken and again along t direction also one point is taken. That results in 1 by 1 integration and 2 by 2 integration two gauss points are taken in S the direction of s and two gauss points are taken in the direction of t, so it results in total 2 by 2, which is going to be four integration point similarly 3 by 3 integration where three points are taken along s direction and three points in the t direction resulting in total three times 39 integration points. The location of these points in each direction and the corresponding weights are same as those we have studied for one dimensional case and they are reproduced.

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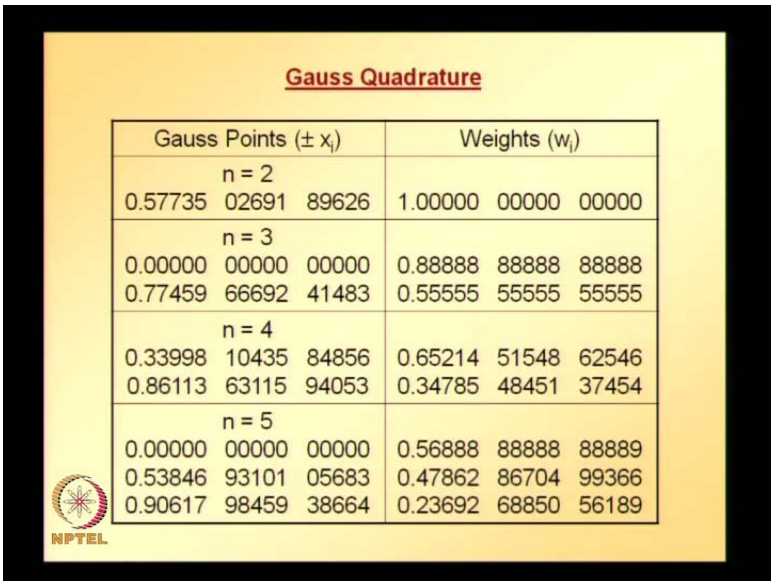

Gauss Quadrature

Gauss Points ($\pm x_i$)	Weights (w_i)
n = 2	
0.57735 02691 89626	1.00000 00000 00000
n = 3	
0.00000 00000 00000	0.88888 88888 88888
0.77459 66692 41483	0.55555 55555 55555
n = 4	
0.33998 10435 84856	0.65214 51548 62546
0.86113 63115 94053	0.34785 48451 37454
n = 5	
0.00000 00000 00000	0.56888 88888 88889
0.53846 93101 05683	0.47862 86704 99366
0.90617 98459 38664	0.23692 68850 56189

We have already looked at this earlier expect that the table starts from n is equal to 2 and n is equal to 1 integration point is 0 the location of integration point is 0 and weight is 2. So here this table give us from n is equal to 2 to n is equal to 10, in this portion of table the points and weights for n is equal to 2, 3, 4, 5 are shown.

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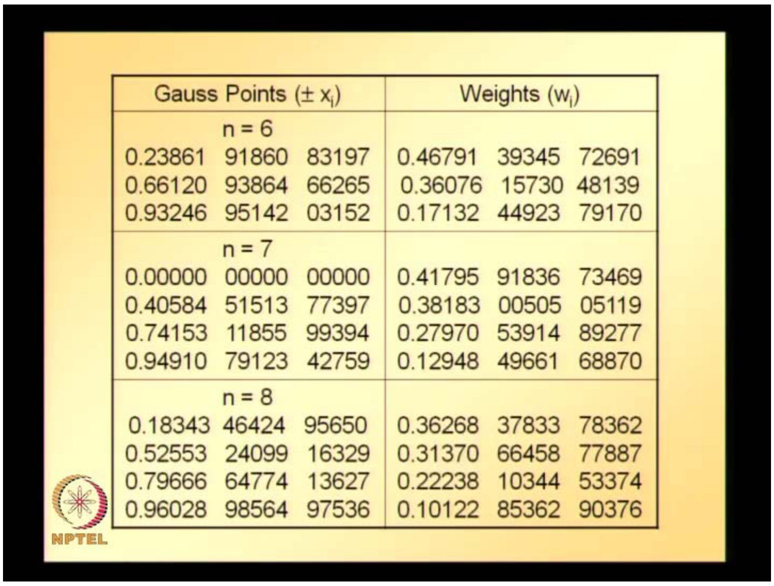

Gauss Points ($\pm x_i$)	Weights (w_i)
n = 6	
0.23861 91860 83197	0.46791 39345 72691
0.66120 93864 66265	0.36076 15730 48139
0.93246 95142 03152	0.17132 44923 79170
n = 7	
0.00000 00000 00000	0.41795 91836 73469
0.40584 51513 77397	0.38183 00505 05119
0.74153 11855 99394	0.27970 53914 89277
0.94910 79123 42759	0.12948 49661 68870
n = 8	
0.18343 46424 95650	0.36268 37833 78362
0.52553 24099 16329	0.31370 66458 77887
0.79666 64774 13627	0.22238 10344 53374
0.96028 98564 97536	0.10122 85362 90376

How to read this table, I already explained when we are looking at one dimensional case and this portion of the table shows from n is equal to 6 to 8.

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Gauss Points (± x_i)	Weights (w_i)
n = 9	
0.00000 00000 00000	0.33023 93550 01260
0.32425 34234 03809	0.31234 70770 40003
0.61337 14327 00590	0.26061 06964 02935
0.83603 11073 26636	0.18064 81606 94857
0.96816 02395 07626	0.08127 43883 61574
n = 10	
0.14887 43389 81631	0.29552 42247 14753
0.43339 53941 29247	0.26926 67193 09996
0.67940 95682 99024	0.21908 63625 15982
0.86506 33666 88985	0.14945 13491 50581
0.97390 65285 17172	0.06667 13443 08688

In this portion of the table shows for 9 number of points and 10 number of gauss points as I mentioned for n is equal to 1, x is equal to 0 and weight is equal to 2.

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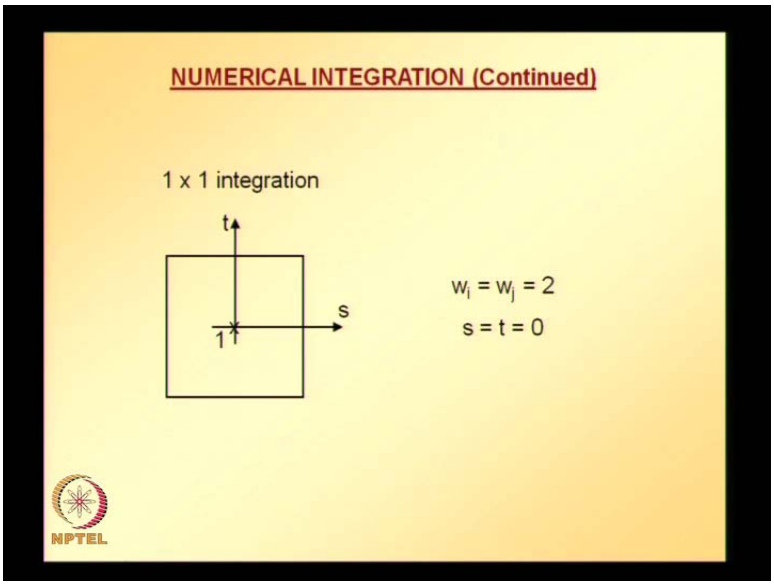

Now let us look at, how to evaluate or how to get the coordinates for 1 by 1 integration. In s direction, the location is at s is equal to 0 in the t direction location is at t is equal to 0. The location of gauss point is going to be 0, 0 because in the s direction it is 0 and in the t direction it is 0 and the weight in the S direction is 2 and weight in the t direction is 2, so total weight is going to be two times to four, so this is for 1 by 1 integration.

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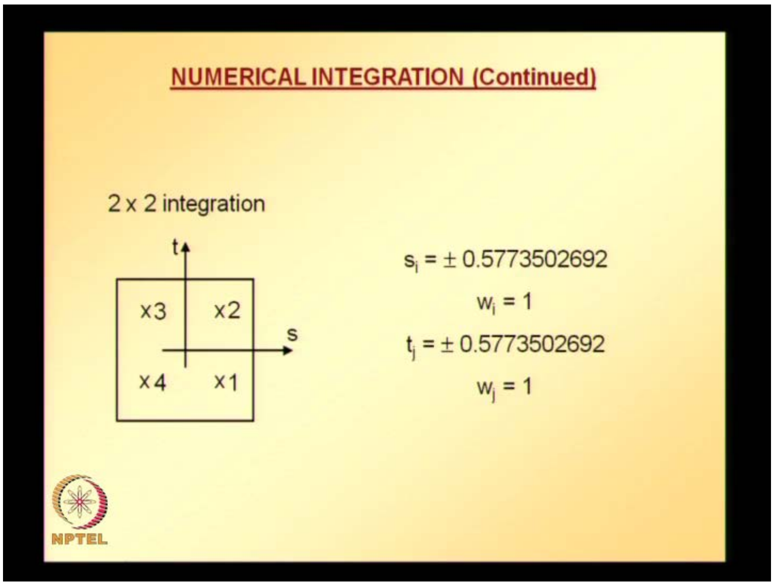

Next is 2 by 2 integration. There are two integration points in s direction and two integration points in the t direction. The coordinates of integration points in the s direction can be obtained from the table that we just looked at taking number of significant digits as the accuracy that one is looking for. So here the integration point coordinates in the s direction are shown and also weight in the s direction at these two points is equal to 1 similarly, integration point coordinates in the t direction are given and weight in the t direction is at the two points is equal to 1, so by combination of this that is s is equal to plus or minus 0.5773502692 and t is equal to plus or minus 0.5773502692 by combination of these we can obtain the coordinates of all integration points and again each integration point weight is going to be weight in the s direction times weight in the t direction, so that is how we can get weights and coordinates of all the four integrations points.

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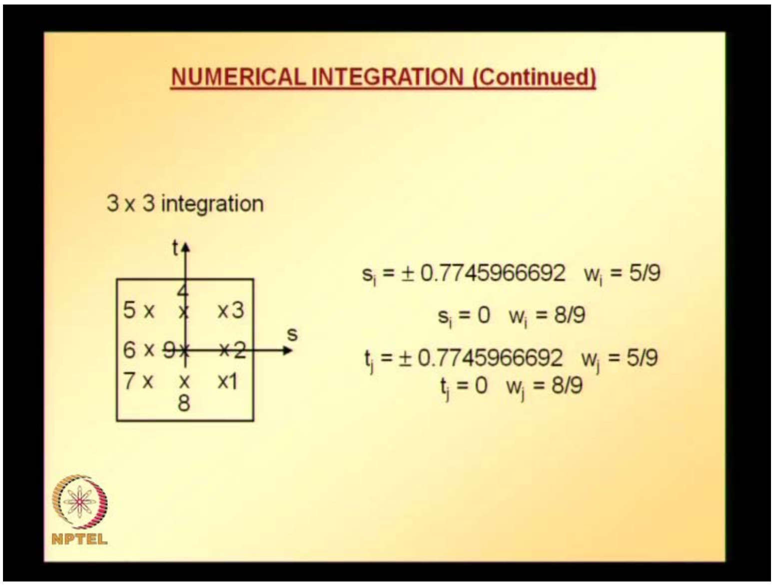

Similarly, 3 by 3 integration from the table read out the coordinates of the integration points in the s direction and the corresponding weights similarly, same thing or you can repeat for the t direction. If there are three points in the t direction as well same the coordinates of the integration points in s direction will be same as in the t direction. If one has different number of integration points in the t direction then we need to read out the corresponding coordinates and weights from the table that is given earlier and doing or calculating the weights as I explained weight in the s direction times weight in the t direction, we get weights at all the points and also by combining these coordinates in s direction and t direction, we can get the coordinates of all points for this two-dimensional integration, so similar procedure can be extended for any order of integration.

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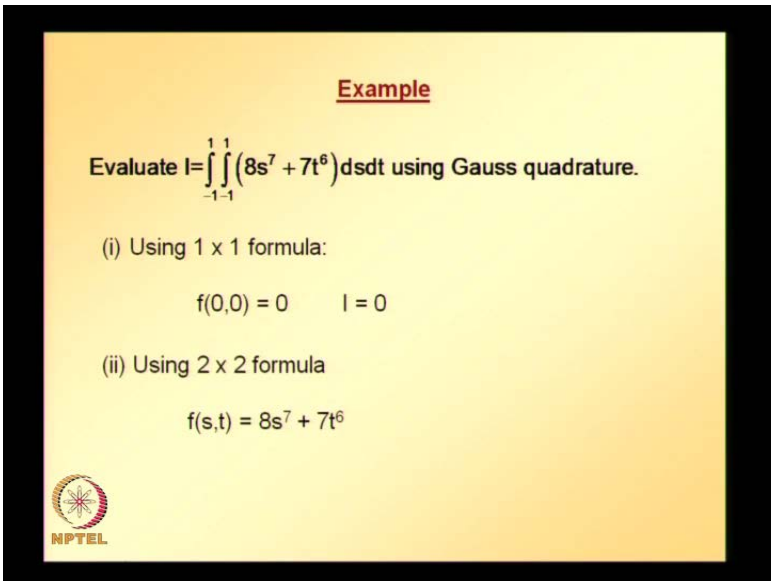

So let us take an example evaluate i is equal to double integral minus 1 to 1, minus 1 to 1 eight times s power 7 plus 7 times t power 6 dsdt using Gaussian quadrature. Let us start with 1 by 1 integration. So in the s direction the coordinate as s is equal to 0 in the t direction coordinate is t is equal to 0 that is the location of integration point and weight is weight in the s direction times weight in the t direction which is weight in the s direction is 2 weight in the t direction is 2 so total weight is 4, but the function value evaluated at s is equal to 0, t is equal to at 0 that is integrant value evaluated at s is equal to 0 and t is equal to 0, it turns out to be 0, so total integral is going to be 0 because the function value is 0 and whatever weight that is even if you multiply with 4, we are going to get 0. In 2 by 2 integration, this is integrant and we need to evaluate this integrant at the four integration points where s and t values can be obtained from the table. Following the procedure that I explained, we need to evaluate this function value at this four integration points and multiply with corresponding weights and sum them up to get the approximate value of integrals.

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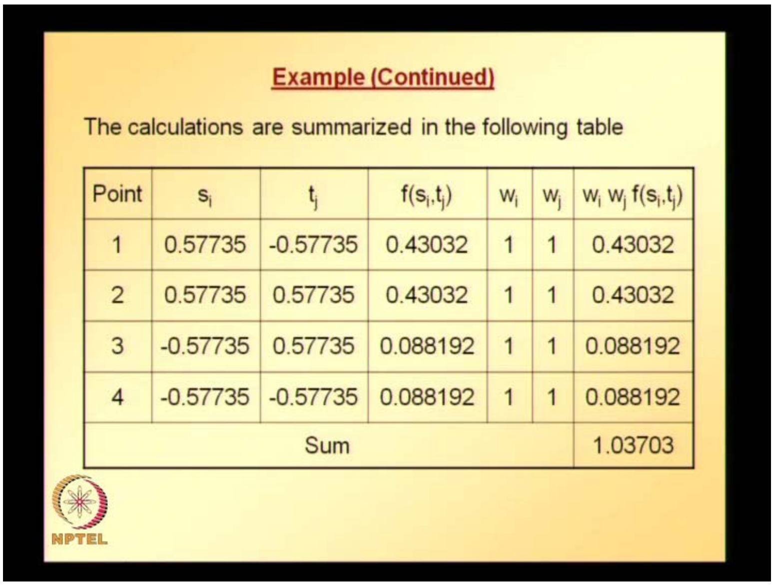

Point	s_i	t_j	$f(s_i,t_j)$	w_i	w_j	$w_i\, w_j\, f(s_i,t_j)$
1	0.57735	-0.57735	0.43032	1	1	0.43032
2	0.57735	0.57735	0.43032	1	1	0.43032
3	-0.57735	0.57735	0.088192	1	1	0.088192
4	-0.57735	-0.57735	0.088192	1	1	0.088192
Sum						1.03703

We can do all these calculations in a table format. The calculations are summarized in the following table. First column shows point, second column shows what is the s coordinate of this particular point, and third column t coordinate of that particular point and the forth column shows the integrant value that is f value evaluated at that particular point substituting s and t values and fifth column shows weight in the s direction, sixth column shows weight in the t direction and seventh column shows weight in the s direction times weight in the t direction times the function value at that particular point, so summing up all the contribution from all the four points that is summing up the values which are in the seventh column we are going to get the value of integral, so this is using 2 by 2 integration two points in the s direction and two points in the t direction.

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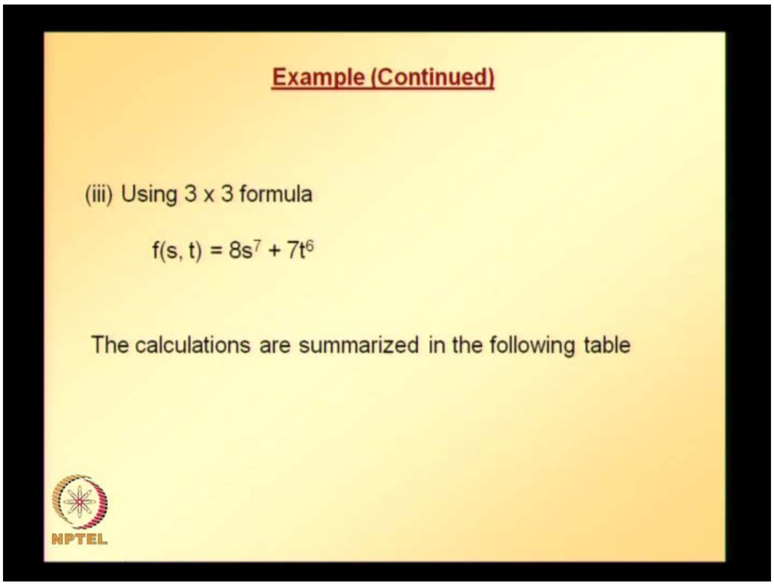

Similar procedure can be extended if one wants to use higher order integration that is 3 by 3 integration integrant or function and calculations are summarized in the table.

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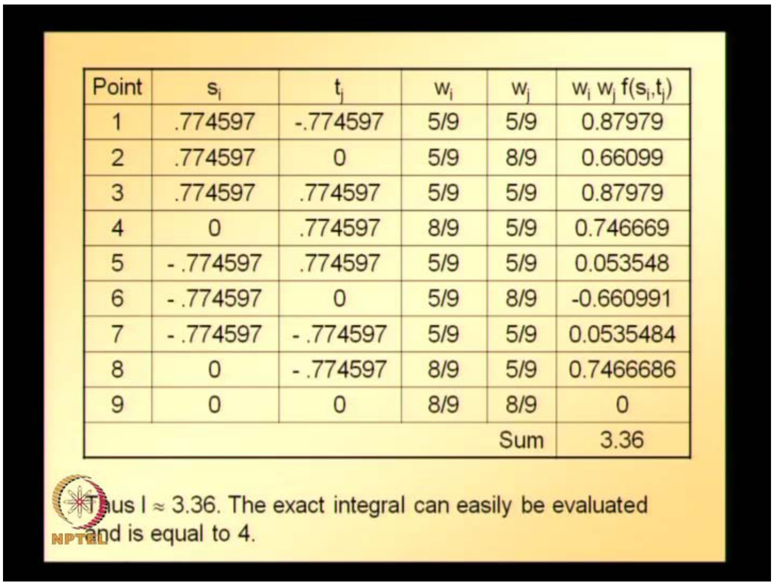

Point	s_i	t_j	w_i	w_j	$w_i\, w_j\, f(s_i,t_j)$
1	.774597	-.774597	5/9	5/9	0.87979
2	.774597	0	5/9	8/9	0.66099
3	.774597	.774597	5/9	5/9	0.87979
4	0	.774597	8/9	5/9	0.746669
5	- .774597	.774597	5/9	5/9	0.053548
6	- .774597	0	5/9	8/9	-0.660991
7	- .774597	- .774597	5/9	5/9	0.0535484
8	0	- .774597	8/9	5/9	0.7466686
9	0	0	8/9	8/9	0
				Sum	3.36

Thus $I \approx 3.36$. The exact integral can easily be evaluated and is equal to 4.

So 3 by 3 results in nine integration points total function value needs to be evaluate at all the nine integration points multiplied with weight in the s direction times and weight in the t direction. The contribution from all the nine integration points can be summed up to get the approximate value of integral and the approximate value of integral is using 3 by 3 integration is 3.36. and whereas this integral that is double integral minus 1 to 1, minus

1 to 1 eight times s power 7 plus seven times t power 6, dsdt we can also perform exact integration because this integrant is very simple s power 7 integral s power 7, d s is s power 8 divided by 8. Similarly integral t power 6, d t is t power 7 divided by 7, we can easily evaluate this integral value exactly and substitute the limits of the integration and if you do that exact integral can be evaluated or can be easily verified to be equal to 4.

If you see this integral in the s direction, we have highest power as 7 and in the t direction highest power as 6 and as we can apply the formula that we looked at when we are discussing one dimensional integral that is n point Gaussian integration can integrate a function of order 2 n minus 1 exactly.

We can use that thumb rule and we can easily back calculate how many points we can use in s direction and how many points we can use in the t direction. So two n minus 1 is equal to 7 results in n is equal to 4 and 2 n minus 1 is equal to 6 results in n is equal to 3.5 which needs to be rounded up to 4. So if one wants to evaluate this integral exactly, 4 by 4 integration needs to be adopted. If one adopts 4 by 4 integration, then the approximate value of integral that we get by doing Gaussian quadrature matches exactly with exact value of the integral which is 4.

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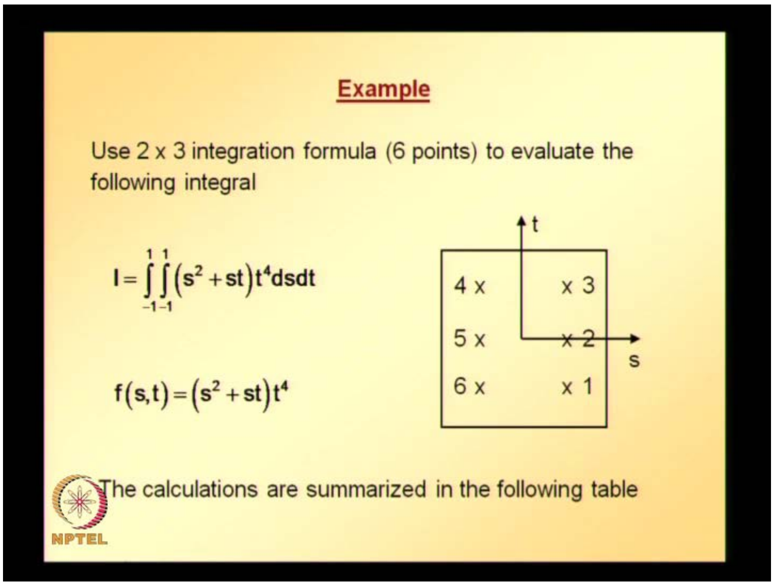

Now let us take another example. Use 2 by 2 integration formula that is total six points to evaluate following integral i double integral minus 1 to 1, minus 1 to 1, s square s t times t power four dsdt, so the function is this as you can see the highest power of s is 2 and the highest power of t is 5 applying the 2 n minus 1 rule that we have for deciding how many points we need to use in s direction and how many points we need to use in the t direction 2 n minus 1 is equal to 2 results in n is equal to 1.5, which can be rounded off to 2 and 2 n minus 1 is equal to 5 results in n is equal to 3, so we need to adopt two integration points in s direction and three integration points in the t direction to evaluate this integral exactly that is what is ask to use that is 2 by 2 integration formula. The approximate solution that we are going to get by adopting Gaussian integration is going to match with the exact results for this particular problem when we adopt 2 by 2 integration. All the calculations these are the locations of integration points. The details of calculations at each of this integration points and their locations and weights are given in the table.

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Point	s_i	t_j	$f(s_i,t_j)$	w_i	w_j	$w_i\, w_j\, f(s_i,t_j)$
1	.57735	-.774597	-.041	1	5/9	-.02278
2	.57735	0	0.0	1	8/9	0
3	.57735	.774597	0.218	1	5/9	0.15611
4	-.57735	.774597	-.041	1	5/9	-.02278
5	-.57735	0	0.0	1	8/9	0
6	-.57735	-.774597	.281	1	5/9	.1561
					Sum	0.26666

Thus I = 0.26666 Exact I = 4/15 = 0.26667

Total six integration points, first column shows the point integration point number, second column shows location in s direction, third column location in the t direction, fourth column function value at that particular location substituting that particular corresponding s and t values and fifth column shows weight in the s directions, sixth column weight in the t direction and seventh column shows weight in the s direction times weight in the t direction times function value evaluated at that particular point. And summing up all the values in the seventh column we get the approximate value of this integral. You can see the approximate value that is obtain using Gaussian quadrature matches very well with the exact solution for this particular problem.

So far, we have seen as a part of this isoparametric quadrilateral element, we have seen how to derive shape functions for four to 9 node quadrilateral elements and also we looked at isoparametric mapping concept for two dimensional elements and also we looked at numerical integration for evaluate the integrals over two dimensional domains. Now, we are ready to develop element equations by taking one, any one of the quadrilateral elements, so now let us look go and take the simplest elements that we can have under quadrilateral element category that is 4 node quadrilateral element.

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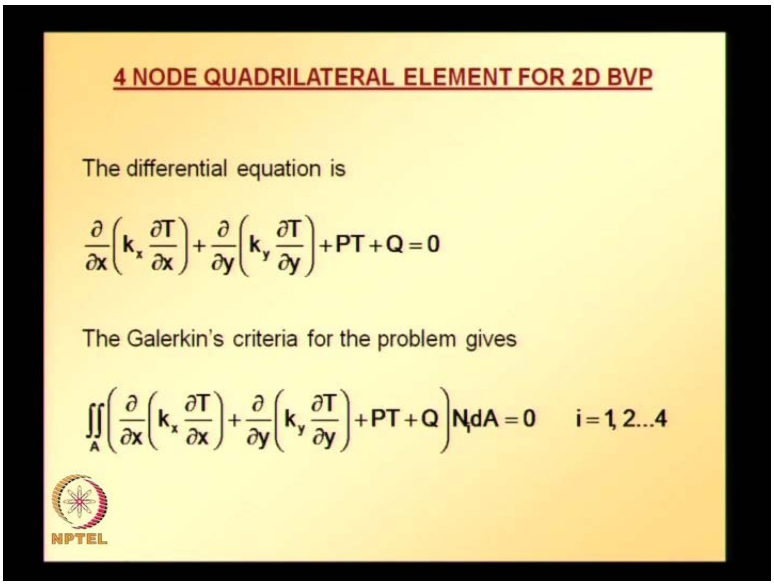

The 4 node quadrilateral element for two dimensional boundary value problems; using isoparametric mapping and the procedure used for triangular element it is straight forward to develop finite element equations for any element. Equations for 4 node quadrilateral element for general two dimensional boundary value problem. we are going to develop the equations for 4 node quadrilateral element for general two dimensional boundary value problem. Let us go back and recall what is the differential equation that we are using for general two dimensional boundary value problem, the differential equation is something like this which (Refer Slide Time: 38:48) we used when we are developing element equations using three 3 triangular element.

Now using Galerkin criteria for the problem gives us this equation basically Galerkin criteria is multiply given differential equation with the weight function. When we are using finite element method shape function is same as weight functions so integrate the given differential equation with shape function and how many shape functions, it depends on what kind of element, one is adopting. So multiply given differential equation with shape function integrate over the problem domain equated to 0 and this is a first step and is equation needs to be further simplified for that we use Green's theorem.

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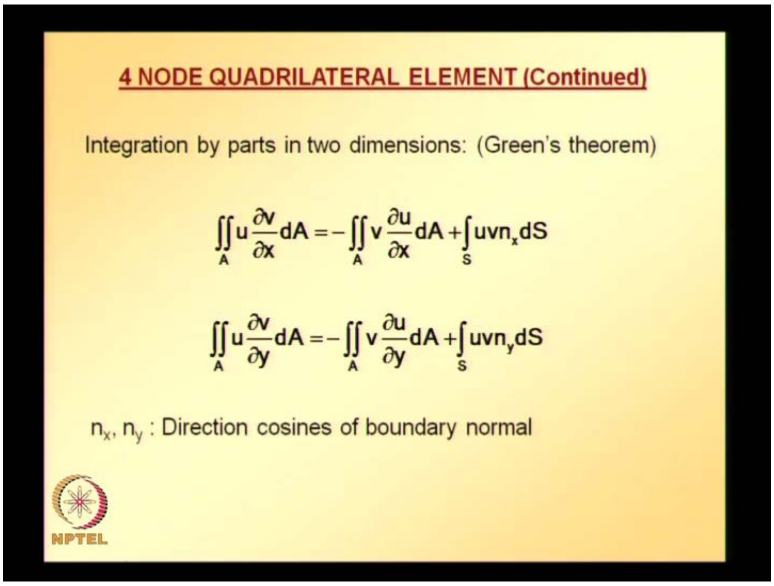

And when we are looking 3 node linear triangular elements already gave you this formulas integration by parts in two dimensions double integral U times partial derivative of V with respect to x d A is equal to minus double integral over area V times partial derivative of U with respect x plus line integral U times V x component of outward normal integrated over the boundary similarly the second equation. These are the two equations that we get using Green's theorem and we will be using these two formulas to simplify the previous equation further x and y are the direction cosines of boundary normal.

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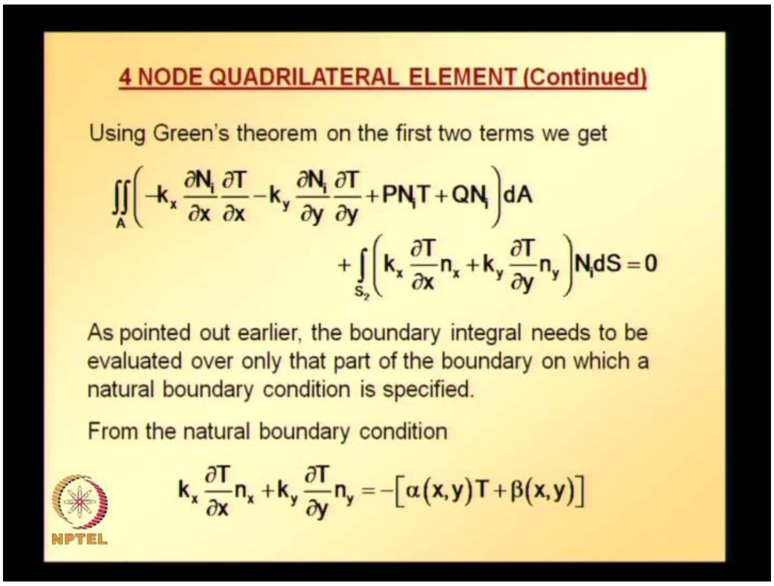

So using these two formulas and simplifying the previous equation. The first two terms of the previous equation can be simplified using Green's theorem formula that we just looked at and that result in this. If you recall we discussed in detail all these when we are dealing with 3 node triangular element except that the only difference that you have between that and here (Refer Slide Time: 42:02) number of shape functions are four whereas earlier number of shape functions are three and the shape function expressions also are different for four node and three node element, except that this part is similar as pointed out earlier boundary integral needs to be evaluated over only the part of boundary on which natural boundary condition is specified.

If you see the second part of the equation, integral needs to be perform only on s 2, s 2 is the part of boundary on which natural boundary condition is specified because the part of the boundary and which essential boundary condition is specified, we do not need to evaluate this integral because it goes as a reaction term in the final equation system that we get, so we ignored this explanation I have given you when we are looking at 3 node linear triangular element.

So let me now we are looking at the second integral, the integrant comes the value of it can be further simplified which comes from the prescribed natural boundary condition. From the natural boundary condition, the line integral that is the boundary integral over s 2 can be further simplified or further rewritten using this boundary condition that is a k x times partial derivative of t with respect times x times. x component of normal plus k y times partial derivative of t with respect to y times .y component of normal is equal to (minus) alpha t plus beta.

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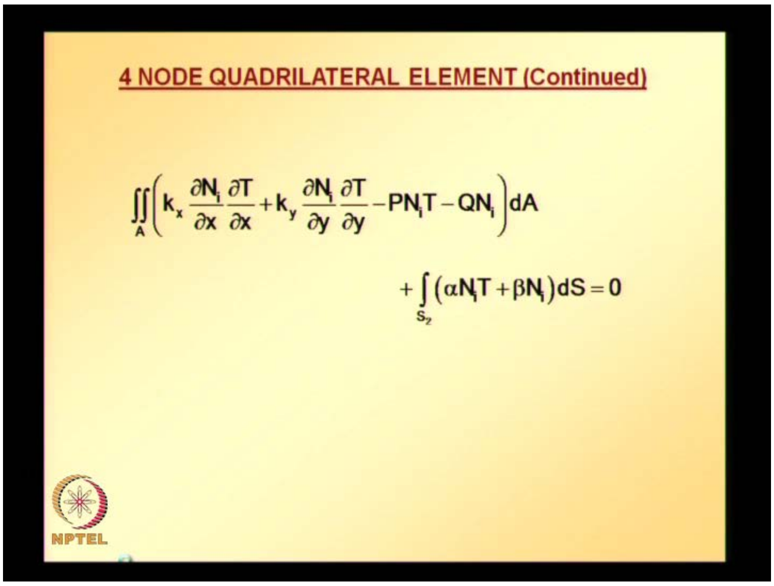

Substituting this quantity into this equation, it can be written and now we need to make substitution of shape function expressions. There are four shape functions for 4 node quadrilateral element.

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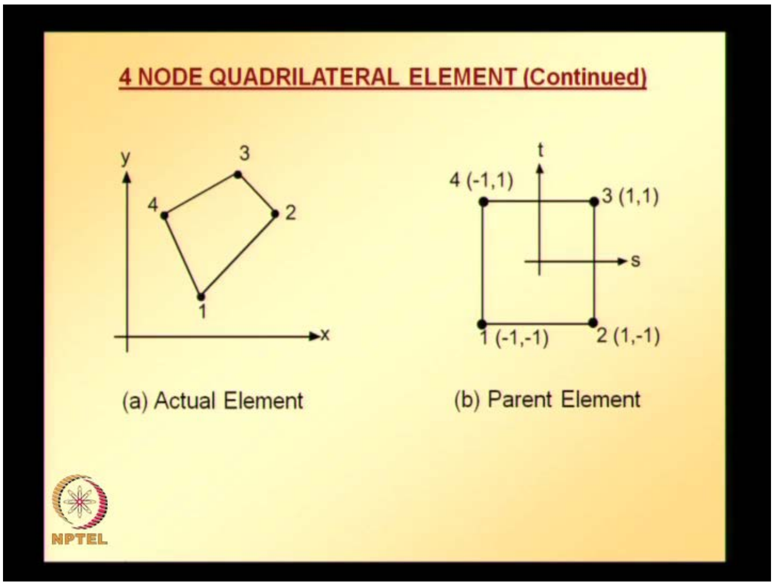

The shape functions need to be in the parent coordinates even though or will be mapping actual element and parent element. Actual Element is shown instead of performing calculations on the actual element we are going to perform calculations on the parent element and that is what isoparametric mapping is.

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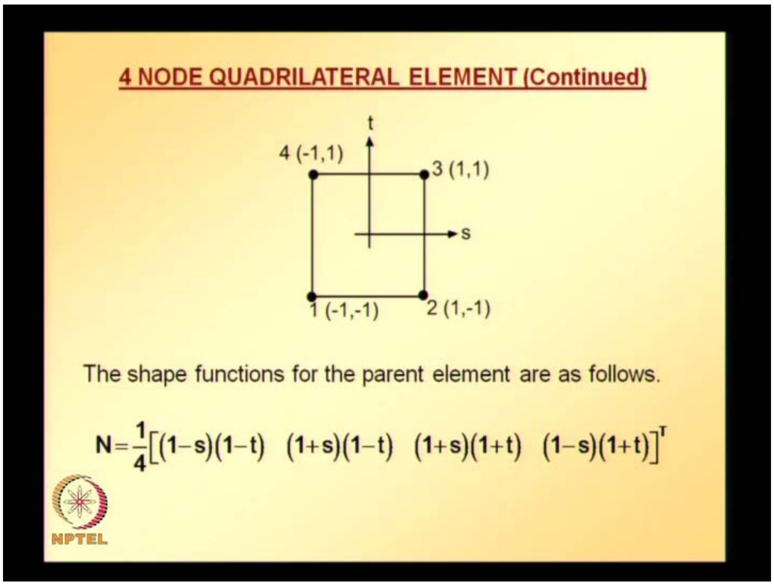

The shape functions for 4 node Parent Element in terms of s and t can we put in a vector and we can take the derivatives of shape functions with respect s and t.

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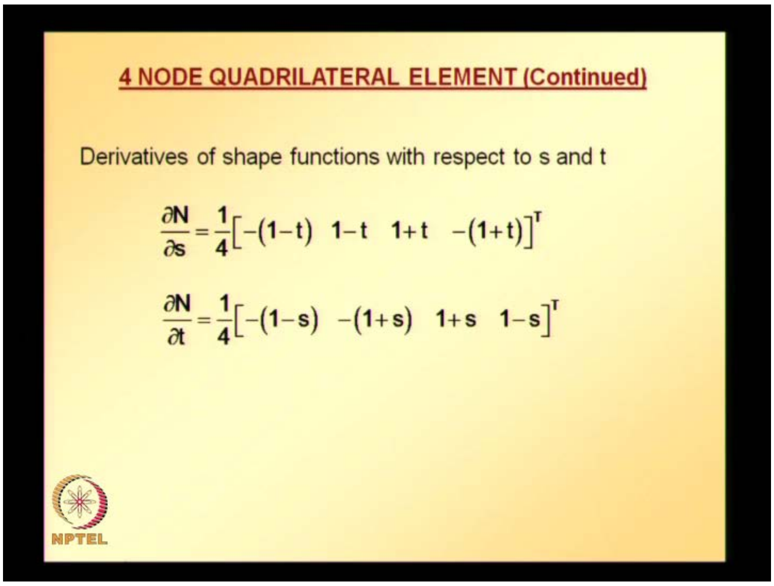

Derivatives of shape functions with respect s and t.

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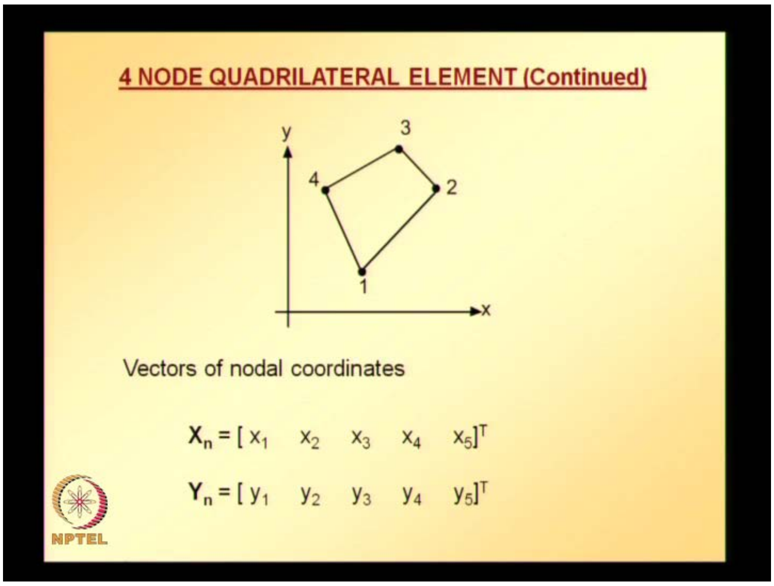

Derivatives of shape functions with respect s and t and vectors of nodal coordinates looking at the actual element. We can write the nodal coordinates here there is a mistake the vector of nodal coordinates consists of only x 1, x 2, x 3, and x 4 similarly, vector of nodal coordinates for Y consists of only y 1, y 2, y 3, and y 4, X Y and y 5 should not appear because of we have only five nodes.

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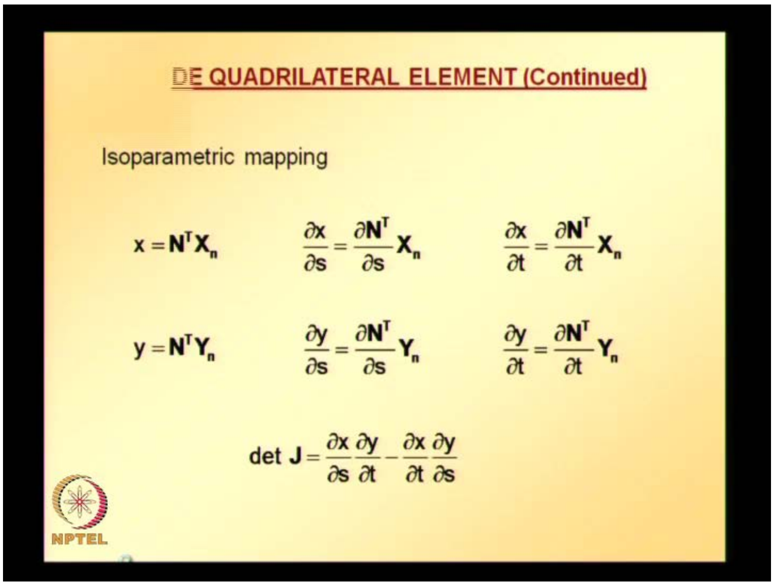

So, using the nodal coordinates x 1, x 2, x 3, x 4, and y 1, y 2, y 3, and y 4, we can write isoparametric mapping relations, which can be compactly written in a matrix and vector form, N transpose comprises of all the shape functions n 1 to n 4 and X N comprises of all the X coordinates of all the nodes x 1 to x 4 and once we have this relation, we can easily take partial derivative of that X with respect to s and with respect to T. Similarly, Y is equal to N transpose Y N again N transpose is vector of shape functions and Y N is vector of Y coordinate of all the nodes y 1 to y 4, y 1, y 2, y 3, and y 4, once we have that relation we can easily take derivatives of y with respect s and t.

Once we have this partial derivative of X with respect to s, partial derivative of X with respect to t, partial derivative of Y with respect s, and partial derivative of Y with respect to t, we can easily calculate what is determinant of Jacobian and derivatives with respect to X and Y can be computed.

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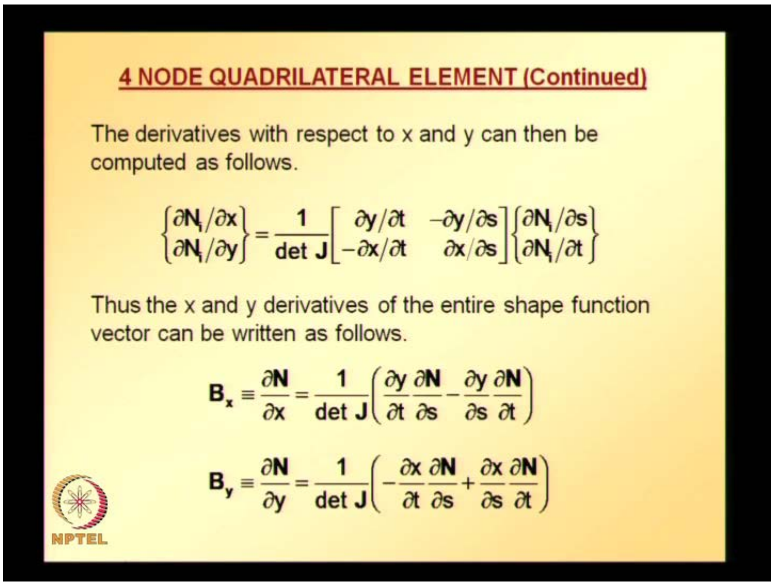

Once we know the determinant of Jacobian we can compute the derivatives of shape function with respect X and Y. Once we know derivatives of shape function with respect s and t, because if you see on the right hand side, we have partial derivative of shape functions with respect s and t and that vector can easily be computed, because shape functions are in terms of parent coordinates s and t . And thus the x y derivatives of entire shape function vector can be written writing shape function derivatives with respect to x as a separate vector B x, we can write from the previous equation, we can write this relation similarly B y.

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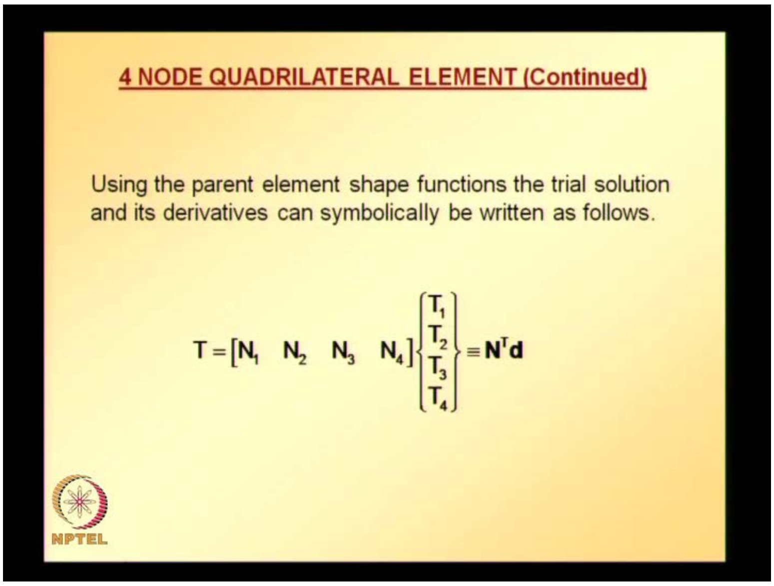

And using the parent element shape functions trial solution and it is derivative. Trial solution can be written like as N 1, T 1 plus N 2, T 2, plus N 2, T 3, plus N 4, T 4 in a matrix and vector form as N transpose d similarly, derivatives of t with respect to X can be written as B x transpose d, derivatives of T with respect to Y can be written as B y transpose d.

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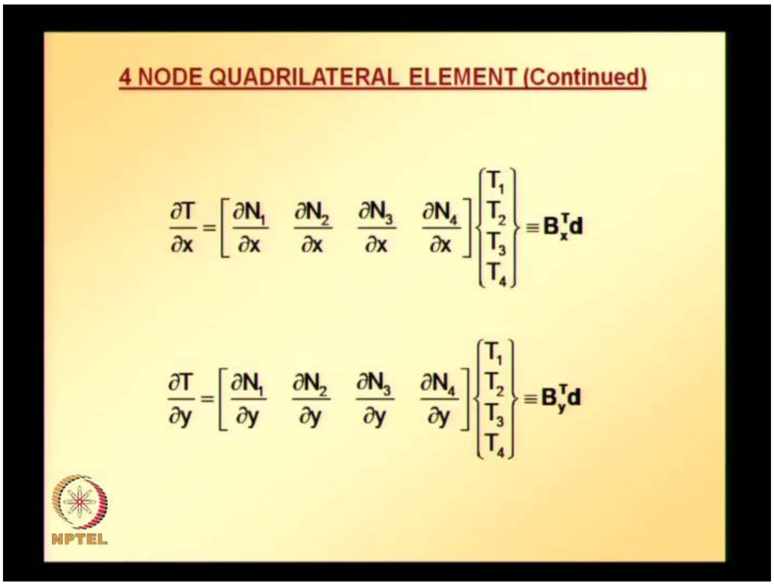

So, we have the trial solution and also derivatives in terms of finite element shape functions. Now we can substitute all this information into the equation.

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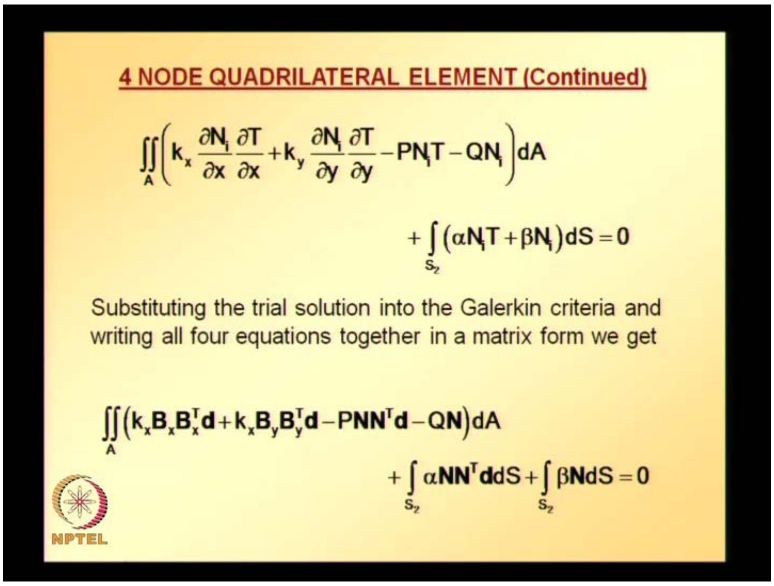

In this equation, thus substituting the trial solution into the Galerkin criteria and writing all four equations together in matrix form we get. There are four equations because there are four shape functions. Substituting the derivatives of trial solution and trial solution in terms of finite element shape functions and derivatives of shape function we get this equation.

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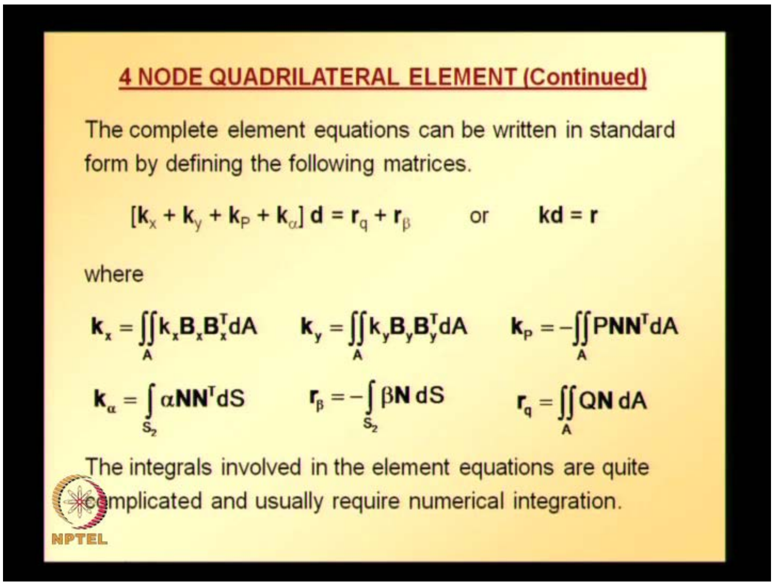

And complete element equations can be written in standard form by defining following matrices. The previous equation can be rewritten in this manner K x, K y, K p, K alpha, K x plus K y plus K p plus K alpha times D is equal to R q plus R beta, which can be compactly written as K D equal to R. Each of this matrices and vectors are defined like this. This is obtained from the previous equation by comparing with previous equation we can write this; and the integrals involved you can easily notice that integrals involved in element equations are quite complicated and usually require numerical integrations. So that is the reason we looked at Gaussian integration in two-dimensional case. In the next class, we will be looking at the details how to evaluate all these area integrals and line integrals.