Finite Element Analysis

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Lecture No. # 25

In the last class, we looked at how to derive the shape functions for rectangular elements and square elements. And we discussed two kinds of elements; one is Lagrangian element and another one is serendipity elements. So, basically here I want to summarize what we have done in the last class. And at the end of the last class we have seen a table in which a general set of functions are given to derive shape functions or to write the shape functions explicitly for four to nine node rectangular element.

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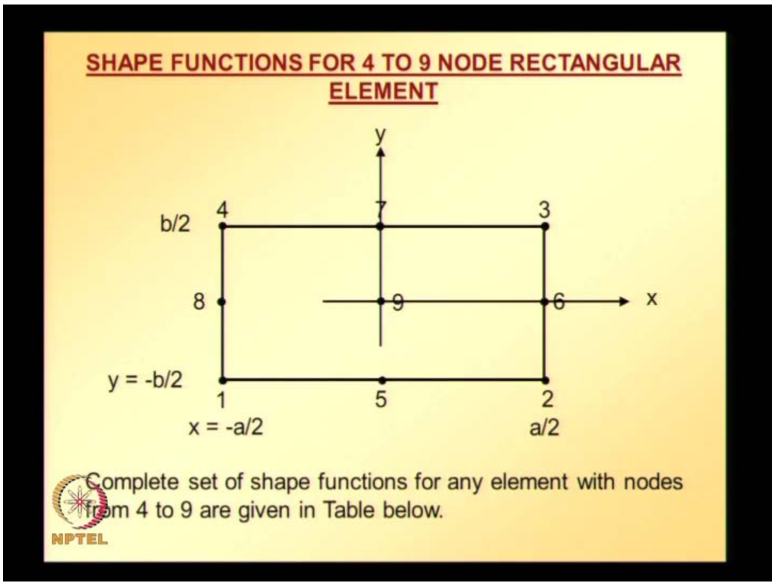

And so, this is the element that we are considering, all the nodes are numbered and the corresponding locations can be easily figured out from the figure that is given. And here node numbering is the way the nodes are numbered is very important, because with this particular node numbering scheme, shape functions for higher order elements can be constructed by adding terms into the shape function for lower order elements.

And now, a complete set of shape functions for any element with nodes from four to nine are given in the table below, and before we read the table to get the shape function expressions for any nodded element starting from four node to nine node certain functions are defined in terms of this coordinates of this element. So, along x direction if you see this element, this element is of length is having length of a unit and along y direction it has b units, and the coordinate system, x y coordinate system is defined with respect to centroid of the element.

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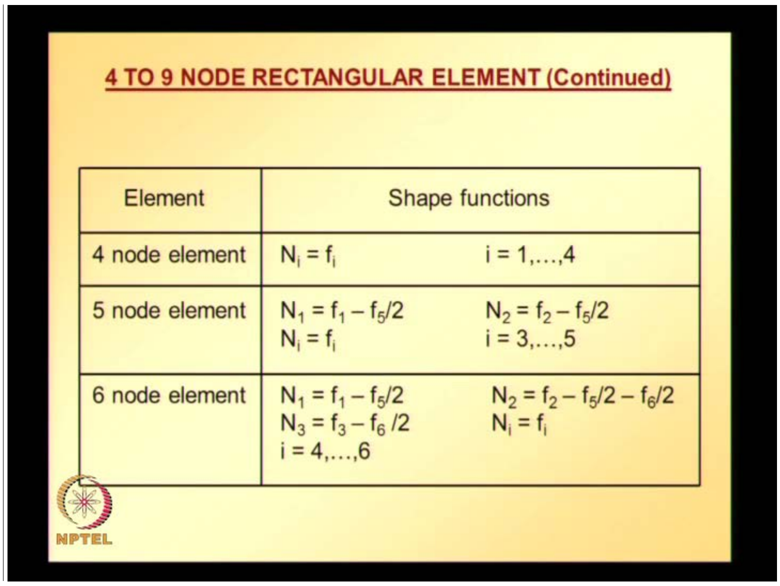

4 TO 9 NODE RECTANGULAR ELEMENT (Continued)

Element	Shape functions	
4 node element	$N_i = f_i$	$i = 1,\ldots,4$
5 node element	$N_1 = f_1 - f_5/2$ $N_i = f_i$	$N_2 = f_2 - f_5/2$ $i = 3,\ldots,5$
6 node element	$N_1 = f_1 - f_5/2$ $N_3 = f_3 - f_6/2$ $i = 4,\ldots,6$	$N_2 = f_2 - f_5/2 - f_6/2$ $N_i = f_i$

So, following notation is used s is equal to two x over a y is equal to or t is equal to t is equal to two y over b. so, s and t are defined like this with this definition of s and t f 1, f 2, f 3, f 4, f 5, f 6, f 7, f 8, f 9 are defined in terms of s and t, where s and t are obtained based on the coordinates of the particular element. So, once f 1 to f 9 are defined like this shape functions of four to nine node rectangular elements can be written in terms of these f 1 to f 9.

So, table shows for four node element how the shape functions shape function expressions are related to these functions similarly, five node element six node element so, once we go through this table we can easily write the explicit expressions for any nodded element starting from four node to nine node rectangular element.

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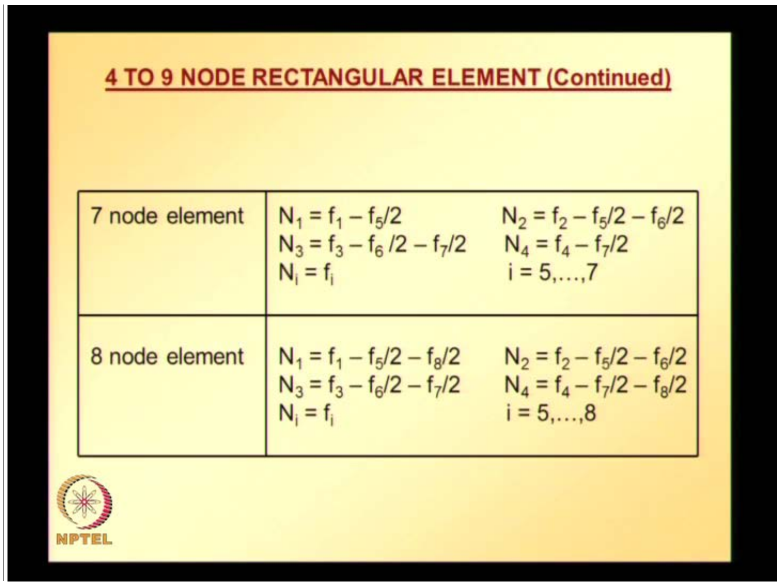

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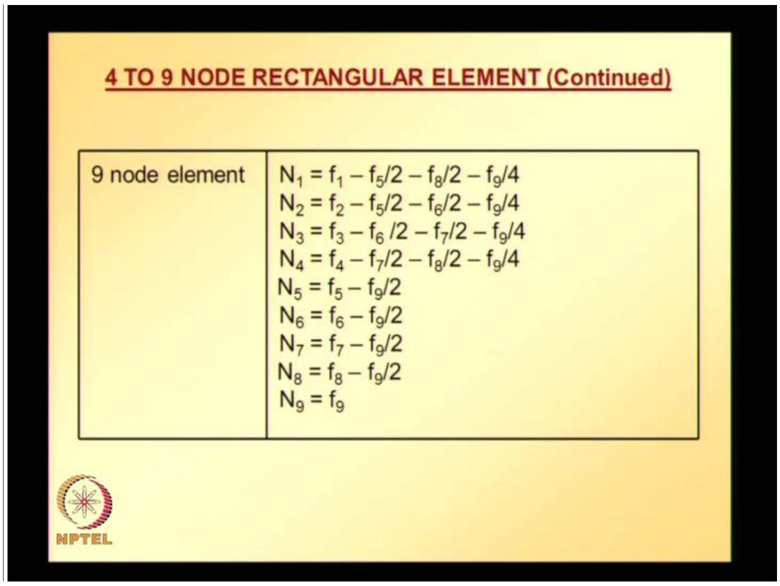

Similarly, seven node eight node element; and nine node element. And elements with number of nodes between four and eight are known as transition elements, and let us see where these transition elements are useful.

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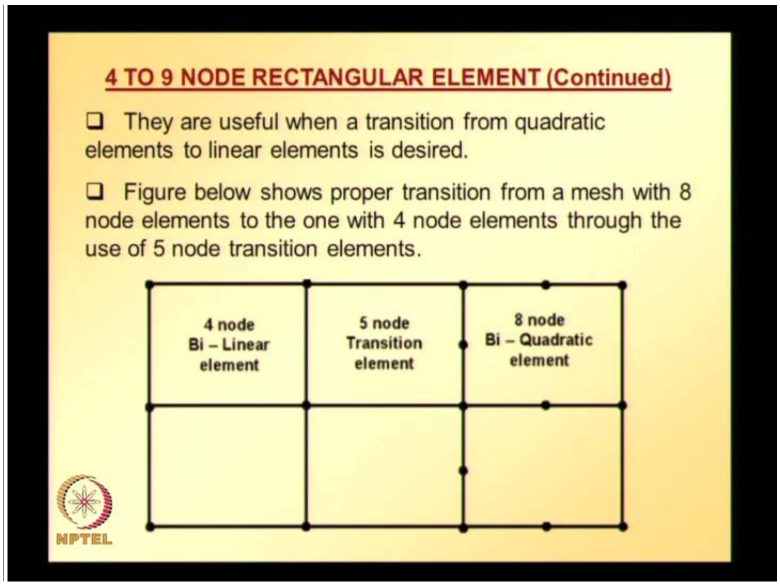

They are useful when a transition from quadratic elements to linear elements is desired. Figure below shows proper transition from a mesh with eight node elements to a mesh with four node elements through the use of five node transition elements. So, here you can see a part of mesh consists of four node elements, and rest consist of eight node elements, and in between the transition from four node to the region, where we have this transition there five node element is adopted.

And all so now, we will see what happens if we do not use this transition elements, instead we can if we have, if you take a mesh, where we have four node elements and, immediately we have eight node elements, what happens?

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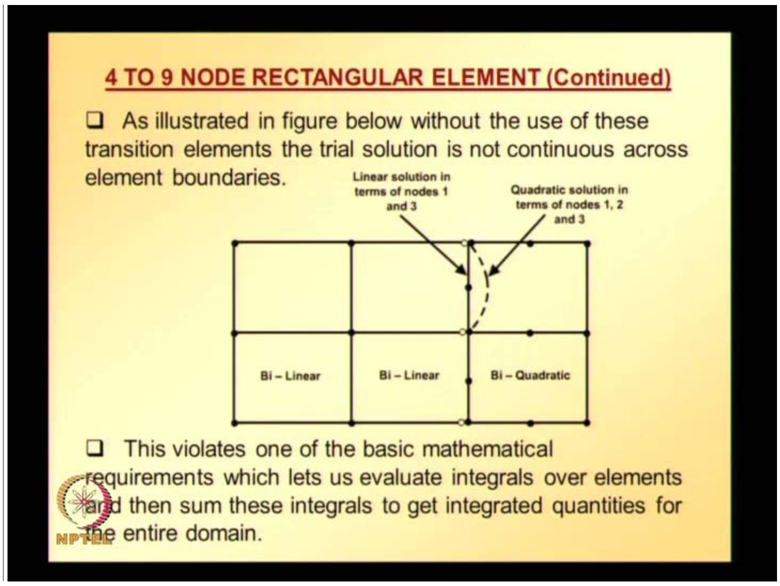

As illustrated in figure below without the use of these transition elements the trial solution is not continuous across element boundaries that can be better, the statement can be better understood by taking this illustration here. So, four node element mesh is directly immediately connected with eight node element mesh. So, if you can see the edge towards which the arrows are pointed out; you have linear solution on one side towards this side of four nodded elements, you have a linear solution in terms of nodes one and three and where as on this side where eight eight node elements are present quadratic solution in terms of nodes one, two, three.

So, this violates one of the basic mathematical requirements, which let us evaluate integrals over elements, and then sum these integrals to get integrated quantities for the entire domain. So, the mathematical requirement that it (Refer Slide Time: 06:39) violates is the solution variation in each of this elements along the same edge is different. So to avoid this kind of mathematical violations, we usually adopt transition elements.

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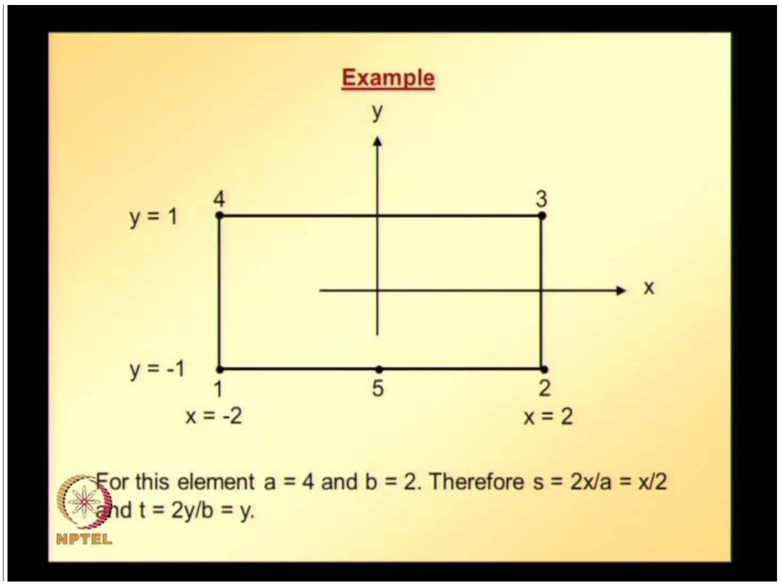

Now, let us look at how to derive shape functions for a five node element. A five node element is shown here. If you compare the node numbering that is given here with the node numbering that we have seen, when we before we looked at the table, which gives us shape functions for four to nine node elements. And if you compare these two node numberings of these two elements, we can see the node numbering is similar to what we have seen earlier.

So, we can directly use the expressions that we have for five node element and plug in the corresponding nodal coordinates, and we can get the shape function expressions for this particular element. For this element, a is equal to four units, b is equal to two units therefore, s is equal to x over 2 and t is equal to y. So, once we have s and t.

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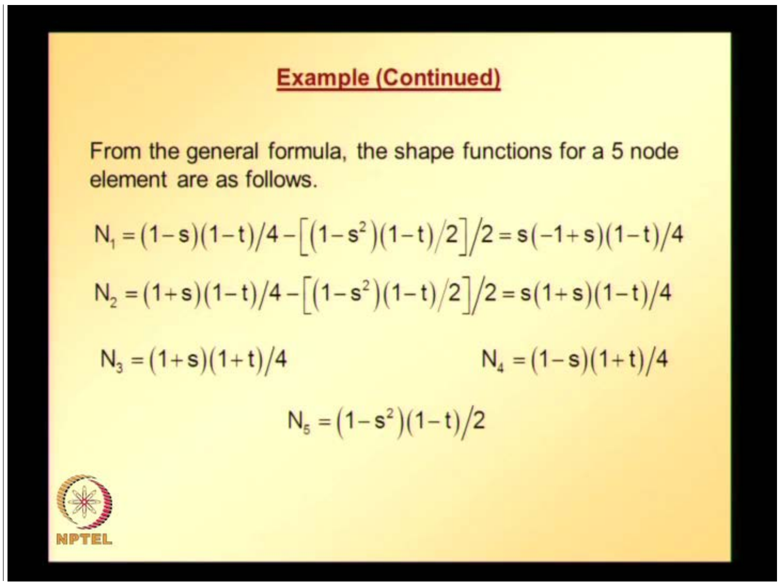

From the general formula, the shape functions for a five node element or as given here N 1, N 2, N 3, N 4, N 5 basically, these expressions are written taking from the table that I just showed you before. So, N_1 to N_5 can be obtained once we substitute s and t in terms of x and y and we just calculated using the information that is given, the nodal coordinate information that is given for this particular element how s is related to x that is s is equal to x over to y is equal-t is same as y or t is equal to y.

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So, substituting that into these expressions n 1 to n 5 we get shape functions for this particular element. Substituting s and t values into the above shape function formulas, the explicit expressions for shape functions for five node rectangular element can be obtained as given here N 1, N 2, N 3 in terms of x and y, N 5. N 1 to N 5. And we can also plot this if one wants to see visualize, how this shape functions look when it is when they are plotted with respect to x and y. x going from minus to 2 minus 2 to 2 y going from minus one to one.

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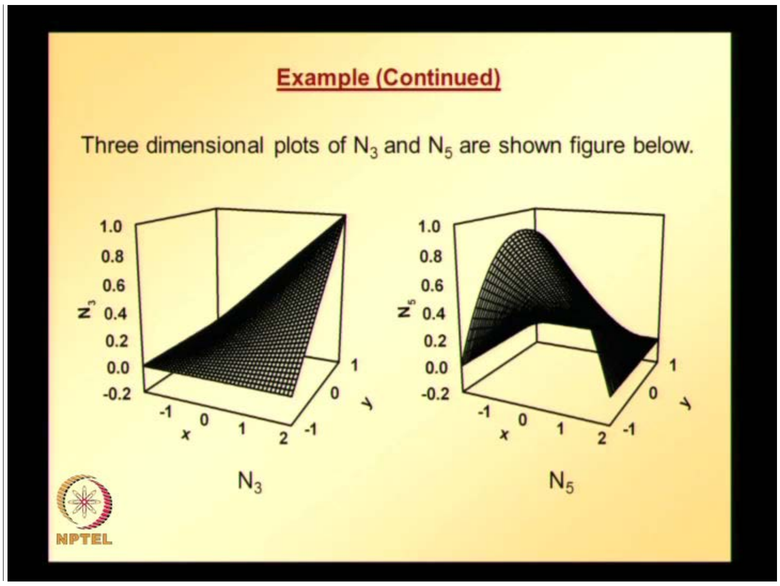

So, three dimensional plots of two of the shape functions namely N 3 and N 5 are shown. And this is how N 3 varies over the domain of that particular element and this is how N_5 varies. So, for any other nodded element that is seven nodded elements. For example, can be easily obtained from the tables.

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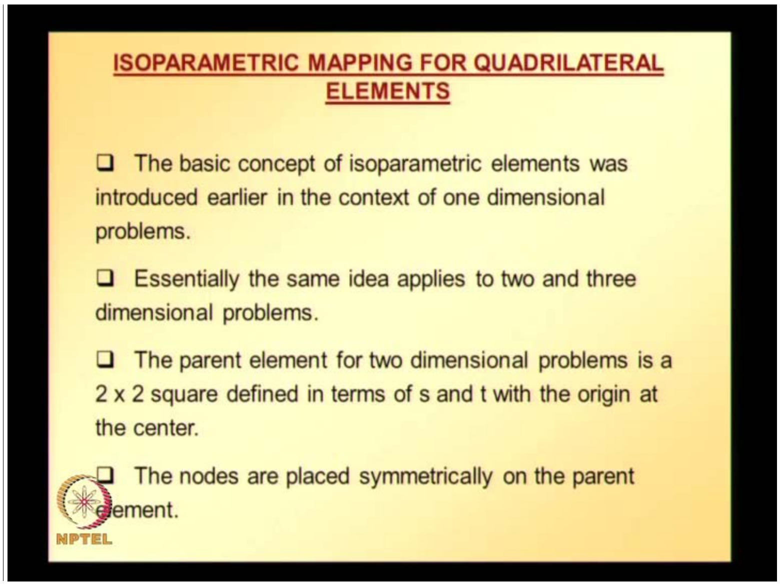

Now let us go to the other concept isoparametric mapping for quadrilateral elements. If you recall, we have already looked at this, when we are dealing with one-dimensional problem. The basic concept of Isoparametric elements was introduced earlier in the context of one dimensional problem. Essentially the same idea applies to two and three-dimensional problems. Only thing is the parent element for two-dimensional problem is a two by two square defined in terms of s and t with origin at the center and s going from minus one to one t going from minus one to one. Nodes are placed symmetrically on the parent element.

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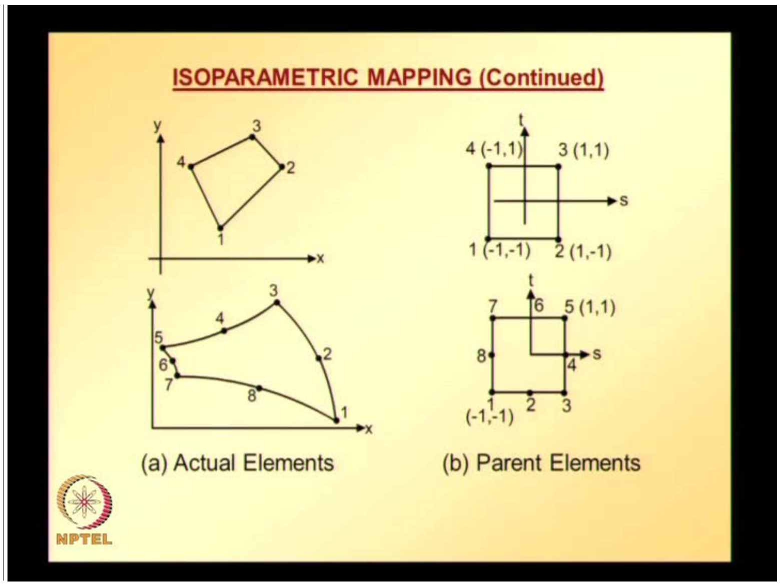

To get an idea, here these are two elements. One is four nodded element and other one is eight nodded element and both are actual elements, and these elements an Isoparametric mapping, we are going to map on to parent element like this. The four nodded element is going to be mapped on to a parent element having 4 nodes one to four and the nodal coordinates are also indicated and an eight nodded element is going to be mapped on to an eight node parent element and the nodal coordinates can be easily figured out from the information that is given in the figure.

So, in Isoparametric mapping for if you recall, what we did for one-dimensional problems shape functions are defined in terms of parent element coordinates.

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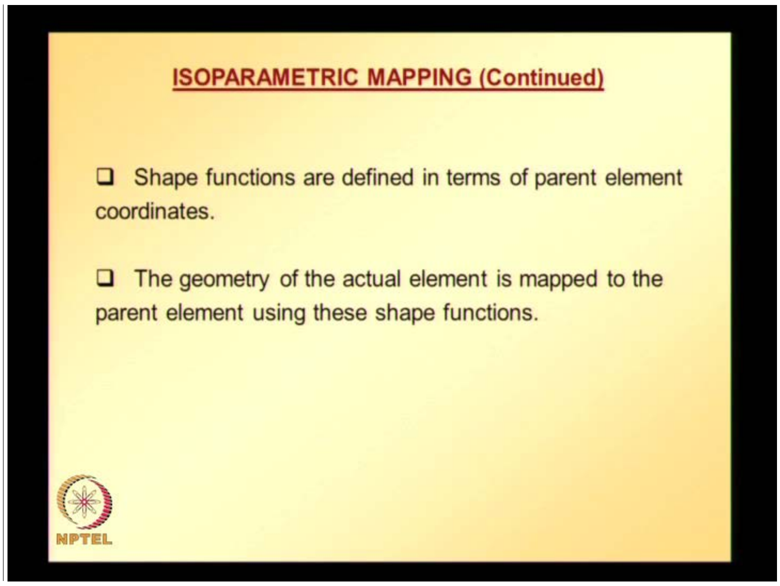

So, same thing we will be doing here also for two-dimensional problems. Shape functions are defined in terms of parent element coordinates. Then geometry of actual element is mapped on to the parent element using these shape functions. If you recall earlier we have written x is equal to sigma N i times x i for one-dimensional problem. Similar kind of expressions we can write for two-dimensional problem except that for two-dimensional problem we have x and y coordinates system.

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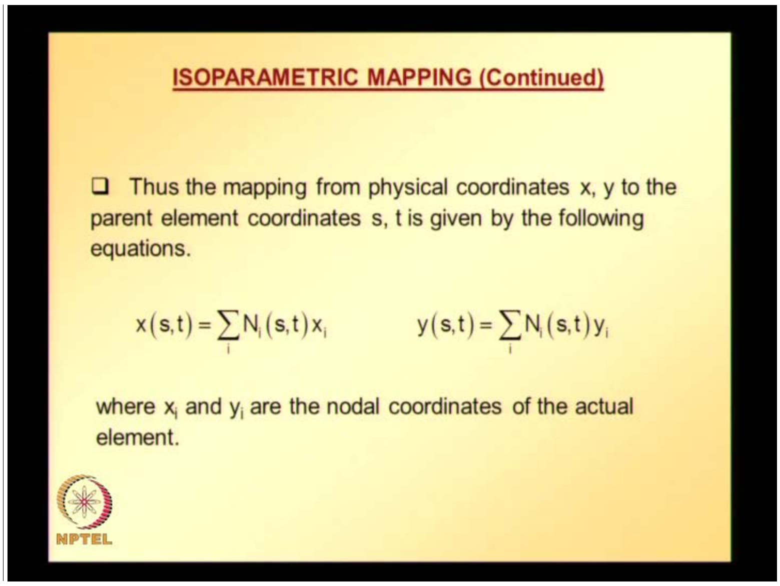

Thus the mapping from physical coordinates x, y to parent element coordinates s and t is given by the following equations; x as a function of s and t is equal to sigma times N i as a function of s and t times x i. Similarly y as a function of s and t is equal to sigma N i as a function of s and t multiplied by y i, where x i and y i are the nodal coordinates of the actual element. So, shape functions are defined in terms of parent element coordinates x i and y i are nodal coordinates of actual elements. So, this is how physical coordinates x y are mapped on to the parent element coordinates s and t.

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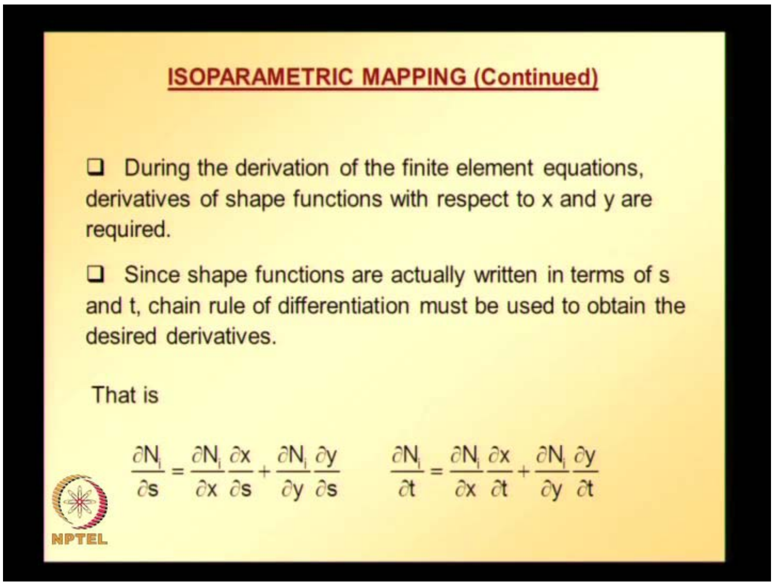

And also during the derivation of finite element equations derivatives of shape functions with respect to x y x and y are required. Since, shape functions are defined in terms of parent element coordinates, we can easily find what is the derivative of shape function with respect to s and t and once we get shape functions derivatives in with respect to s and t we can find shape function derivatives with respect x and y using chain rule.

Since shape functions are actually written in terms of s and t, chain rule of differentiation must be used to obtain the desired derivatives. That is partial derivative of N i N i, i takes values depending on the number of nodes that a particular element has, so, partial derivative of shape function with respect to s can be written as partial derivative of shape function with respect to x times partial derivative of x with respect to s plus partial derivative of shape function with respect to y times partial derivative of y with respect to s.

Similarly, partial derivative of shape function with respect to t is equal to partial derivative of shape function with respect to x times partial derivative of x with respect to t plus partial derivative of shape function with respect to y times partial derivative of y with respect to t. These two equations can be written or we can put these two equations in a matrix and vector form.

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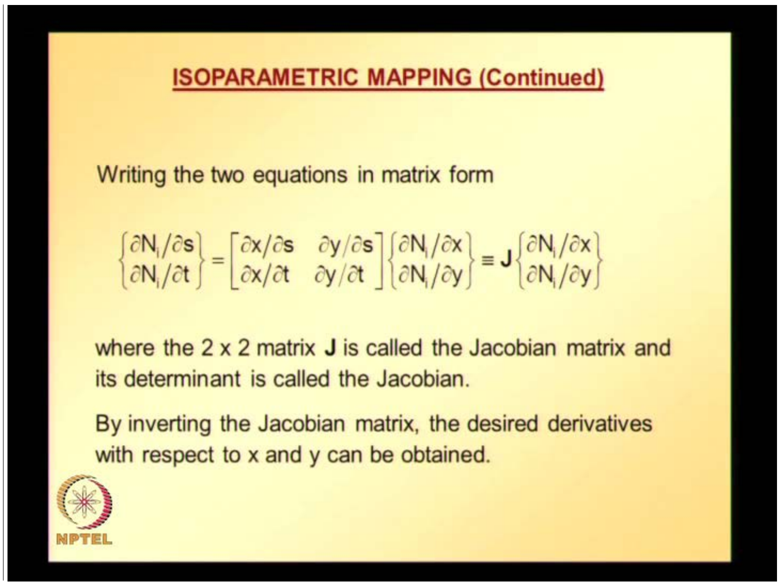

Writing the two equations in matrix form we get this, where a new quantity or a quantity which you are already familiar with, which is called Jacobian, you are familiar with Jacobian for one-dimensional elements. Jacobian there for is defined as dx over ds if you can recall for, two-dimensional elements Jacobian becomes a matrix that is why J is written in matrix notation that is bold letter, where two by two J two by two matrix J is called Jacobian matrix and it is determinant is called Jacobian. We have this relation if you now the derivatives of shape function with respect to s and t we can easily find what is the derivative of shape function with respect x and y and vice versa.

And here if you see this equation if we know the derivatives of shape function with respect to x and y; and this relation gives shape function derivatives with respect s and t; and if he invert this by inverting Jacobian matrix, the derivatives with respect x and y can be obtained. So, this is the inverse relation that is derivatives of shape function with respect x and y are expressed in terms of derivatives of shape function with respect s and t, for which we require inverse of Jacobian, which is given there.

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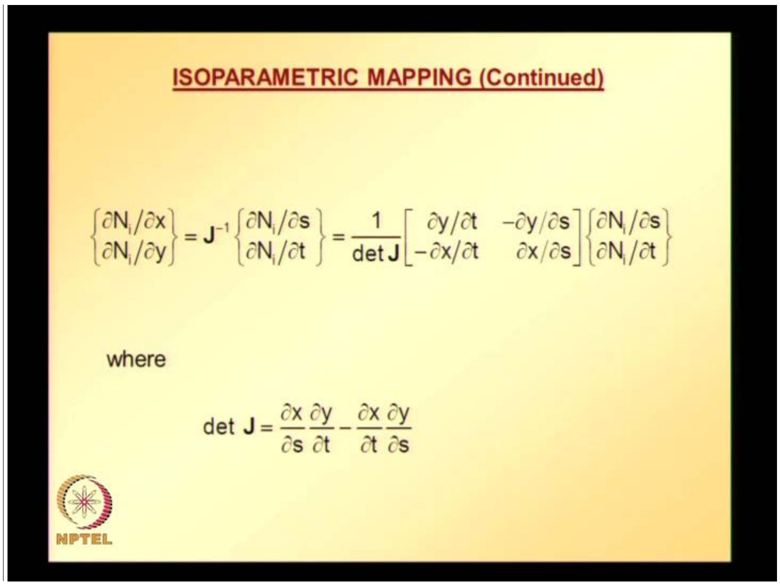

Where determinant of J, determinant of Jacobian is defined like this, partial derivative of x with respect to s times partial derivative of y with respect to t minus partial derivative of x with respect to t times partial derivative of y with respect to s. So, basically what we did is, we mapped the physical coordinates x y to parent element coordinates s and t, and from there we got all this. For that, mapping to be valid any point in the actual element should correspond to a unique point in the parent element.

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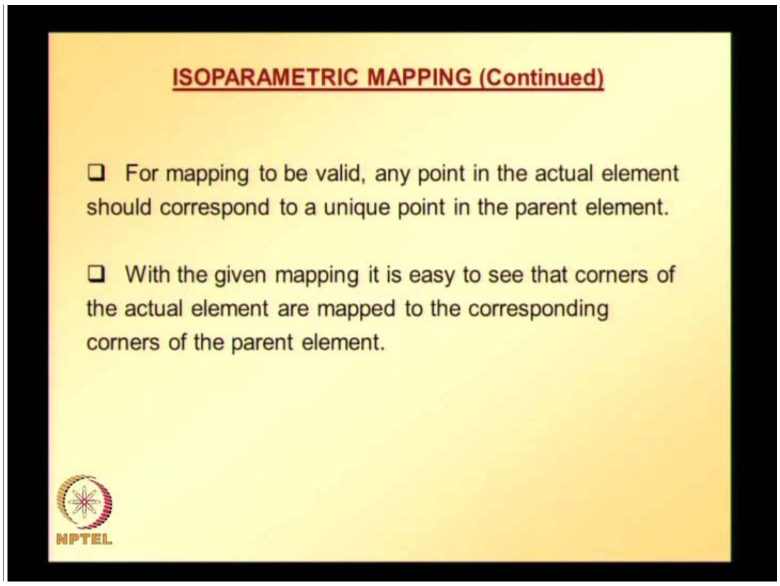

So, for the mapping from physical coordinate system x y to the parent element coordinate system s and t to be valid any point in the actual element should correspond to a unique point in the parent element. With the given mapping, it is easy to see that corners of actual element are mapped to the corresponding corners of parent element.

To illustrate this let us take let us go back to the figure which we have already seen and verify whether the statement is correct or not. What this statement says is corners of actual element are mapped to the corresponding corners of parent element.

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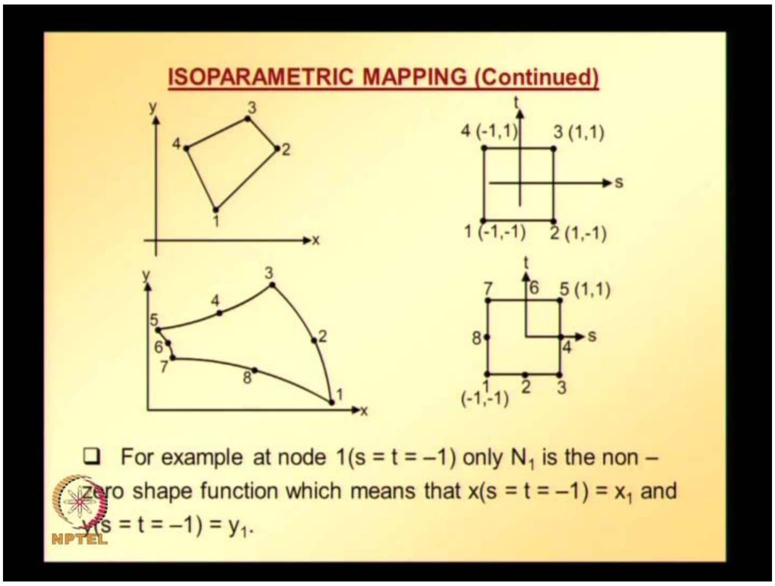

Let us check whether this statement is true or not so, these are the two actual elements one is four node element, another is eight node element. These elements are mapped on to this parent element, corresponding parent element four node element is mapped on to four node parent element eight node element is mapped on to eight node parent element. And if you take node one in any of these elements, node one corresponds to s is equal to minus 1 t is equal to minus 1 and at node one only N_1 is non-zero shape functions which means that s recall how s x is defined x is N i sigma x i.

So, all other shape function values at node one are zero except node one which is equal to one so, x becomes same as x 1 and y becomes same as y 1 so, that is what that statements says for example, at node one only N 1 is non-zero shape function and actually it is non-zero and a its value is equal to one (Refer Slide Time: 23:39), which means that x evaluated at s is equal to minus 1 t is equal to minus 1 is x 1` and y evaluated at s is equal to minus 1 t is equal to minus 1 is y 1. So, this verifies or this illustrates that corners of actual element are mapped on to corresponding corners of parent element.

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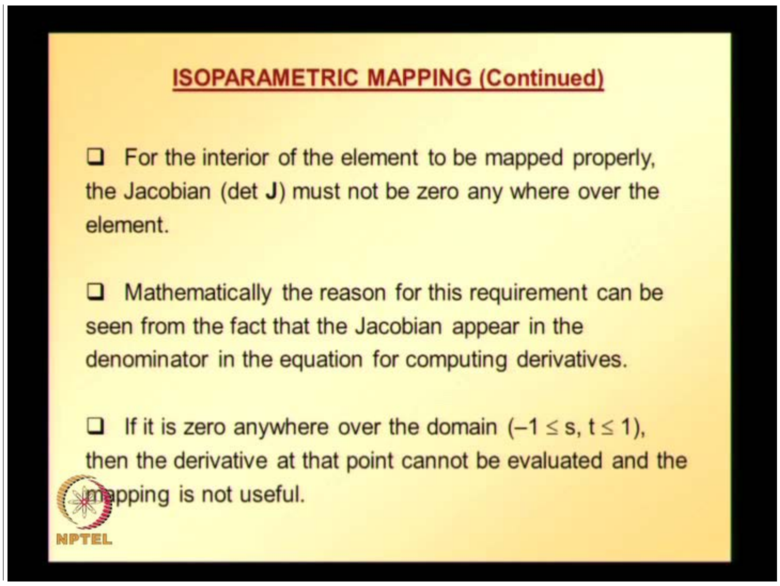

For the interior of the elements, so corner nodes no problem they are mapped, but for interior of element to be mapped properly Jacobian that is determinant of Jacobian must not be zero any where over the element. So, this is the requirement. Mathematically the reason for this requirement can be seen from the fact that Jacobian appears in the denominator in the equations for computing derivatives if you recall the relation to get the derivatives of shape function with respect x y.

We require to find what is one over determinant of j, so that is what this statement says mathematically the reason for this can be seen from the fact that Jacobian appears in the denominator in the equation for computing derivatives with respect to x and y. So, if that is that is equal to zero that is determinant of J is equal to zero anywhere over the domain of the parent element is s going from minus 1 to 1 t going from minus one to one than the derivative at that point cannot be evaluated and mapping is not useful because one over zero is infinity. So, we cannot find derivative at that point and mapping is going mapping is not going to be useful.

And we will illustrate these issues related to mapping by considering some numerical examples, now let us take an example numerical example check the validity of Isoparametric mapping for four node quadrilateral element shown here.

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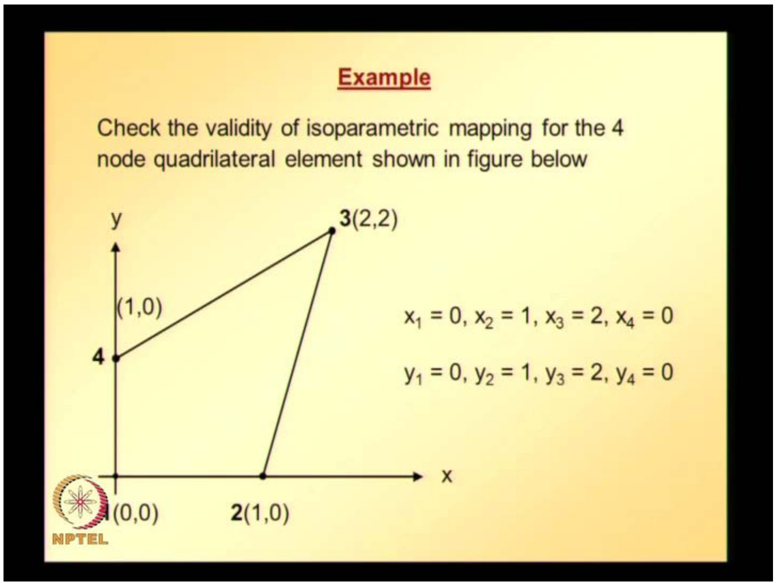

This is the four node quadrilateral element in the figure the nodal coordinates are given. And we can easily figure out what are x 1 to x 4 and y 1 to y 4 and before we proceed with writing the expressions for mapping, we need to know, what are the shape functions for a four node parent element in terms of s and t?

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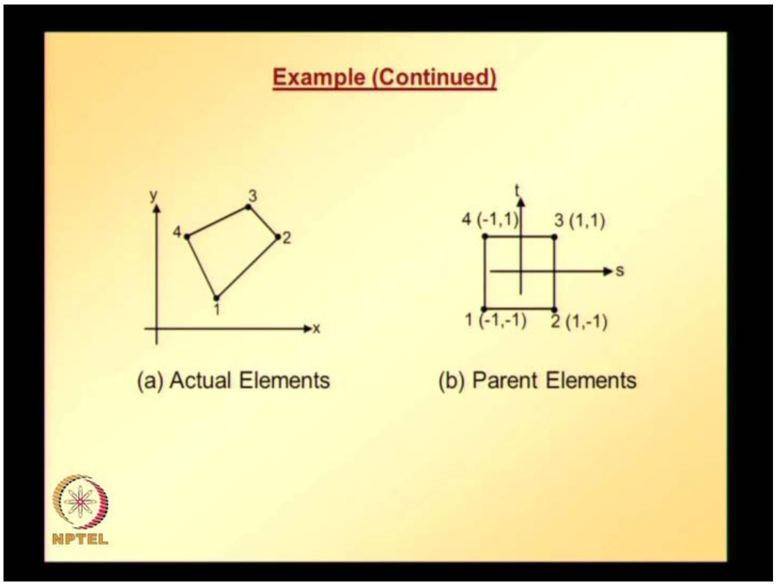

So, consider a four node quadrilateral element shown here in the first figure the coordinates of node one to node one or x 1 y 1 and for node two x 2 y 2 similarly, for other nodes. The square element shown in first figure or figure a is mapped on to the parent element, which is shown in figure b this square element so, this square element shown in figure b is used as a parent element for the actual element shown in figure a mapping from x y coordinates of the actual element to s and t coordinate is given by shape functions written in terms of parent element.

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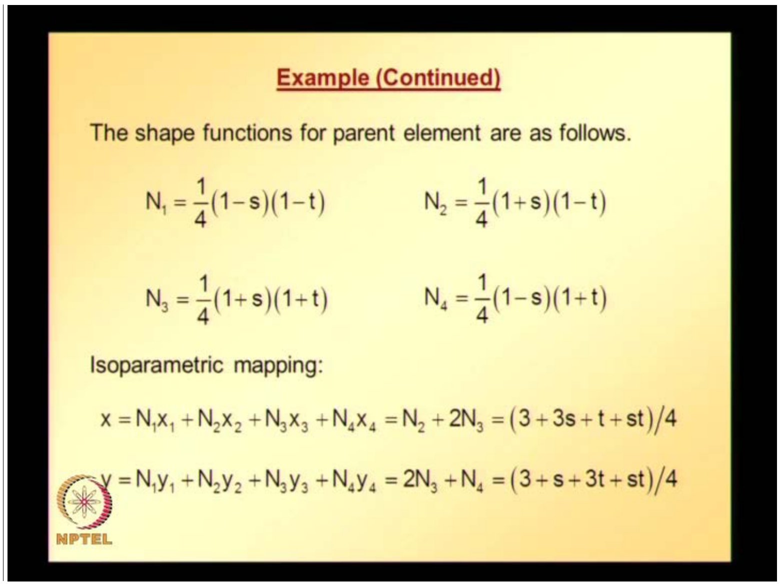

So, now going back to the problem that we are looking at the shape function for parent element are these N 1 to N 4 in terms of s and t and we already noted down what are x 1 to x 4 y 1 to y 4. So, we can easily write what are the expressions for Isoparametric mapping x how it is related to s and t is given by x is equal to N 1 x1 plus N 2 x 2 plus N 3 X 3 plus N 4 x 4 where N 1 to N 4 they are shape functions of the parent element which are given and x_1 to x_4 are the nodal coordinates of the actual element.

Similarly y is equal to N 1 y 1 plus N 2 y 2 plus N 3 y 3 plus N 4 y 4 where N 1 to N 4 or the parent element shape functions y 1 to y 4 are the actual element nodal coordinate substituting this information we get the relationship between y and s and t. So, once we get these two expressions x in terms of s and t y in terms of s and t we can easily take partial derivatives of x with respect to s partial derivative of x with respect to t. Similarly partial derivative of y with respect to s partial derivative of y with respect to t and find what is J and once we get j we can find what is determinant of J and one of the requirement is determinant of J should not be equal to 0 over the domain of the parent element which is going which goes from s minus one to one t going from minus one to one.

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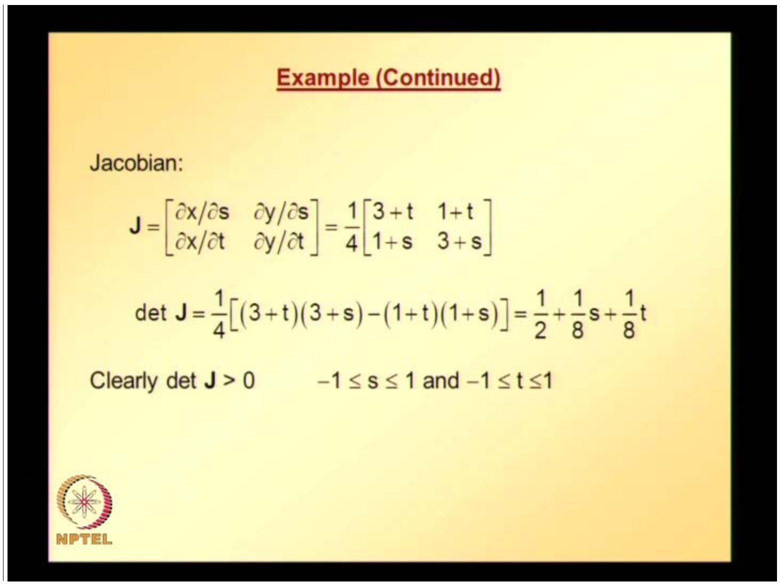

So, over this domain determinant of J should not be zero or clearly we can verify that determinant of J expression that we obtain we can easily check that is indeed is greater than zero for the domain of the parent element as going from minus one to one t going from minus one to one. So, this mapping is good.

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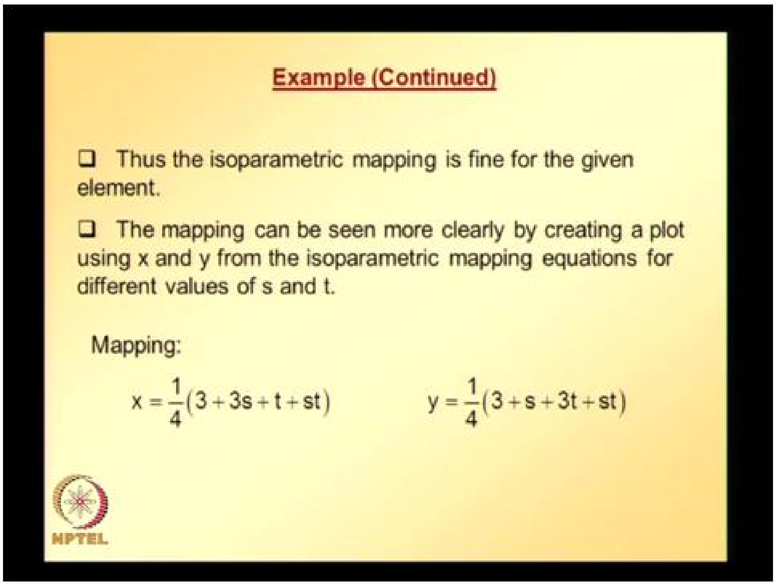

Thus isoparametric mapping is fine for the given element, and this is one way of checking mapping that is once determinant of j expression is obtained checking whether determinant of J is greater than zero over the parent element domain that is one way of checking whether mapping is fine or not. The other we have checking mapping is we obtain or we already have x in terms of s and t y in terms of s and t, we can plot this mapping can also be seen more clearly by creating a plot using x y from isoparametric mapping equations for different values of s and t.

So, we have x in terms of s and t, y in terms of s and t, we can plot this over the parent element domain as going from minus one to one, t going from minus one to one using some plotting software with little bit programming one can use mathematical or mat lab to plot this to get this plot. So, we have these expressions plot these expressions between s going from minus one to one, t going from minus one to one.

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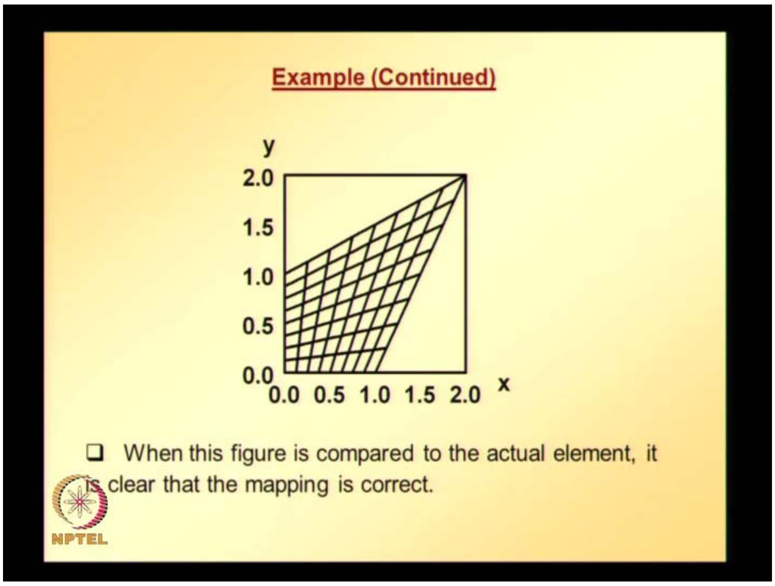

So, if you do that we get a plot like this. And here you can easily see that if you compare this plot with the actual element, this is exactly same as actual element. And when this figure is compared to the actual element, it clearly shows that mapping is correct. So, this is how one can check validity of Isoparametric mapping. So, we have taken a four nodded element and we verified for the element we have taken mapping turned out to be fine. But now let us take another element and see when mapping is going to be not good or when there is going to be some problem with mapping.

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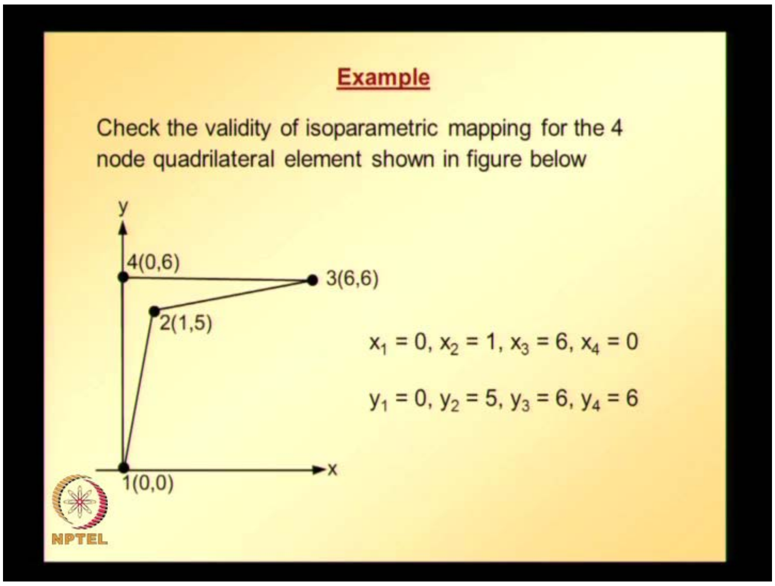

So, now let us take an example, check the validity of Isoparametric mapping for four note quadrilateral element, element is looking like this. So, before we proceed or we need to follow same steps as we did for the last example. So, before we proceed further we just make a note of all x coordinates and y coordinates of all the nodes of parent element x 1 to x 4, y 1 to y 4 and using parent element shape functions that is shape functions corresponding to a four node element in parent coordinate system s and t; and using these coordinates x y of all the nodes, we can write isoparametric mapping x in terms of s and t, y in terms of s and t.

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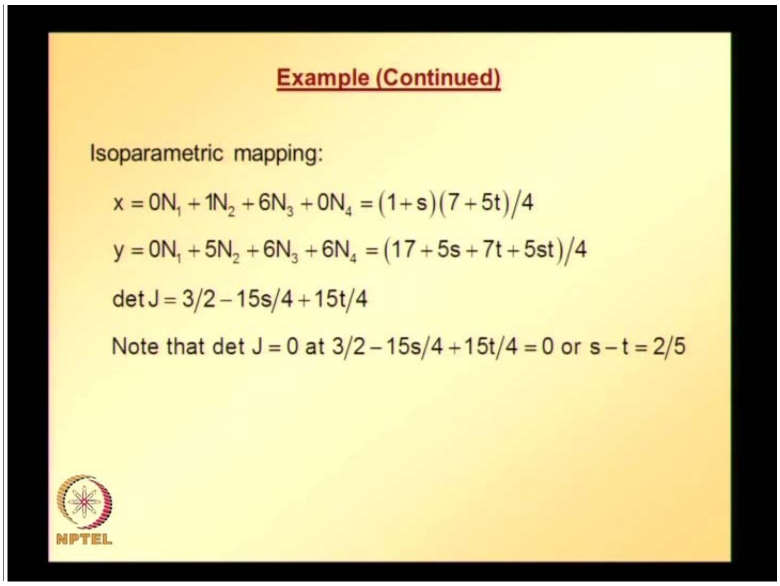

Once we get these two expressions we can easily take partial derivatives with respect to s and t, and we can find what is determinant of J and determinant of J is given by this. And now our job is to check whether determinant of J is greater than zero over the parent element domain as going from minus one to one t going from minus one to one. Instead of that, we can check when determinant is going to be zero determinant of J is going to be zero by looking carefully at this at this expression.

So, now let us equate determinant of J two zero note that determinant of J is equal to zero when three over two minus 15 s over four plus 15 t over four is equal to zero. So, that result in the line s minus t is equal to two over five. And if you plot, if you over lie this line that is s minus t is equal to two over five on the parent element, you can easily see that this line is going to cut across the domain of the parent element, which is going from s minus one to one, t from minus one to one. So you can easily verify when you plot this one we over lie this line over the parent element this line is going to cut across that particular element

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So, in that case at some point on that element or some point inside the element Jacobian or determinant of J is going to be zero. That is this is another way of checking earlier we checked whether determinant of J is greater than zero over the parent element domain and here we are actually checking, where determinant of J is equal to zero along what line and we are over line that line on the parent element domain and that is also another way of checking whether mapping is good or not.

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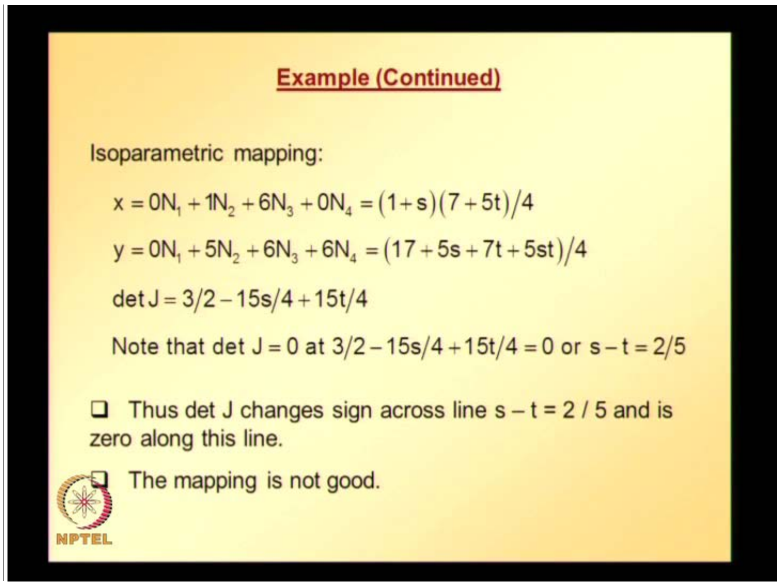

So, here in this case it turns out that mapping is not good because determinant of J changes sign across the line which is going to cut the element domain. So, it is going to be determinant of J is going to be zero along this line s minus t is equal to two over five. So, mapping is not good. And also we can plot this is one way of checking mapping that is using determinant of J the other way is we can plot x and y using mat lab or mathematical with little bit programming.

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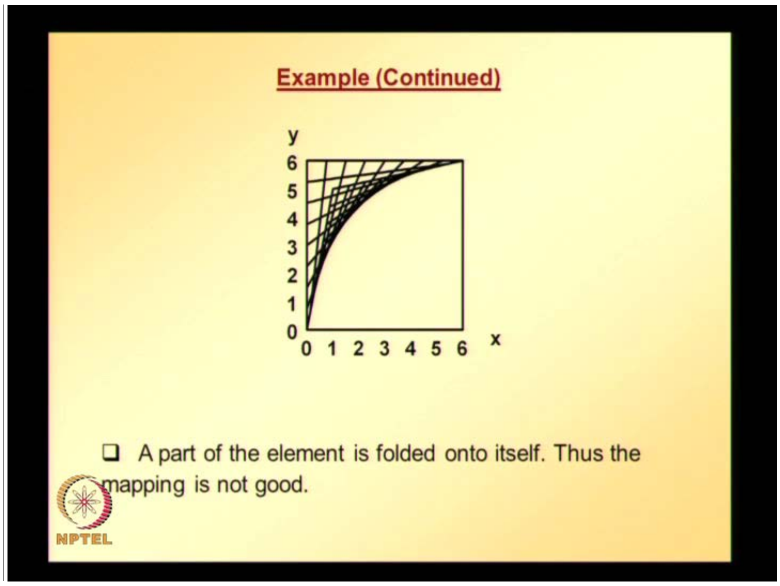

So, if you plot x and y over the element domain s going from minus one to one t going from minus one to one, you can easily see that a part of element is folded onto itself so, this kind of thing is not acceptable, because every point in the actual element maps are required to be mapped onto a unique element in the parent element, and when this kind of folding is happening that means, it is violating that requirement. So, mapping is not good so, this is another way of checking by plotting x and y with respect to s and t. So, this particular element mapping is not good.

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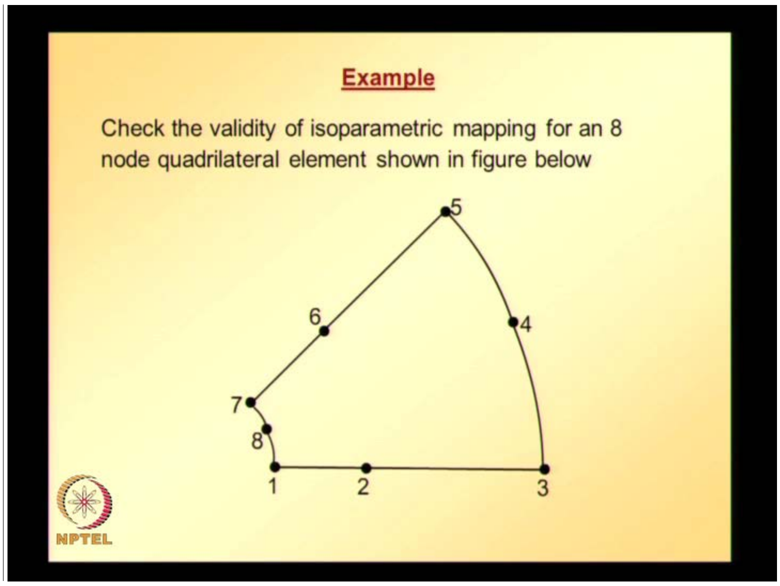

So, now let us see how to check validity of isoparametric mapping for eight node elements. This is the element, x y coordinates of all the nodes are given here they are put in a vector x n and y n all the x coordinates y coordinates.

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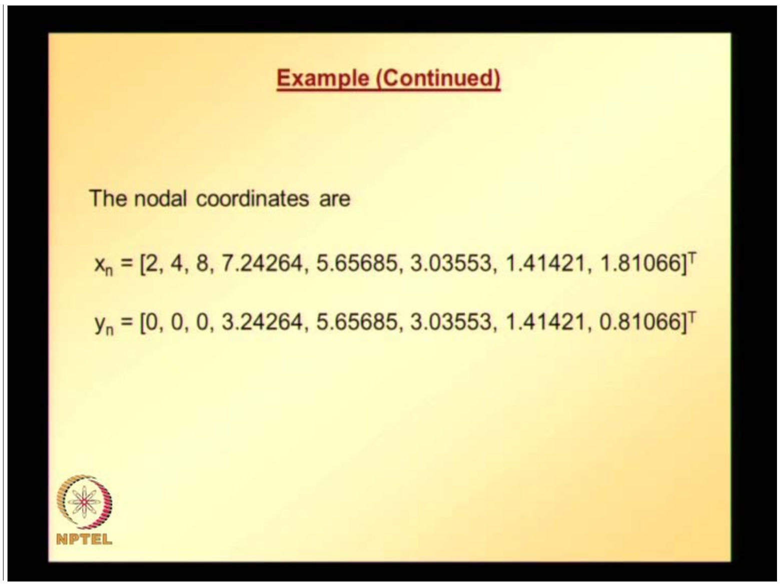

The procedure for checking or to obtain the isoparametric mapping remain same whether it is eight node element or four node element so same steps, which we followed for four node element; except that, now we need to take shape functions of eight node parent element before we proceed with writing isoparametric mapping expressions.

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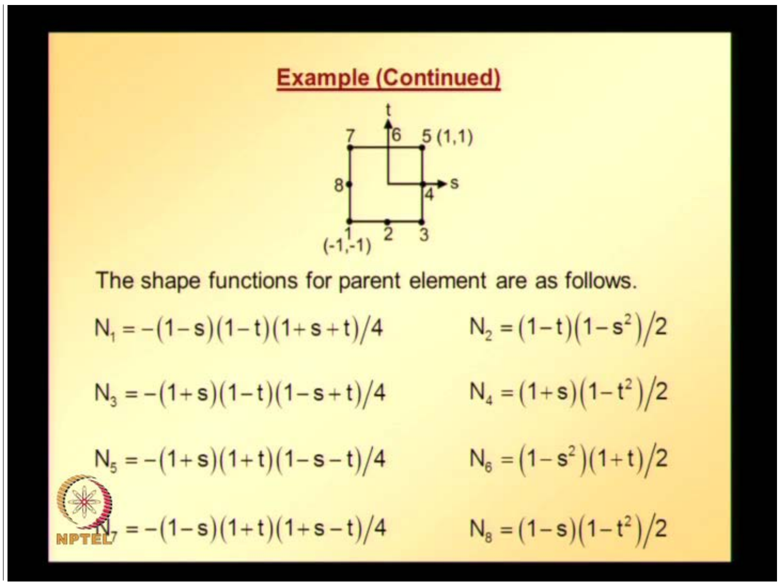

So, we need to know, what are the shape functions for eight node parent element shape function expressions we need to know and how can we get these you just go back to the table, where shape functions for four to nine node element are given. So, from there, you can easily get this information the shape function for this eight node parent element in terms of s and t. So, we have all the expressions for the shape functions we also know the coordinates x y coordinates of the actual element. So, we can easily use this information both the information and we can write expressions for isoparametric mapping.

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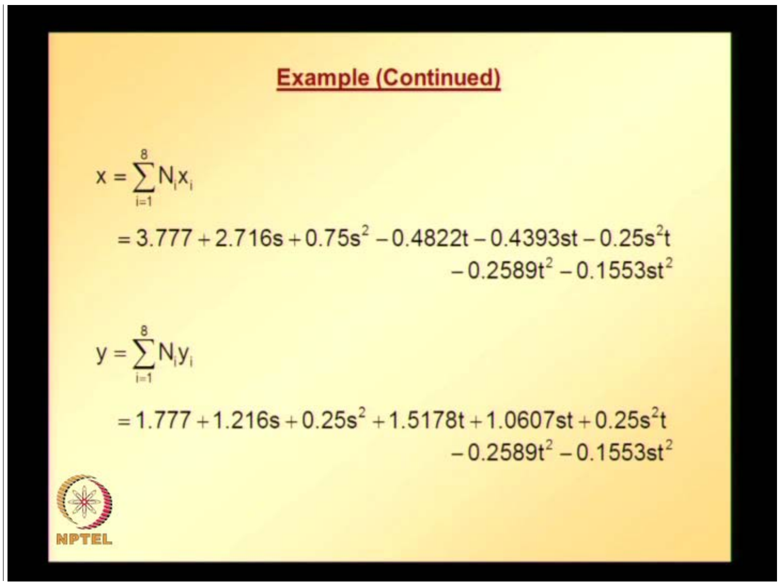

x is equal to sigma N i x i. i goes from 1 to 8 and substituting N 1 to N 8 x 1 to x 8 and simplifying we get this expression; similarly, y in terms of s and t. So, we got x in terms of s and t y in terms of s and t these are the expressions for Isoparametric mapping once we obtain these two expressions, we can easily take partial derivatives of x with respect s and t, y with respect s and t. And we can easily find what is determinant of J following, same procedure as what we did for four node element by evaluating derivatives the Jacobian matrix can easily be written and once we simplify we get this expression.

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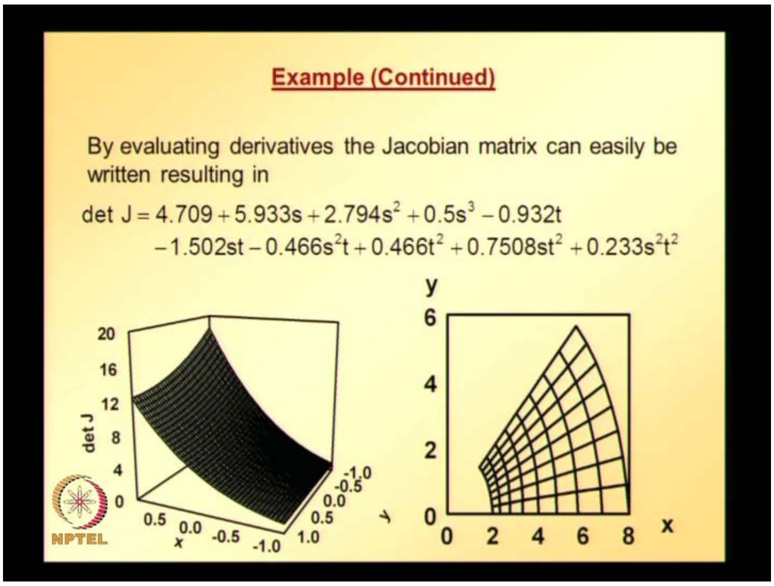

Now we can see determinant of J expression is looking more complicated. So, in this case, it is better we plot this with s going from minus 1 to 1, t going from minus 1 to 1. A surface plot of Jacobian can be plotted with respect to s and t with s going from minus one to one t going from minus one to one. So, this is surface plot of Jacobian and it can be easily verified from the plot that determinant of J is greater than zero over the entire parent element domain that is s going from minus one to one, t going from minus one to one.

So, mapping is fine. In the other way of checking is we can also plot x y because we know we obtained already expressions of x and y with respect s and t. So, we can plot that and if you compare this figure with the actual element we can see, this is a similar to actual element or exactly looking same as actual element so, mapping is fine. So, we have two evidences one is determinant of J greater than zero and the other one is plot of x y, so both these evidence show that mapping is good.

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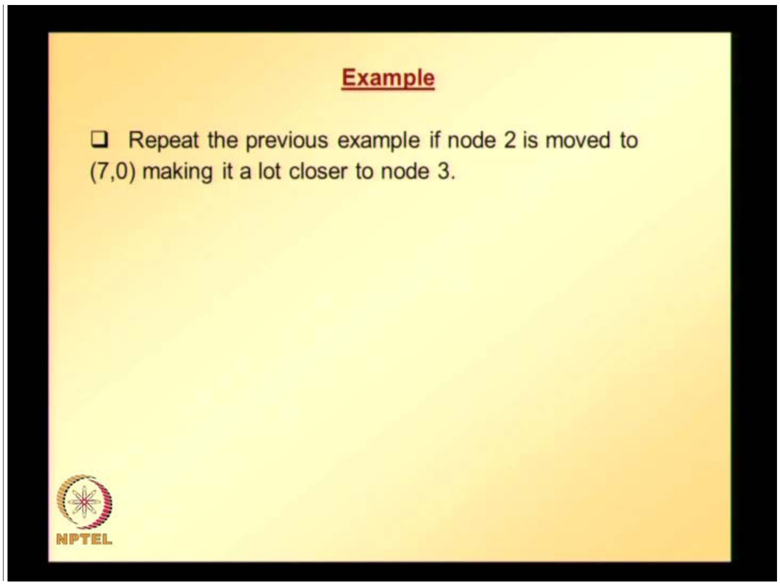

So, now let us see what happens if we tweak this element little bit. So, the next example is repeat the previous example if node two is moved to location seven zero that is x coordinate is seven y coordinate is zero making it making node two closer to node three. So, whatever the vectors of x coordinates and y coordinates, we have in that earlier node two is located at four zero location now it is moved to seven zero location (Refer Slide Time: 45:50) except that all the other nodes are the locations of other nodes remain same.

So, the nodal coordinates get modified like this, so with this change, we can repeat the entire process following same steps as those in previous example determinant of J Jacobian matrix, which is determinant of J is given by this one. So, same procedure what we need to do is we need to plot this determinant of J with s going from minus one to one, t going from minus one to one or we can plot x y, because we already know x y in terms of s and t with s going from minus one to one, t going from minus one to one.

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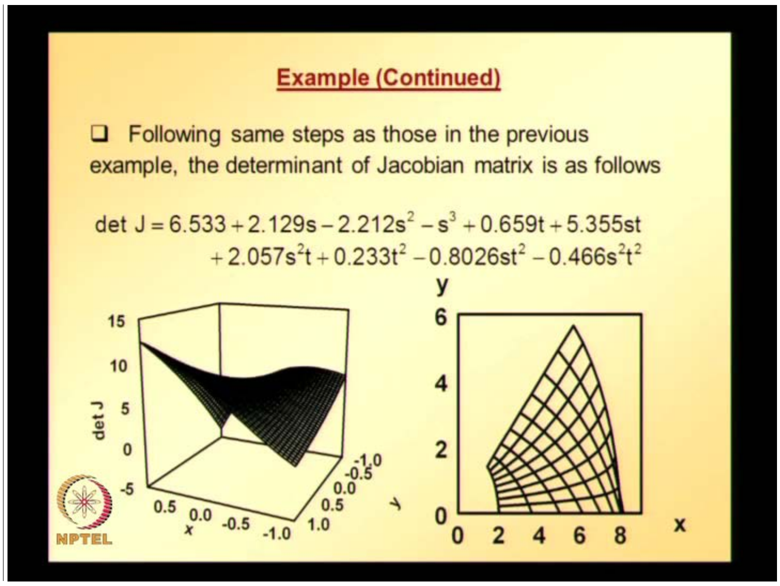

So, determinant of J plot looks like this. You can see from this plot, determinant of J is changing sign from positive to negative, so at some location over the domain of element determinant of J is going to be zero is something is changing from positive to negative or negative to positive over a particular element domain that means, at some point in that element that particular quantity takes value equal to zero.

(Refer Slide Time: 46:53) So, this plot clearly shows that there is a change in sign over the parent element domain so, the mapping is not good and same can also be verified by plotting x y in terms of s and t and if you see in the right hand side bottom corner if you see at where x is almost equal to eight you can see a fold there.

A fold is can be observed so, that shows that mapping is not good the plot of element produced from mapping clearly shows that there is some problem with mapping. So, here wherever mapping is not good that is we have taken two sets of examples; one is corresponding to four node element, one is corresponding to four node quadrilateral element, the other set of examples corresponds to eight node element. And we have seen with one particular placement of nodes mapping is good; and if I change the nodal location little bit, a mapping there is a problem with mapping is observed. Similarly, when a four nodded element is if the sides of that particular elements or the angle between the sides of elements is very small or very large also we have seen, there is some problem with mapping (Refer Slide Time: 46:53).

So, in the next class, we will see, what are the guidelines for proper modeling with Isoparametric elements; guidelines for placement of nodes and also angle between the edges of element.