

Finite Element Analysis
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Lecture No. # 24

So in today's class, we will look at quadrilateral elements; and we will first look at derivation of shape functions for both rectangular elements and square elements with the different number of nodes; and then we look at similar to 1-D elements. We will look at isoparametric mapping concepts for quadrilateral element and also its limitation, and also we will discuss about numerical integration and **two-dimensional** for two-dimensional elements, and also derivation of element equations for two-dimensional boundary value problem using quadrilateral element with different number of nodes.

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ISOPARAMETRIC QUADRILATERAL ELEMENTS

- Higher order elements and elements with curved boundaries are effective when good approximate solutions are required with relatively few elements.
- Theoretically shape functions for any of these elements can be developed by starting from a polynomial of an appropriate degree and then expressing the coefficients in the polynomial in terms of nodal parameters.
- This was the procedure used in the development of the linear triangular element.
- This approach, however, becomes tedious and impractical for higher order elements.



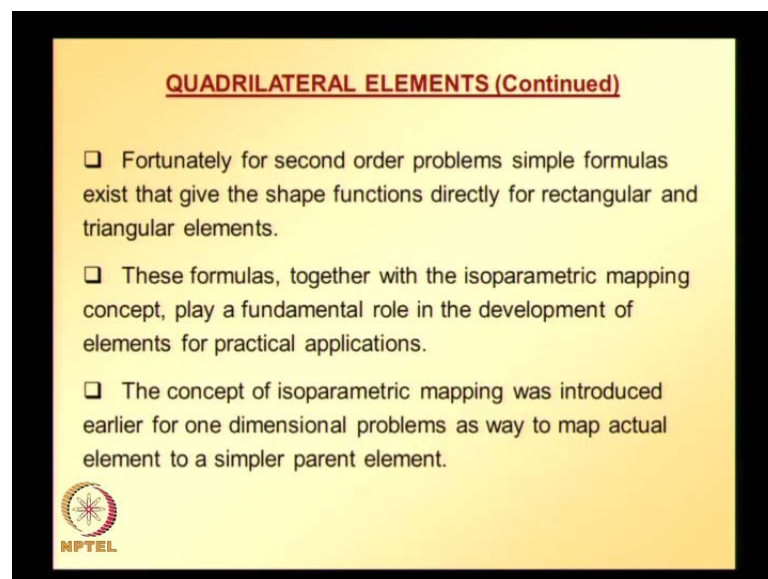
So, now higher order elements and elements with curved boundaries are effective, when good approximate solutions are required with relatively few elements. So, that is the basic idea behind these quadrilateral elements. Theoretically shape functions for any of these elements can be developed by starting from a polynomial of appropriate degree, and then expressing the coefficients in the polynomial in terms of nodal parameters or

the nodal values, similar to the way we did for one-dimensional elements both 2 node elements and 3 node elements and also similar kind of approach we also adopted for deriving shape functions for 3 node triangular elements which are linear.

This **this** was the procedure used in the development of linear triangular element. This approach however, becomes tedious and impractical for higher order elements that is starting from a polynomial of appropriate degree and then trying to find the coefficients of this polynomial by substituting the nodal values or nodal parameters and corresponding nodal coordinates and solving these coefficients and substituting back these coefficients in to the polynomial and grouping terms containing nodal parameters common nodal parameters.


So, that is how we derived for that is how we derived shape functions for 1-D elements and also linear triangular element, but that approach becomes tedious or impractical for higher order elements that it as the number of nodes for a particular element increases we need to choose a polynomial having as many number of coefficients as the number of nodes for that particular elements. So, solving these coefficients and substituting back and grouping terms having the nodal parameters common nodal parameters becomes tedious for higher order elements.

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QUADRILATERAL ELEMENTS (Continued)

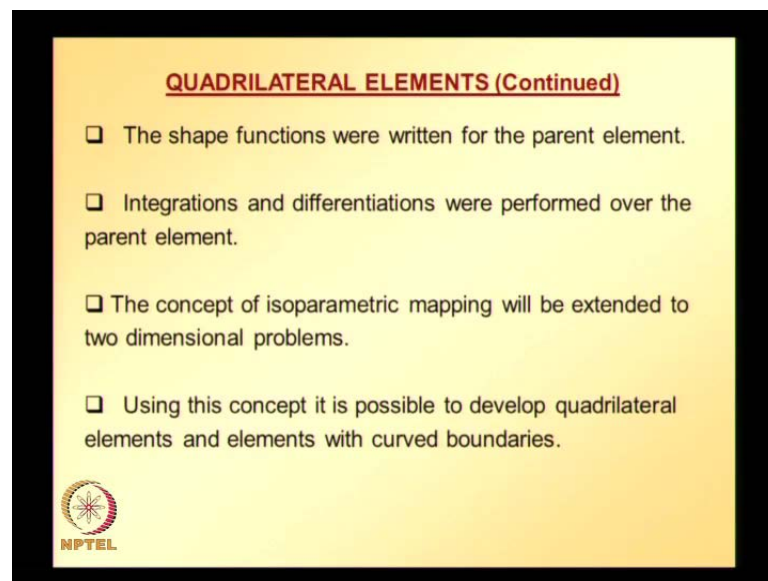
- ❑ Fortunately for second order problems simple formulas exist that give the shape functions directly for rectangular and triangular elements.
- ❑ These formulas, together with the isoparametric mapping concept, play a fundamental role in the development of elements for practical applications.
- ❑ The concept of isoparametric mapping was introduced earlier for one dimensional problems as way to map actual element to a simpler parent element.


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Fortunately, for second order problems simple formulas exist that give shape functions directly for rectangular and triangular elements. So, we will be discussing some of these approaches how to get these simple formulas in this class. So, this lecture presents shape function formulas for higher order rectangular and triangular elements and these formulas together with isoparametric mapping concept play a fundamental role in **development of shape** development of elements for practical applications, because these higher order elements are element with curved boundaries are really required for solving some of the practical problems.


The concept of isoparametric mapping was introduced earlier for one-dimensional problem as a way to map actual element to simpler parent element, basically **this is** this was done for one-dimensional elements to integrate some of the matrices and vectors that we get.

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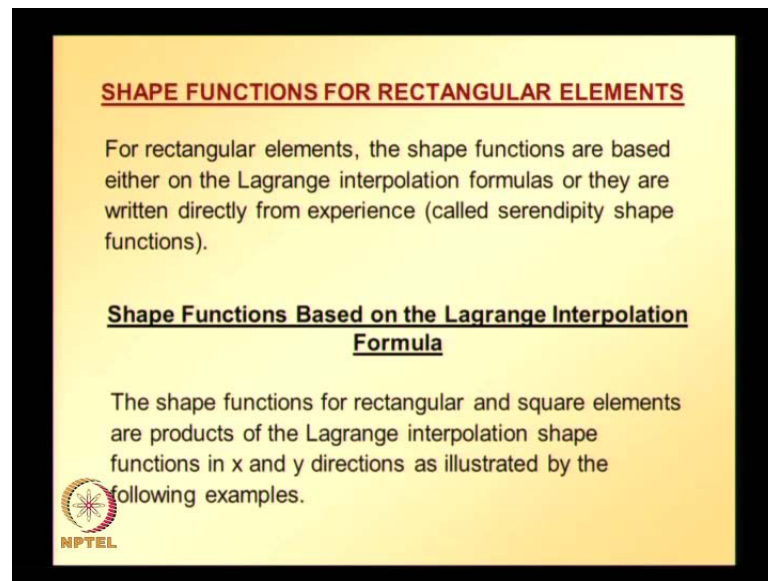
QUADRILATERAL ELEMENTS (Continued)

- The shape functions were written for the parent element.
- Integrations and differentiations were performed over the parent element.
- The concept of isoparametric mapping will be extended to two dimensional problems.
- Using this concept it is possible to develop quadrilateral elements and elements with curved boundaries.


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Shape functions this is how we did; shape functions were written for the parent element. Integrations and differentiations were performed over parent element. So, in this lecture, the concept of isoparametric mapping will be extended to two-dimensional problems. Using this concept, it is possible to develop quadrilateral elements and element with curved boundaries.

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SHAPE FUNCTIONS FOR RECTANGULAR ELEMENTS

For rectangular elements, the shape functions are based either on the Lagrange interpolation formulas or they are written directly from experience (called serendipity shape functions).

Shape Functions Based on the Lagrange Interpolation Formula

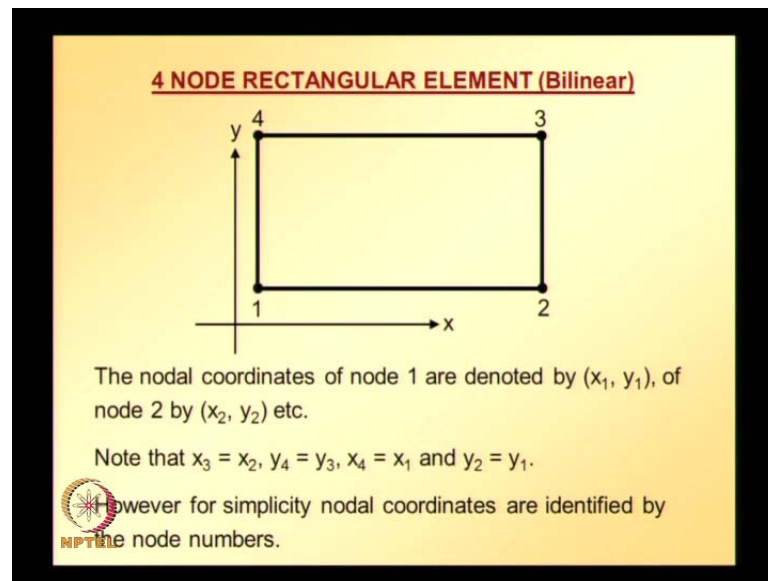
The shape functions for rectangular and square elements are products of the Lagrange interpolation shape functions in x and y directions as illustrated by the following examples.



So, now let us start with derivation of shape functions for rectangular elements. For rectangular elements, the shape functions are based either on Lagrange interpolation formula or they are written directly from experience. The elements which can be the shape functions of which can be obtained using Lagrange interpolation formula are classified as Lagrange elements, and the other elements they are classified as serendipity elements.

So, now shape functions based on Lagrange interpolation formula. The shape functions for rectangular and square elements are products of Lagrange interpolation shape functions in x and y directions as illustrated in the following examples. So, basically we need to write **Lagrange interpolation formula** Lagrange interpolation shape function in x direction, Lagrange interpolation shape function in the y direction, multiply these two, then we get the shape function for the particular rectangular or square element. This is how the procedure goes based on Lagrange interpolation formula.

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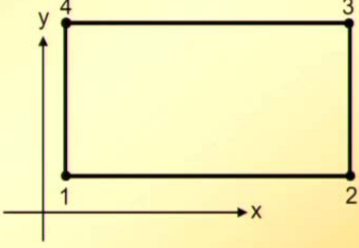


So, now let us take a 4 node rectangular element, A 4 node rectangular element is shown in the figure. The coordinates of node 1 are denoted by x_1, y_1 and of node 2 are denoted by x_2, y_2 , similarly node 3, node 4 etcetera and also note that for this particular element that is shown in the figure x coordinate of node 2 is same as x coordinate of node 3. Similarly, y coordinate of node 4 is same as y coordinate of node 3. Similarly, x coordinate of node 4 is same as x coordinate of node 1 and y coordinate of node 2 is same as y coordinate of node 1. So nodes can be denoted using the coordinates or for simplicity nodal coordinates are identified by the node numbers.


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4 NODE RECTANGULAR ELEMENT (Continued)

If $T(x, y)$ is the field variable and T_1 etc. are the nodal variables, then the trial solution in terms of shape functions is expressed as follows.



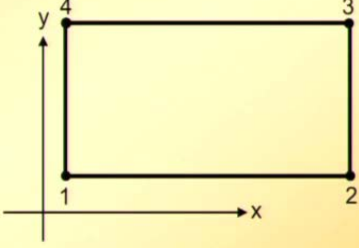
Along 1-2 $y = y_1$ (constant), therefore shape functions must be functions of x only.

 $T_1(x, y_1) = [n_1(x) \quad n_2(x)] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$


So, now let us see how to derive shape functions for this particular element. If T is the field variable and T_1, T_2, T_3, T_4 etcetera are the nodal variables, then the trial solutions in terms of shape functions is expressed as; first let us see only **along line 1 2** along 1 2 you can see from the figure y is going to be constant, y is equal to y_1 ; therefore shape functions must be function of x only. So, now we are going to write shape functions along line 1 2. So, that is denoted with T_1 that is field variable variation along line 1 2 is denoted with T_1 , and it is going to be function of x alone n_1, n_2 are going to be Lagrange interpolation functions T_1, T_2 are the field variable values at node 1 and node 2 and from the knowledge of one-dimensional elements we already know how to get n_1 and n_2 .

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Using one dimensional Lagrange interpolation formula

$$n_1(x) = \frac{x - x_2}{x_1 - x_2} \qquad n_2(x) = \frac{x - x_1}{x_2 - x_1}$$


Along 4-3 $y = y_3 = y_4$

 $T_{II}(x, y_3) = [n_4(x) \quad n_3(x)] \begin{Bmatrix} T_4 \\ T_3 \end{Bmatrix}$

Using one-dimensional Lagrange interpolation formula, we know n_1 is equal to this n_2 is equal to the value or the quantity that is given there. So, we know how the field variable is varying along 1-2. Now, let us look at alongside 4-3 and from the figure it can be easily noticed that y value along 4-3 lie is equal to y_3 is equal to y_4 and field variable along 4-3 is denoted with T , roman letter II and that can be written in terms of shape functions of node 4 and node 3 in the manner that is shown there, that is T_2 is equal to n_4 times T_4 plus n_3 times T_3 , which can be written in matrix and vector form in the manner that is shown.

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4 NODE RECTANGULAR ELEMENT (Continued)

Again from one dimensional Lagrange interpolation formula

$$n_4(x) = \frac{x - x_3}{x_4 - x_3} \quad n_3(x) = \frac{x - x_4}{x_3 - x_4}$$

In the y direction

$$T(x, y) = [n_1(y) \quad n_4(y)] \begin{Bmatrix} T_1(x, y_1) \\ T_2(x, y_2) \end{Bmatrix}$$

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Again, from one-dimensional Lagrange interpolation formula T n_4 and n_3 can be written like this. So, we have seen how the field variable T is varying alongside 1-2 and also alongside 4-3. Now, let us look at take one of the sides, which is along y direction in y direction, so along 1-2, we know T_1 and along 4-3, we know T_2 from the previous equations. So, **you** once we know the value of field variable alongside 1-2 and 4-3 in the y direction variation of T in the y direction that is alongside 1-4 or 2-3 can be written as T is equal to $n_1 T_1$ plus $n_4 T_2$, which can be written in matrix and vector form in the way that is shown there.

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4 NODE RECTANGULAR ELEMENT (Continued)

where $n_1(y) = \frac{y-y_4}{y_1-y_4}$ $n_4(y) = \frac{y-y_1}{y_4-y_1}$

The expressions for sides 1-2 and 4-3 can be written in the matrix form

$$\begin{Bmatrix} T_I(x, y_1) \\ T_{II}(x, y_2) \end{Bmatrix} = \begin{bmatrix} n_1(x) & n_2(x) & 0 & 0 \\ 0 & 0 & n_3(x) & n_4(x) \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$

Therefore

$$T(x, y) = \begin{bmatrix} n_1(y) & n_4(y) \end{bmatrix} \begin{bmatrix} n_1(x) & n_2(x) & 0 & 0 \\ 0 & 0 & n_3(x) & n_4(x) \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$


And now substituting T_1 T_2 and n_1 n_4 can be obtained by writing one-dimensional Lagrange interpolation in y direction, so that is how n_1 and n_4 are obtained. Expressions for side 1-2 and 4-3 can be written in matrix form, that is T_I T_{II} are written together in a matrix and vector form. So, substituting T_I T_{II} vector in to the previous equation we get this one. So, carrying out multiplications of n_1 n_4 vector with the matrix containing n_1 n_2 0 0 0 0 n_3 n_4 .


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4 NODE RECTANGULAR ELEMENT (Continued)

or

$$T(x, y) = \begin{bmatrix} n_1(x)n_1(y) & n_1(y)n_2(x) & n_4(y)n_3(x) & n_4(x)n_4(y) \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$

or

$$T(x, y) = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$


We get this which can be compactly written like this. So capital N 1 is defined as small n 1 as a function of x times small n 1 as a function of y, similarly capital n 2, which is shape function corresponding to node 2 is equal to small n 1 as a function of y times small n 2 as a function of x. Similarly, shape function of node 3, which is denoted with capital N 3 it is equal to small n 4 as a function of y times small n 3 as a function of x. Similarly, shape function at node 4, which is denoted with capital N 4 is equal to small n 4 as a function of x times small n 4 as a function of y. So we can write what is capital N 1, capital N 2, capital N 3.

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4 NODE RECTANGULAR ELEMENT (Continued)

where

$$N_1(x, y) \equiv n_1(x)n_1(y) = \frac{x-x_2}{x_1-x_2} \frac{y-y_4}{y_1-y_4}$$

$$N_2(x, y) \equiv n_2(x)n_1(y) = n_2(x)n_2(y) = \frac{x-x_1}{x_2-x_1} \frac{y-y_3}{y_2-y_3}$$

$$N_3(x, y) \equiv n_3(x)n_4(y) = n_3(x)n_3(y) = \frac{x-x_4}{x_3-x_4} \frac{y-y_2}{y_3-y_2}$$

$$N_4(x, y) \equiv n_4(x)n_4(y) = \frac{x-x_3}{x_4-x_3} \frac{y-y_1}{y_4-y_1}$$


And substituting what is small n 1 as a function of x small n 1 as a function of y we get N 1 like this, which is basically derived based on Lagrange interpolation formula in x direction multiplied by Lagrange interpolation formula in y direction at node 1. Similarly, shape function of node 2, shape function of node 3 and shape function of node 4. It can be easily observed that all of these shape functions, let say N 1 N 1 is going to be equal to 1 at x is equal at x is equal to x 1 and y is equal to y 1 and it is going to be equal to zero at all other locations. Similarly, N 2 is going to be 1 at x is equal to x 2 y is equal to y 2 and it is going to be zero at other nodal position. Similarly, N 3 and N 4 are going to be N 3 is going to be equal to 1 at node 3 and it is going to be equal to zero at rest of the nodes. Similarly, N 4 is going to be 1 at node 4 and it is going to be equal to zero at rest of the nodes.

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4 NODE RECTANGULAR ELEMENT (Continued)

Thus shape functions for rectangular elements are product of Lagrange interpolations in the two coordinate directions. Note that

$$N_1(x,y) = \frac{x-x_2}{x_1-x_2} \frac{y-y_4}{y_1-y_4}$$

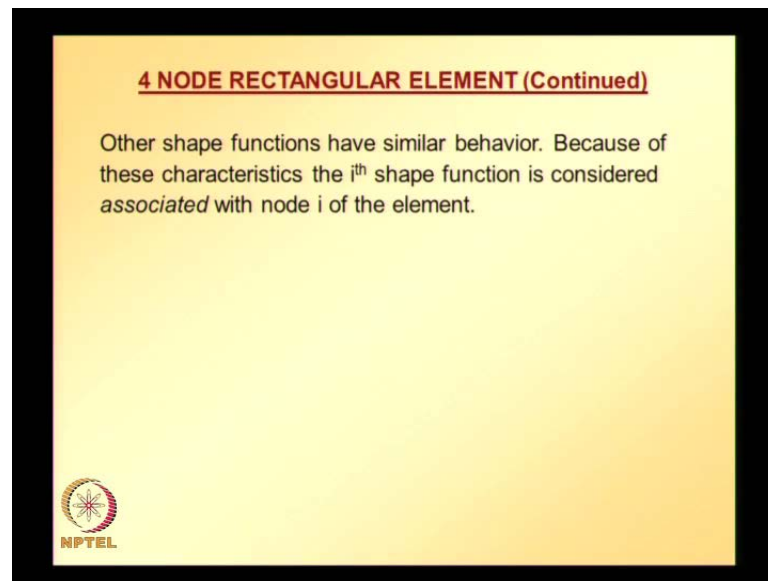
is :

- 1 at node 1
- 0 at other nodes
- Linear function of x along 1-2 and linear function of y along 1-4
- Zero along 2-3 and 3-4




So, the shape functions for rectangular elements are product of Lagrange interpolations in two coordinate directions. So, that is how we derived and this N_1 and note that this is equal to 1 at node 1 and zero at other nodes and it is linear function of x along 1-2 side 1-2 and linear function of y alongside 1-4 and 0 alongside 2-3 and 3-4, because node 1 is not part of side 2-3 and 3-4, so it is going to be shape function of node 1 is going to be zero alongside 2-3 and 3-4. So, these properties not only node shape function of node 1, but other shape functions of other nodes also satisfy these properties. So, shape function of a node it is going to be zero alongside to which it is not going to be part and shape function of a particular node is going to be equal to 1 at its own location and it is going to be equal to 0, at all other locations, at all other nodal locations.

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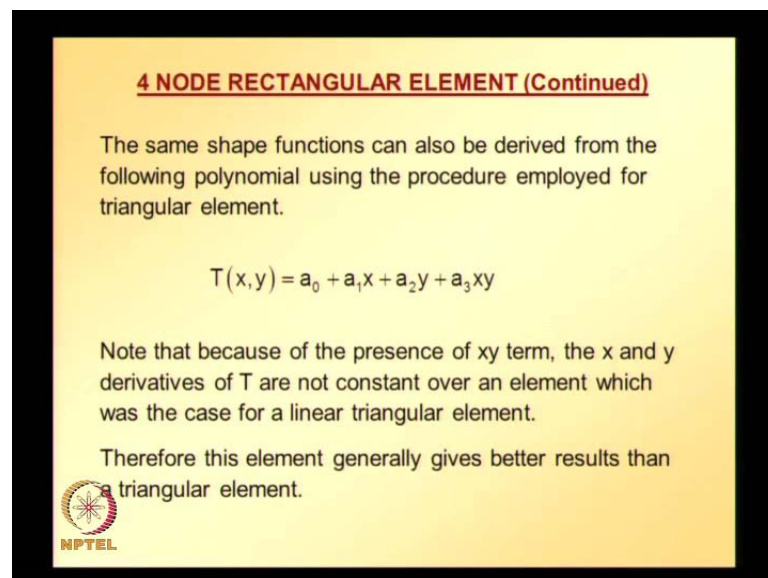
4 NODE RECTANGULAR ELEMENT (Continued)

Other shape functions have similar behavior. Because of these characteristics the i^{th} shape function is considered *associated* with node i of the element.



So, other shape functions have similar behavior, because of these characteristics the i^{th} shape function is considered associated with node i of the element.

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4 NODE RECTANGULAR ELEMENT (Continued)

The same shape functions can also be derived from the following polynomial using the procedure employed for triangular element.

$$T(x, y) = a_0 + a_1x + a_2y + a_3xy$$

Note that because of the presence of xy term, the x and y derivatives of T are not constant over an element which was the case for a linear triangular element.

Therefore this element generally gives better results than a triangular element.

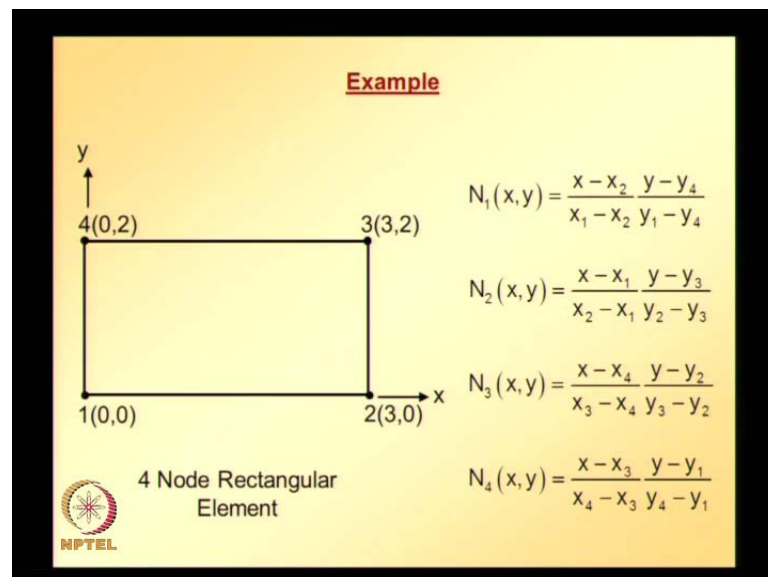


And the shape functions that we derived based on Lagrange interpolation formula the same shape functions can also be derived **from the** from starting with a polynomial and that polynomial is given here. So, starting with this polynomial we can derive same shape functions, similar to the procedure that we adopted for triangular element. And if you see this polynomial note that because of presence of term $x y$, the x and y derivatives

of T are not constant and if you recall shape function are the shape function derivatives of shape functions of a linear triangular element of constant and also derivatives of T field variable T are also constant for linear triangular element, because if you recall the polynomial that we used for deriving shape functions for linear triangular element do not contain this x y term.

But now for this 4 node quadrilateral element one way is one way of deriving shape function is starting with a Lagrange interpolation formula or the other way is by starting from a polynomial like this and if you start with polynomial like this you can see there is a presence of this x y term. Basically, please note that this 1 x y x y all these terms are coming from Pascal's triangle, so because of the presence of this x y term in this expression the derivatives of T with respect x and y are not constant, which was the case for triangular element linear triangular element. So, therefore this element generally gives better results than a triangular element. So, we have seen how to derive shape functions for a 4 node rectangular element using Lagrange interpolation formula.

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So, we can solve an example like this for shape functions a 4 node rectangular element is shown all the coordinates of all the nodes are also shown x y coordinate system x y axis are also indicated clearly in the figure and these are the shape function expressions that we derived, so now if somebody is interested in writing shape functions for each of these nodes N 1 to N 4, simply we need to plug in the corresponding coordinates

coordinate values into these expressions for N 1, N 2, N 3, and N 4. So x 1 is equal to 0, y 1 is equal to 0, x 2 is equal to 3, y 2 is equal to 0, x 3 is equal to 3, y 3 is equal to 2, x 4 is equal to 0, y 4 is equal to 2 substituting these quantities in to N 1, N 2, N 3, and N 4 expressions, we can get the shape functions.

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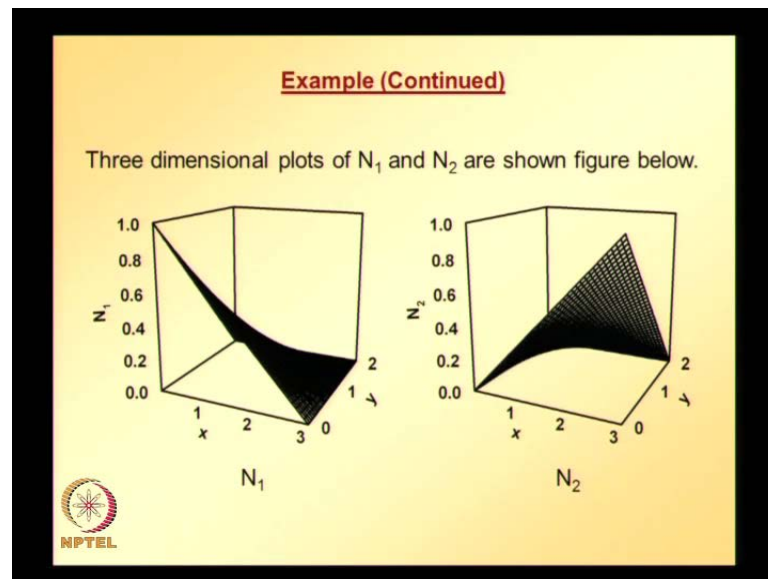
Example (Continued)

Substituting the numerical values of nodal coordinates into the above shape function formulas, the explicit expressions for shape function for this rectangular element are as follows.

$$N_1 = \frac{(-3+x)(-2+y)}{6} \qquad N_2 = \frac{-x(-2+y)}{6}$$
$$N_3 = \frac{xy}{6} \qquad N_4 = \frac{-y(-3+x)}{6}$$


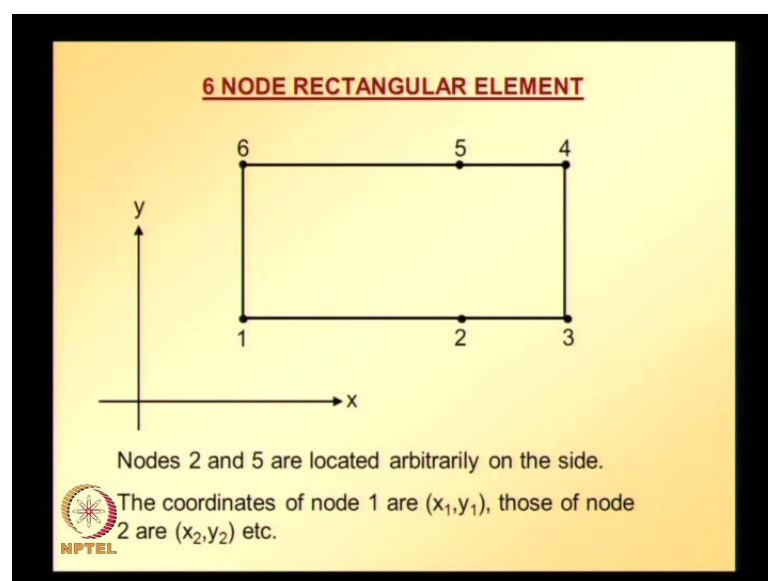
Substituting the numerical values of nodal coordinates in to the above shape function formulas, the explicit expressions for shape function for this rectangular element are as follows. So, this is N 1 after simplification N 2, N 3 and N 4. To visualize how the shape functions varies as a function of x and y over the domain of that particular element we can actually plot N 1, N 2, N 3 and N 4 as a function of x and y with x varying from 0 to 3 and y varying from zero to 2.

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So, similar kinds of plots are shown here for N_1 and N_2 , three-dimensional plots of N_1 and N_2 are shown and these plots can be obtained using any of the commercial software like MATLAB or Mathematica by just giving the expression for shape function and also the range over which plot is required that is x going from 0 to 3 y going from 0 to 2. So, this is how we can derive shape functions for 4 node rectangular element using Lagrange interpolation formula.

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So, now let us take a 6 node rectangular element like this and here also shape functions can be written in the manner are following the procedure that we adopted for 4 node rectangular element writing shape function expressions along x direction and shape function expressions along y direction multiplying both we get shape functions of each of the nodes. So, the coordinates of node 1 are x_1, y_1 , and those of node 2 are x_2, y_2 , similarly other nodes, and you can see here in this 6 node rectangular element, which is shown node 2 and node 5 are interior to side 1 2 and side 6 4 and these nodes 2 and 5 are located arbitrarily on the side 1 3 and 6 4.


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6 NODE RECTANGULAR ELEMENT (Continued)

Following same reasoning as for a four node element, it is obvious that the shape functions have quadratic variation in x and linear variation in y directions.

$$N_1(x, y) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} \times \frac{y - y_6}{y_1 - y_6}$$

$$N_2(x, y) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} \times \frac{y - y_5}{y_2 - y_5}$$

$$N_3(x, y) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} \times \frac{y - y_4}{y_3 - y_4}$$


So, now let us write the shape functions for all the nodes of this six node rectangular element. **Following same reason** following same reasoning as for four node element it is obvious that shape functions have quadratic variation in x direction and linear variation in y direction. If you see this six node rectangular element alongside 1 3 we have three nodes alongside 1 6 we have only two nodes, so two nodes in y direction gives us linear variation in y direction three nodes in x direction gives quadratic variation in x direction.

So, following the procedure that we adopted for rectangular a four node rectangular element, we can derive in a similar manner shape functions for all six nodes of this particular element. Here N 1 is shown N 1 the shape function of node one is Lagrange interpolation formula in x direction, which is going to be quadratic, because there are three nodes in x direction times Lagrange interpolation in y direction, which is going to

be linear, because there are two nodes in y direction. The product of those two gives us shape function for node 1.


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$$N_4(x, y) = \frac{(x - x_6)(x - x_5)}{(x_4 - x_6)(x_4 - x_5)} \times \frac{y - y_3}{y_4 - y_3}$$

$$N_5(x, y) = \frac{(x - x_6)(x - x_4)}{(x_5 - x_6)(x_5 - x_4)} \times \frac{y - y_2}{y_5 - y_2}$$

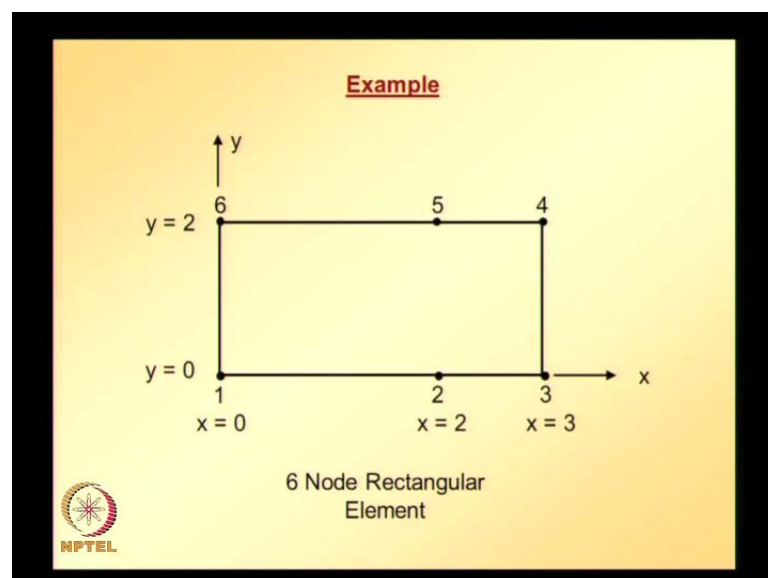
$$N_6(x, y) = \frac{(x - x_5)(x - x_4)}{(x_6 - x_5)(x_6 - x_4)} \times \frac{y - y_1}{y_6 - y_1}$$

With these shape functions, the trial solution is

$$T(x, y) = [N_1(x, y) \quad N_2(x, y) \quad \dots \quad N_6(x, y)] \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix}$$


Similarly, node 2, node 3, node 4, node 5 and node 6 and once we have all the shape functions the trial solution can be written like this T is equal to N 1 times T 1 plus, N 2 times T 2 plus, N 3 times T 3 plus, N 4 times T 4 plus, N 5 times T 5 plus, N 6 times T 6, which can be written in a matrix and vector form the way that is shown.

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So, now let us take an **example** numerical example with all the coordinate values given. So, here in the figure a 6 node rectangular element is shown, **x y coordinates of all nodes are also can be easily obtained** x y coordinates of all the nodes can also be easily obtained using the information that is given in figure that is x 1 is equal to 0, y 1 is equal to 0, x 2 is equal to 2, y 2 is equal to 0, x 3 is equal to 3, y 3 is equal to zero and x 4 is going to be 3 and y 4 is going to be 2, x 5 is going to be 2, y 5 is going to be 2, x 6 is going to be 0, y 6 is going to be 2, so with all this information **we can** what we can do is we can plug in these coordinates of these nodes in to the expressions that we have for shape functions N 1 to N 6 and we can get the shape function values of all the nodes and also we can write the trial solution.

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Example (Continued)

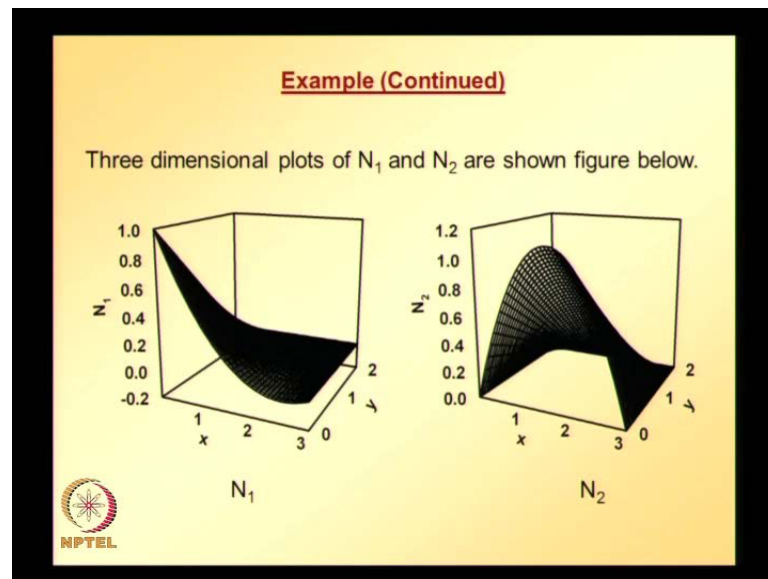
Substituting the numerical values of nodal coordinates into the shape function formulas, the explicit expressions for shape functions for this rectangular element are as follows.

| | |
|------------------------------------|------------------------------|
| $N_1 = (3 - x)(-2 + x)(-2 + y)/12$ | $N_2 = (-3 + x)x(-2 + y)/4$ |
| $N_3 = (2 - x)x(-2 + y)/6$ | $N_4 = (-2 + x)xy/12$ |
| $N_5 = (3 - x)xy/4$ | $N_6 = (-3 + x)(-2 + x)y/12$ |



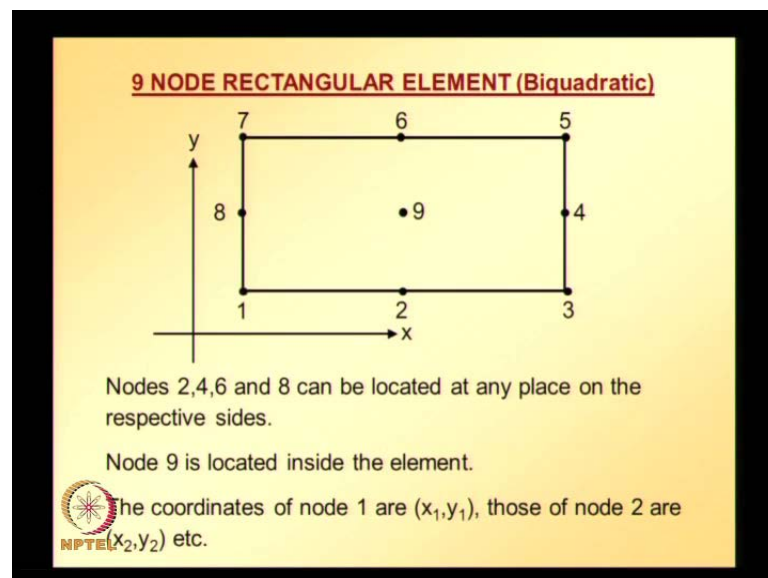
Substituting the numerical values of nodal coordinates into the shape function formulas explicit expressions for shaped functions for 6 node rectangular element are given here N 1, N 2, N 3, N 4 and N 5 and to visualize how the shape functions looks or **how they vary** how they vary along x and y directions, we can plot three-dimensional plots of N 1 N 2 are shown in figure below.

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And similarly shape function of other nodes can be plotted. So this is a 6 node rectangular element based on Lagrange interpolation formula.

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So, now let us look at another Lagrange element, which is 9 node rectangular element. A 9 node rectangular element is shown here also x y axis are shown in the figure and nodes 2, 4, 6, 8 can be located at any place on respective sides and node 9 is located inside the element and **coordinates of node 1 are** coordinates of node 1 are $x_1 y_1$ and similarly for the other nodes and here you can see shape functions varies quadratically **in x direction**

both in x direction y direction, because we have three nodes along x direction and three nodes along y direction.

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9 NODE RECTANGULAR ELEMENT (Continued)

Here shape functions vary quadratically in both directions.

$$N_1(x, y) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} \times \frac{(y - y_8)(y - y_7)}{(y_1 - y_8)(y_1 - y_7)}$$

$$N_2(x, y) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} \times \frac{(y - y_9)(y - y_6)}{(y_2 - y_9)(y_2 - y_6)}$$

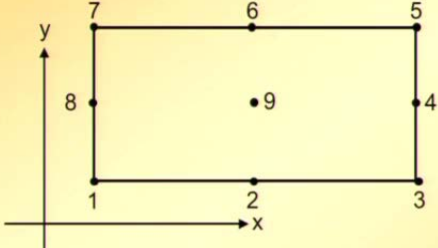
etc.




Here shape functions vary quadratically in both directions. So, writing Lagrange interpolation formula in x direction multiply with Lagrange interpolation formula in y direction we can write shape function expressions for all the nine nodes N_1 Lagrange interpolation x direction times Lagrange interpolation y direction.

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9 NODE RECTANGULAR ELEMENT (Continued)



Note that: $N_1(x, y) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} \times \frac{(y - y_8)(y - y_7)}{(y_1 - y_8)(y_1 - y_7)}$



Similarly, N_2 and the rest and the properties that we have seen for 4 node quadrilateral element the shape functions for these nine node rectangular elements also satisfies. So **that is** if you see this node 1 this is the expression for shape function of node 1 and it can be easily verified that N_1 is going to be equal to 0 at all other nodes except node 1 **where is** where it is equal to 1 and also it can be verified that N_1 is going to be 0 alongside 3-4-5 and alongside 7-6-5 and alongside 1-2-3 this expression is quadratic function in x and alongside 1-8-7 is quadratic function of y.

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9 NODE RECTANGULAR ELEMENT (Continued)


Note that:
$$N_1(x,y) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} \times \frac{(y-y_8)(y-y_7)}{(y_1-y_8)(y_1-y_7)}$$

= 0 at all other nodes except node 1 where it is equal to 1

= 0 along edges 3-4-5 and 7-6-5

Along 1-2-3: Quadratic function of x

Along 1-8-7: Quadratic function of y.

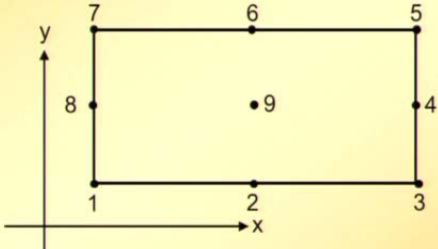
 Similar observations can be made for other shape functions.

It is 0 at all other nodes, except node 1 where it is equal to 1, zero along edges 3-4-5 and 7-6-5 and along edge 1 or side 1-2-3 it is going to be quadratic function of x and alongside are edge 1-8-7 it is a quadratic function of y and not only for **shape** shape function of node, one similar observations can be made for other shape functions. So, here when we are deriving shape function expression for this 9 node Lagrange element, basically we use Lagrange interpolation formula, instead of that we can also start with a polynomial having 9 coefficients and we can derive same shape functions.


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9 NODE RECTANGULAR ELEMENT (Continued)

The same shape functions can also be derived from the following polynomial using the procedure employed for triangular element.

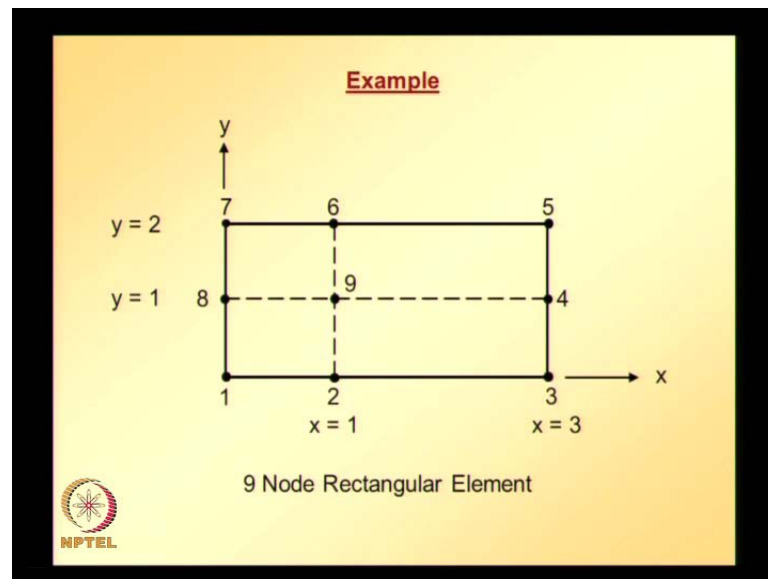


$T(x, y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^2y + a_7xy^2 + a_8x^2y^2$

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Same shape functions can also be derived from the following polynomial using procedure employed for linear triangular element, so this is the element for which we need to derive shape functions. There are 9 nodes, so we need to start with a polynomial having **9 coefficients** 9 coefficients like this and we can adopt the procedure that we adopted for deriving shape functions for linear triangular element; and once we do that, we get same shape functions as we obtained using Lagrange interpolation formula. But only thing is this procedure is going to be tedious, and also it is going to be cumbersome since we need to solve for nine coefficients and we need to group terms containing same nodal parameters or nodal values to get the shape function expressions. Now, we have the shape function expressions, explicit expressions based on Lagrange interpolation formula for this nine node element, we can write shape function for any element once we knew the nodal coordinates.

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So, now let us take an example here a 9 node element is shown x y axis are also shown; and also from the information that is given, we can easily figure out what are the x y coordinates of each of the nodes. So, once we have that information, we can plug this information into the explicit expressions for shape functions that we obtained using Lagrange interpolation formula to get the shape function expressions.

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Example (Continued)

Substituting the numerical values of nodal coordinates into the above shape function formulas, the explicit expressions for N_1 , N_2 and N_9 shape functions for this rectangular element are as follows.

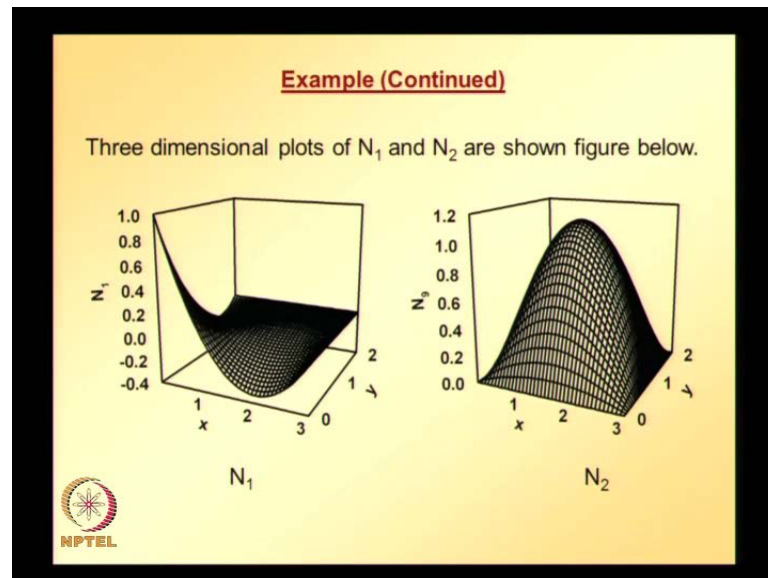
$$N_1 = (-3+x)(-1+x)(-2+y)(-1+y)/6$$
$$N_2 = (3-x)x(-2+y)(-1+y)/4$$
$$N_9 = (-3+x)x(-2+y)y/2$$

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Substituting the numerical values of nodal coordinates in to the above shape function formulas explicit expressions for N_1 , N_2 and N_9 shape functions for this rectangular

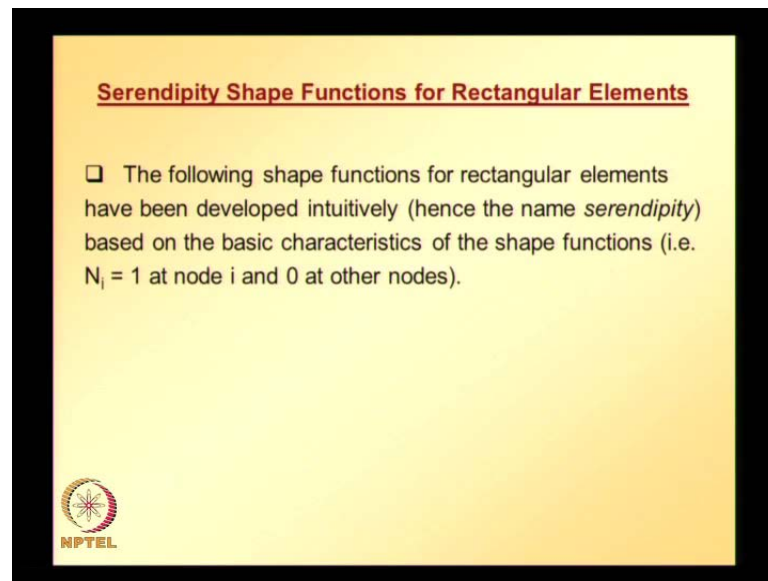
element. Here even though N_1 , N_2 and N_9 are shown we can easily write or we can easily simplify the substituting the nodal coordinates in to the previous explicit formulas and we can get the node shape function expressions for other nodes as well and to visualize how is shape function N_1 and N_2 varies we can even plot.

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So, here three-dimensional plot of N_1 and N_2 are shown for this particular 9 node element. So, we have seen 4 node rectangular element, 6 node rectangular elements and 9 node rectangular element and we have also seen how to derive shape functions of all these elements using Lagrange interpolation formula. So, these are one the elements for which we can adopt Lagrange interpolation formula to derive the shape functions, but there are some other elements for which we need to adopt some other procedure, so those set of elements are called serendipity elements.

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Serendipity Shape Functions for Rectangular Elements

- The following shape functions for rectangular elements have been developed intuitively (hence the name *serendipity*) based on the basic characteristics of the shape functions (i.e. $N_i = 1$ at node i and 0 at other nodes).


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So, now let us look at those elements serendipity shape functions for rectangular elements. Following shape functions for rectangular elements have been developed intuitively hence the name serendipity based on basic characteristics of shape functions that is N_1 is equal to 1 at node 1 and 0 at other nodes or N_i is equal to 1 at node i and 0 at other nodes. So, instead of using Lagrange interpolation or sometimes it is not possible to use Lagrange interpolation formula to derive shape functions for certain rectangular elements containing certain number of nodes, in that case we need to derive shape functions intuitively without violating the conditions that shape function of node i is going is equal to 1 at node i and equal to 0 at other nodes.

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Serendipity Shape Functions for Rectangular Elements

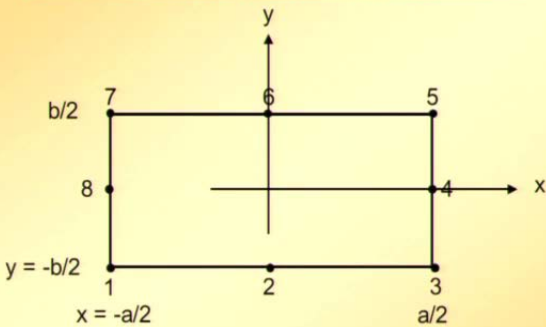
- The following shape functions for rectangular elements have been developed intuitively (hence the name *serendipity*) based on the basic characteristics of the shape functions (i.e. $N_i = 1$ at node i and 0 at other nodes).
- The elements based on these shape functions are very popular.
- Their main advantage is that all nodes are placed on the element sides and thus there are no interior nodes.




Using that basic characteristics of shape functions and intuitively, if you can derive the shape functions and that is what serendipity element. Elements based on these shape functions are very popular, their main advantage is that all the nodes are placed on element sides and thus there are no interior nodes.

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8 NODE QUADRATIC SERENDIPITY ELEMENT



Note that nodes are at the corners and at mid-sides and that the origin of the coordinate system is at the element centroid.



So, now let us look at 8 node serendipity element and it is a quadratic element, so it is called 8 node quadratic serendipity element and x y axis are indicated in the figure. And also based on the information that is given in the figure, we can easily figure out what

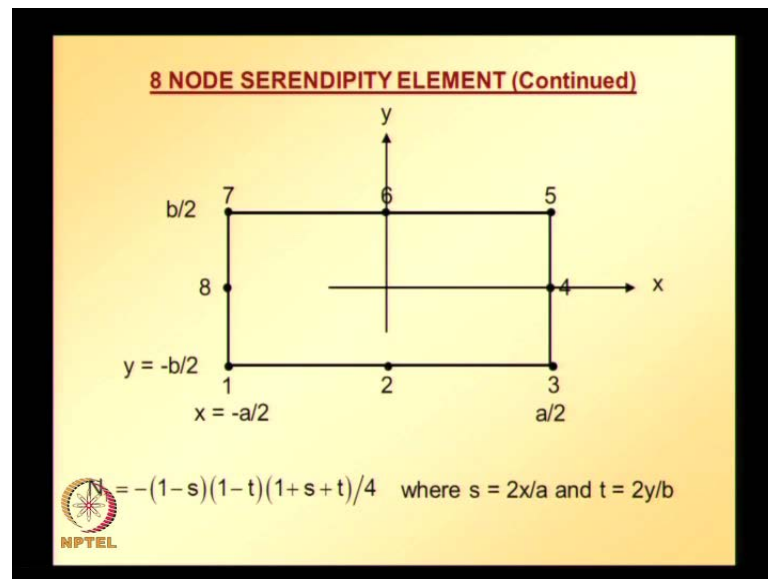
are the x y coordinates of each of the nodes. And if you compare this element with the 9 node element that we have just seen, you can see we can you can notice that only the middle interior node, which is 9th node is missing, so that is the only difference.

So, here before I show the expressions for this element shape function expressions for this element, let us see if you want to derive shape function of node 1 and you can see from the figure node 1 is going to be 0 along edge 3-6 and node 1 should also be zero along edge 7-6-5, in addition to edge 3-4-5 and if you can include the equation of line 3-4-5 equation of line of edge 3-4-5 and equation of line of edge 7-6-5 in to the shape function expression of node 1, so then node 1 is going to be zero along edge 3-4-5 and it is going to be 0 along edge 7-6-5.

Similarly, node 1 has to be 0 along or at node 2 and 8. So, if you can come up or if you can get the equation of line, which passes through node 2 and 8 that can be easily derived based on the nodal coordinates of node 2 and node 8, we can easily write what is the equation of straight line that passes through nodes 2 and 8. If you can include that equation of that line also in to the shape function of node 1 then we get the shape function including all the equations of sides 3-4-5 and 7-6-5 and also equation of line passing through node 2 and 8. If you include all these into the shape function expression for node 1 and normalize it we are going to get finally the shape function expression explicit expression for this 8 node quadratic serendipity element for node 1.

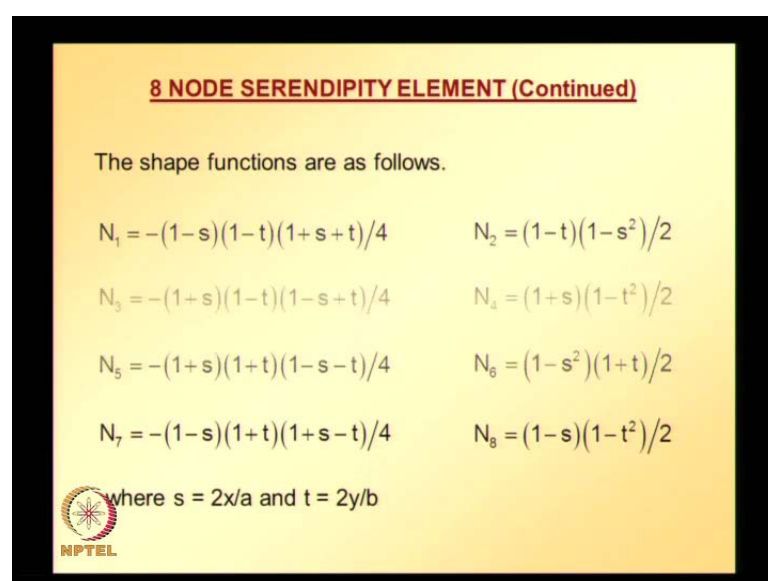
Similarly, we can derive for node 2, node 3, node 4, node 5, node 6, node 7 and node 8. So, based on that, we can easily write the shape functions for this element. **Notes** Note that nodes are at the corners nodes are at the corners and at the mid sides and the origin of the coordinate system is at the element centroid for this element.

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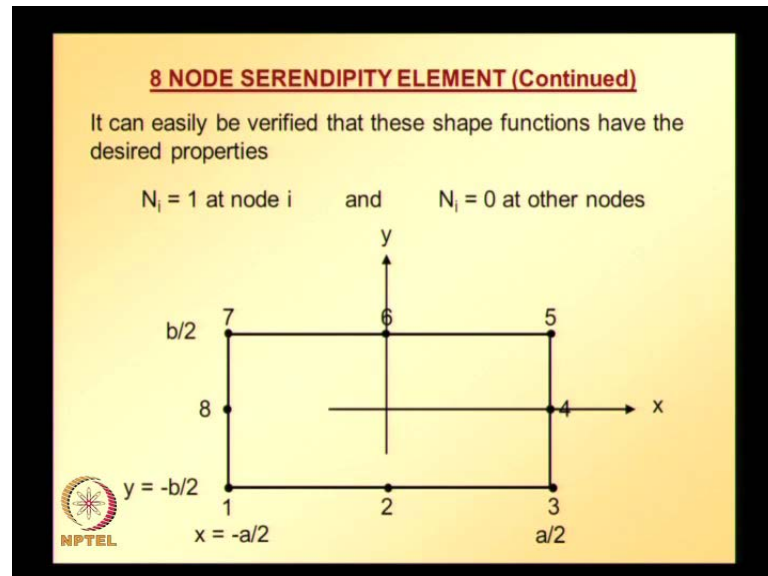
So, based on the procedure that I mentioned, when we are writing shape function for node 1 include equation of line passing through sides 3-4-5 and also include equation of line passing through 7-6-5 and equation of line passing through 2 and 8 and normalize it then we are going to get shape function expression of node 1. Here it is written in terms of s and t, where s and t are define **s** s is equal to 2 x over a, t is equal to **2 over** 2 y over b.

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So, this is how we can write shape functions for rest of the nodes. So, adopting the explanation or the procedure that I mentioned, one can easily write shape functions for rest of the nodes explicit expressions for all the nodes. All the 8 nodes are given here and it can be easily verified that each of these nodes is equal to 1 at its own position and it is equal to 0 at the other nodal locations.

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It can be easily verified that the shape functions have the desired properties that is **N_i** **is equal to 1 at node i and N_i is equal to zero at other nodes.** So, here we used some kind of reasoning or we have developed whatever expressions that I have shown we have developed for this element intuitively by making sure that it is shape function at a particular node is equal to zero at other nodal location, **we have derived the element** we have derived the nodal shape functions explicit expressions of nodal shape functions intuitively; instead of that we can also start with **a polynomial** taking a polynomial having 8 number of coefficients.


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8 NODE SERENDIPITY ELEMENT (Continued)

These shape functions can be derived from the following polynomial using the method for triangular elements.

$$T(s, t) = a_0 + a_1s + a_2t + a_3s^2 + a_4st + a_5t^2 + a_6s^2t + a_7st^2$$

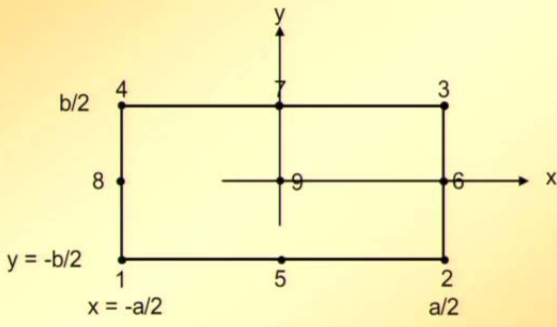
Three dimensional surface plots of these shape functions are similar to those of the biquadratic Lagrangian shape functions.




The shape functions can be derived from the polynomial from the following polynomial using method that we adopted for linear triangular elements. So, since there are 8 nodes, we need to start with a polynomial having 8 coefficients. So, this is the polynomial with which we can start and adopt the procedure that we did for linear triangular elements and finally, we can get same shape functions as we have seen and the three-dimensional surface plots of these shape functions are similar to those of biquadratic lagrangian shape functions.

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SHAPE FUNCTIONS FOR 4 TO 8 NODE RECTANGULAR ELEMENT



Complete set of shape functions for any element with nodes from 4 to 9 are given in Table below.



So, now we have derived shape functions for 4 node element, 6 node element, 9 node element and 8 node element, so we can write shape functions for all elements with nodes having nodes from 4 to 9, so here we will write a general set of shape functions to define shape functions for any rectangular elements with nodes range in from 4 to 9. The complete set of **function** shape functions will be given **in the** in a table and but the ordering of nodes or the ordering of element node numbers is very important.


So, the expressions that **we** I am going to show you are valid only for this node numbering. With this particular node numbering scheme shape functions for higher order elements are constructed by adding terms in to shape function for lower order elements. Here there is a typing mistake in the title it should be shape functions for 4 to 9 node rectangular element, so complete set of shape functions for any element with nodes from 4 to 9 are given in table below.

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4 TO 8 NODE RECTANGULAR ELEMENT (Continued)

The following notation is used: $s = 2x/a$ and $t = 2y/b$

| | |
|------------------------|------------------------|
| $f_1 = (1-s)(1-t)/4$ | $f_2 = (1+s)(1-t)/4$ |
| $f_3 = (1+s)(1+t)/4$ | $f_4 = (1-s)(1+t)/4$ |
| $f_5 = (1-s^2)(1-t)/2$ | $f_6 = (1+s)(1-t^2)/2$ |
| $f_7 = (1-s^2)(1+t)/2$ | $f_8 = (1-s)(1-t^2)/2$ |
| $f_9 = (1-s^2)(1-t^2)$ | |




And the following notation is used for the expressions that are given in the table s is defined like the way it is done earlier s is equal to $2x/a$ and t is equal to $2y/b$ and $f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8$ and f_9 are defined like this. The shape functions expressions for all nodes of an element having 4 to 9 nodes is expressed in terms of these f 's, so this definition of f_1 to f_9 is very important to read the table.

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4 TO 8 NODE RECTANGULAR ELEMENT (Continued)

| Element | Shape functions | |
|----------------|---|--|
| 4 node element | $N_i = f_i$ | $i = 1, \dots, 4$ |
| 5 node element | $N_1 = f_1 - f_5/2$ $N_i = f_i$ | $N_2 = f_2 - f_5/2$ $i = 3, \dots, 5$ |
| 6 node element | $N_1 = f_1 - f_5/2$ $N_3 = f_3 - f_6/2$ $i = 4, \dots, 6$ | $N_2 = f_2 - f_5/2 - f_6/2$ $N_i = f_i$ |




So, **table** in the table expressions for shape functions of all nodes for 4 to 9 node rectangular elements are given, 4 node element, 5 node element, 6 node.

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4 TO 8 NODE RECTANGULAR ELEMENT (Continued)


| | | |
|----------------|---|---|
| 7 node element | $N_1 = f_1 - f_5/2$ $N_3 = f_3 - f_6/2 - f_7/2$ $N_i = f_i$ | $N_2 = f_2 - f_5/2 - f_6/2$ $N_4 = f_4 - f_7/2$ $i = 5, \dots, 7$ |
| 8 node element | $N_1 = f_1 - f_5/2 - f_8/2$ $N_3 = f_3 - f_6/2 - f_7/2$ $N_i = f_i$ | $N_2 = f_2 - f_5/2 - f_6/2$ $N_4 = f_4 - f_7/2 - f_8/2$ $i = 5, \dots, 8$ |



7 node, 8 node.

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| 4 TO 8 NODE RECTANGULAR ELEMENT (Continued) | |
|--|---|
| 9 node element | $N_1 = f_1 - f_5/2 - f_8/2 - f_9/4$ $N_2 = f_2 - f_5/2 - f_8/2 - f_9/4$ $N_3 = f_3 - f_6/2 - f_7/2 - f_9/4$ $N_4 = f_4 - f_7/2 - f_8/2 - f_9/4$ $N_5 = f_5 - f_9/2$ $N_6 = f_6 - f_9/2$ $N_7 = f_7 - f_9/2$ $N_8 = f_8 - f_9/2$ $N_9 = f_9$ |



And 9 node element and please note that elements with number of nodes between 4 and 8 are known as transition elements. These elements are useful **when transition** when transition from quadratic to linear element is desired. So using this table, we can derive shape functions for any noded elements starting from 4 node to 9 node element and we will continue in the next class.