Finite Element Analysis

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Lecture No. # 22

In today's lecture we will be looking at general two-dimensional boundary value problems, and similar to one dimensional boundary value problem - general one dimensional boundary value problem we will be deriving element equations.

We can do that using either Rayleigh-Ritz method by coming up with equivalent functional for the given boundary value problem or we can also derive the element equations using Galerkin criteria. Irrespective of whether we adopt Rayleigh-Ritz method or Galerkin criteria we require to decide what kind of element, we want to use for solving two-dimensional boundary value problems, and accordingly we need to express the trial solution before we proceed to get the element equations.

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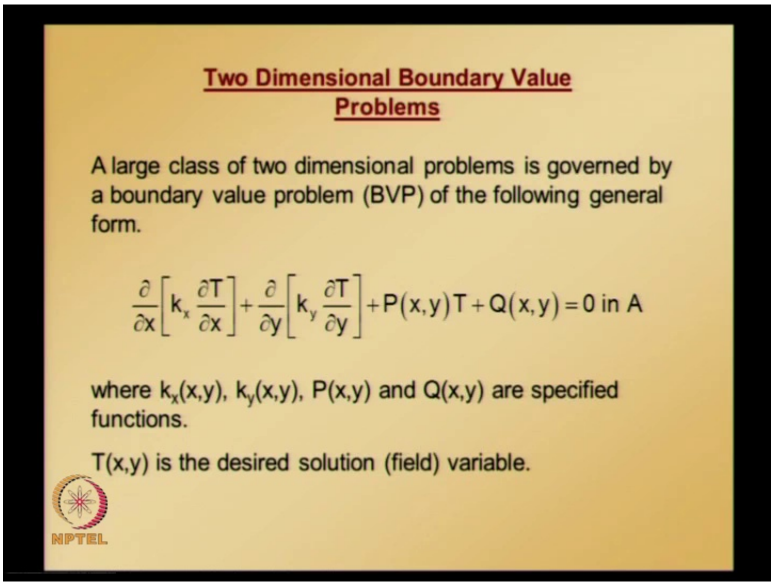

Let us look at the problem statement a large class of two-dimensional boundary value problems is governed by a boundary value problem of the following general form. Where k x, k y, P Q are some specified functions. They can be functions of spatial coordinates or constants. So, this is a general two dimensional boundary value problem which we require to solve over the domain A, which is an arbitrarily shaped two dimensional domain bounded by some surface S and T is the field variable which we are after or T x is the desired solution.

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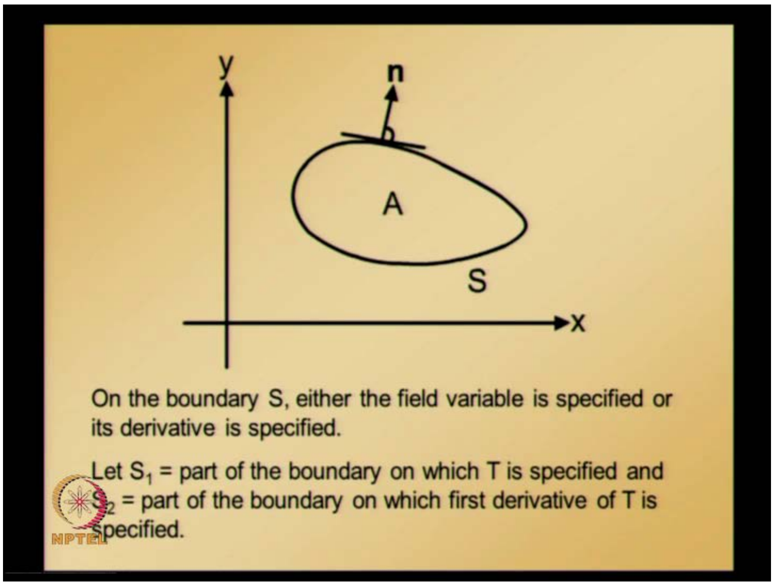

We need to find solution of this differential equation over the domain A, where A is arbitrarily two-dimensional domain bounded by a surface S as shown in figure, where coordinates are X Y axis are indicated, and n is a unit outer normal on the boundary and this S which is the boundary of domain A. On this boundary either field variable T or its derivative can be specified, and boundary S can be divided into two parts. Let S 1 be the part of boundary on which T is specified that is field variable value specified, and S 2 is a part of boundary on which first derivative of T is specified.

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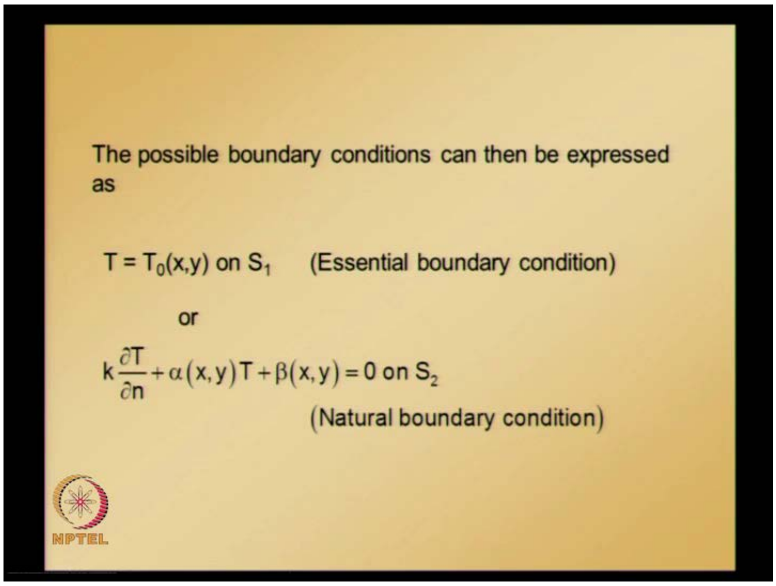

If you agree for this, the two possible boundary condition can be expressed as T is equal to T 0 on the part of boundary over which T is specified, and this is essential boundary condition because order of the equation is zero eth order. And if you see the second boundary condition either this can be specified either essential boundary condition can be specified or natural boundary condition can be specified. Natural boundary condition is the boundary condition which contains first derivative of T, and if you recall the general two dimensional boundary value problem that we are trying to solve is a second order differential equation. We require two boundary conditions to solve out of which one is essential another is natural.

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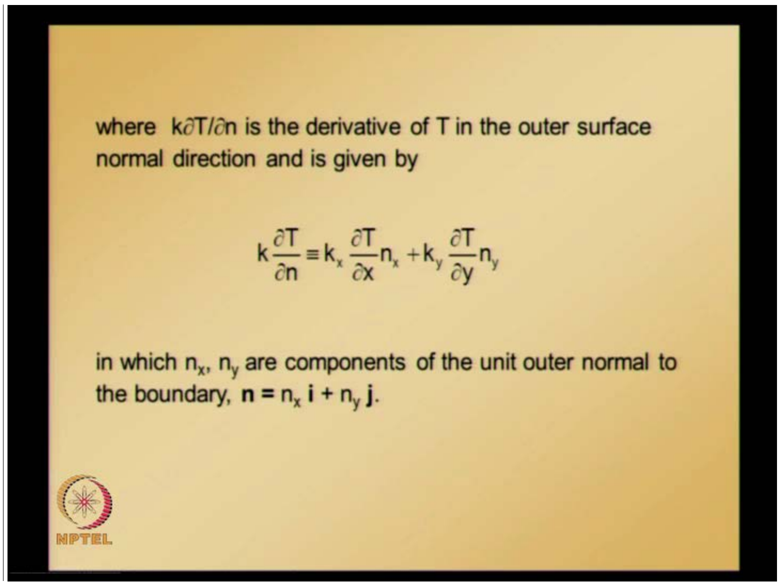

So, any of these boundary conditions may be specified and where k times partial derivative of T with respect to unit outer normal to the boundary is a derivative of T in the outer surface normal direction that is given by this one. This is how k times partial derivative of T partial derivate of field variable with respect to unit outer normal is defined, where n x n y are components of unit outer normal to the boundary and the unit outer normal n can be written as n x i plus n y j.

This is the problem statement and we require to solve this over the domain A, which is bounded by surface S on which either essential boundary conditions or natural boundary conditions may be specified. And if you see or if you compare this with general one dimensional boundary value problem, you can see this is a just an extension of one dimensional boundary value problem.

And the finite element formulation follows same steps as before for one-dimensional boundary value problem with one exception. In general one-dimensional boundary value problem, there is no choice of shape of element all the elements that we are chosen are line elements even though we looked at both two node element, and three node element.

However in two dimensions there are many possibilities for element shapes like you can have triangle, rectangle, quadrilaterals, square or triangle with curve edges, and quadrilateral with curved edges or curved sides. So, we need to look at when we are trying to develop element equations for general two-dimensional boundary value problem we need to look at the possible element shapes as well.

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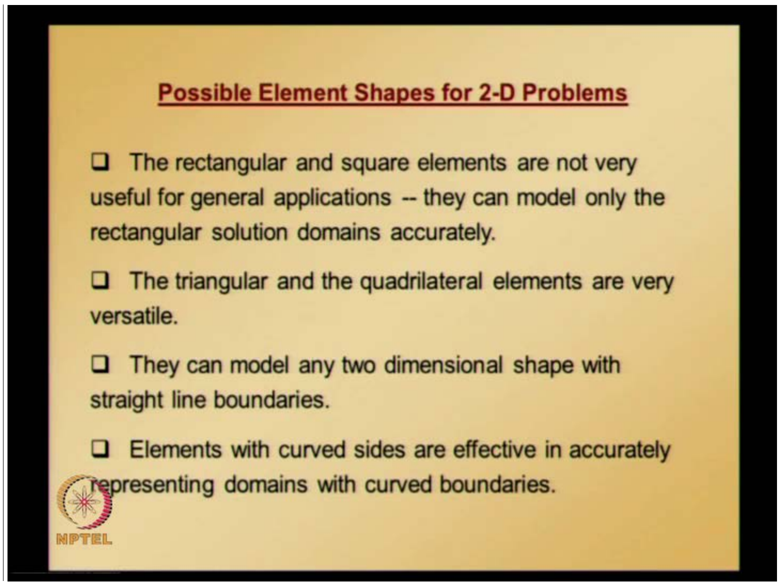

Possible element shapes for 2 - D problems the rectangular, and square elements are not very useful for general applications, they can model only rectangular solution domains accurately. The triangular and quadrilateral elements are very versatile, they can model any two dimensional shape with straight line boundaries. Elements with curved sides are effective in accurately representing domains with curved boundaries.

In this lecture we will derive triangular element shape functions as triangular element is the simplest to formulate. As it is based on linear polynomial trial solutions which will be seen or which will be clearer when we actually start derived element shape functions for triangular three node triangular element. Three node triangular element is based on linear polynomial trial solutions in two dimensions, the required integrations for assembling the element equations are fairly simple to carry out in closed form. The quadrilateral element and element with curved boundaries are formulated using isoparameteric mapping which will be seen in the later lectures.

As a part of this lecture we will be developing or we will be looking at equivalent functional for the given two dimensional boundary value problem statement, and equivalent functional is required if finite element solution is based on Rayleigh-Ritz method. And we also look at how to derive element shape functions for three node triangular element. The procedure helps us to understand or to generalize it later for any noded 2 D element. And also we look at the Galerkin criteria for two-dimensional boundary value problem to derive the element equations.

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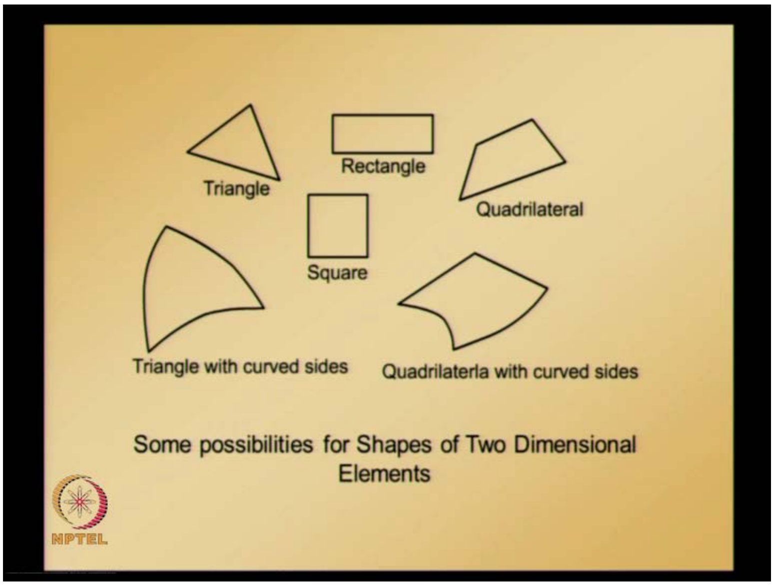

Now, let us look at equivalent functional for general two-dimensional boundary value problems statement that we already noted. And before that these are the possible shapes for two dimensional elements triangle, rectangle, square, quadrilateral, triangle with curved edges and quadrilateral with curved edges or curved sides.

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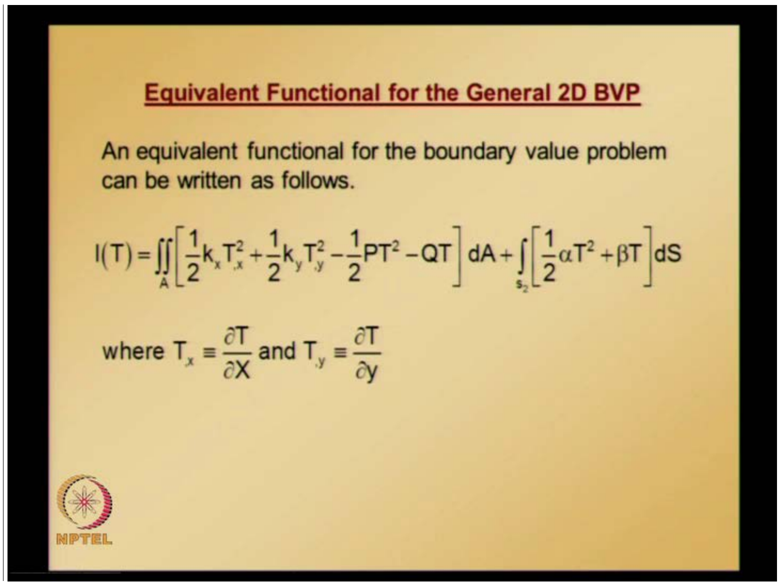

Equivalent function for the boundary value problem that we already noted can be written as given, where I is the equivalent function and k , subscript x is first derivative of k with respect x. Similarly T subscript x is partial derivative of T with respect to x. Similarly, k y T , y. Where T, y is partial derivative of T with respect x and T, x is partial derivate of T with respect x T, y is partial derivative of T with respect to y. Note that the second integral is evaluated only over S 2 the part of boundary over which natural boundary conditions are specified. If you recall S 1 is the part of the boundary on which essential boundary conditions are specified, and S 2 is the part of the boundary on which natural boundary conditions are specified.

So, the second integral needs to be evaluated on the part of boundary over which natural boundary conditions are specified, and this is the equivalent functional for the boundary value problem that we noted and this needs to be verified; whether this is the functional for the given problem or not. So, here we will be verifying that and this procedure can be adopted for any other kind of problem where equivalent functional is given for a particular problem, and whether somebody if somebody is interested in verifying whether that given functional is corresponding to a particular problem or not the procedure is going to be similar.

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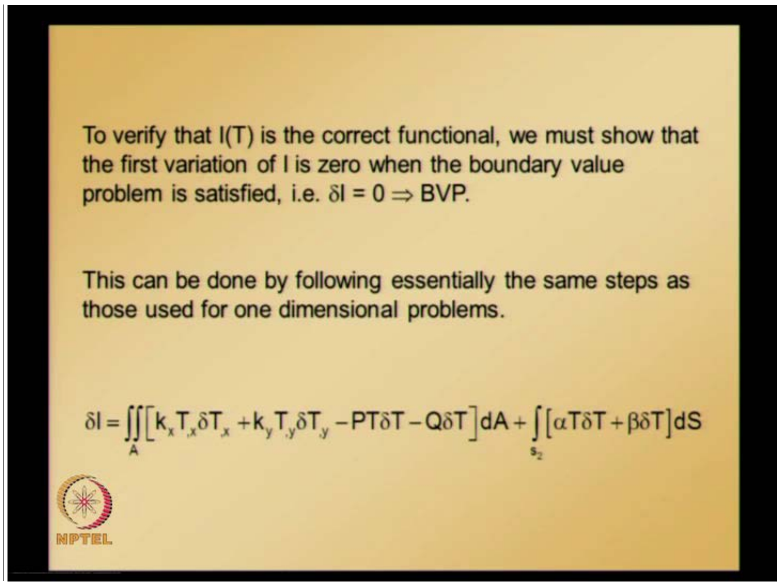

To verify that indeed the equivalent functional which is given the previous slide is correct functional, we must show that the first variation of I is zero when the boundary value problem is satisfied. So, equivalent functional is given, so we need to check whether variation of I is equal to zero or not. If it is satisfied then it is going to be the equivalent functional for the given problem, and this can be done by following, essentially the same steps as those used for one dimensional problem which we already did.

First step is we need to take variation of I equivalent functional is given, so variation of I can be written like this. By using variational identities that we learnt when we started out with this variational approach or Rayleigh-Ritz method in the earlier lectures. Now in this equation interchanging the order of differentiation and variation in the first two terms, because order of differentiation and variation can be interchanged.

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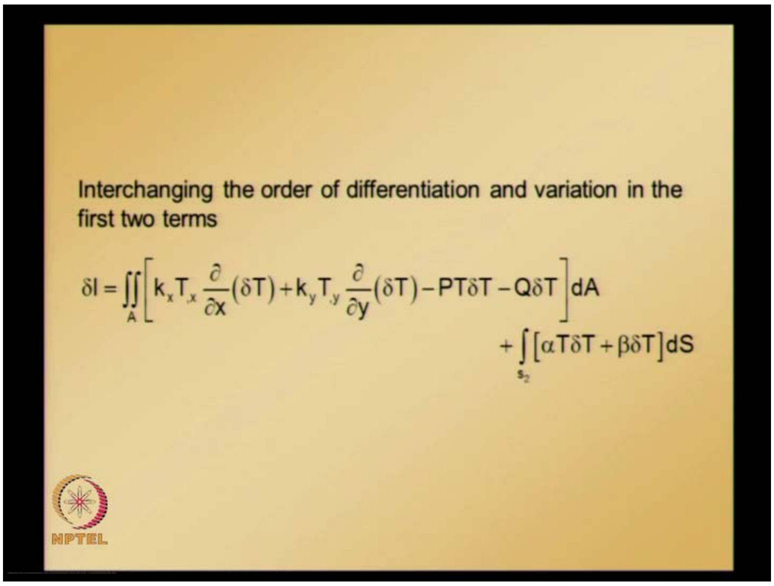

Interchanging order of differentiation, and variation in the first two terms we get this. Basically what we did is variation of derivative of T is written as derivative of variation of T, and now we need to look at what is called Green's theorem to further simplify this equation.

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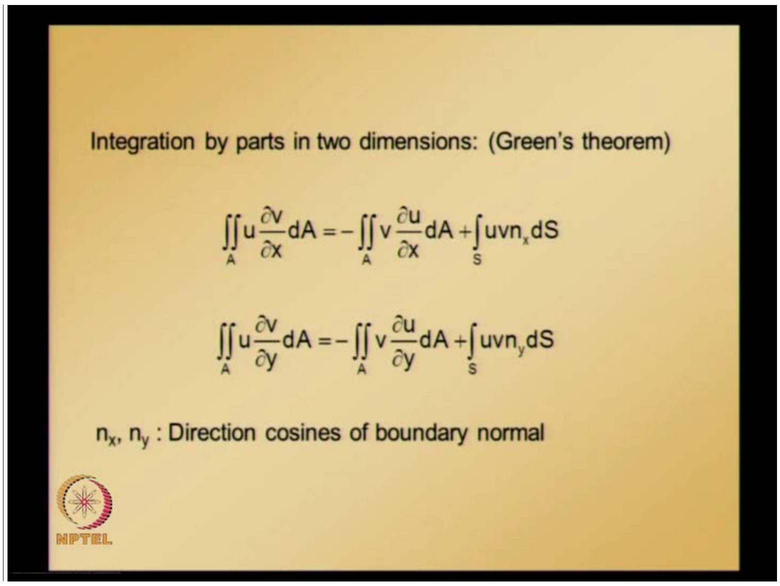

Now, let us look at two formulas or two identities to simplify the first two terms. These are the two identities integration by parts in two dimensions which is also called Green's theorem, integral u times partial derivative of v with respect x integrated over A can be written as minus v times partial derivative of u with respect x integrated over A plus u times v times x component of unit outer normal to this surface integrated over the surface. Similarly, we can write the second equation also which is u times partial derivative of v with respect to y integrated over A is equal to minus v times partial derivative of u with respect to y integrated over A plus u times v times y component of unit outer normal integrated over surface. So, using these two formulas up to identities the first two terms of previous equation can be further simplified.

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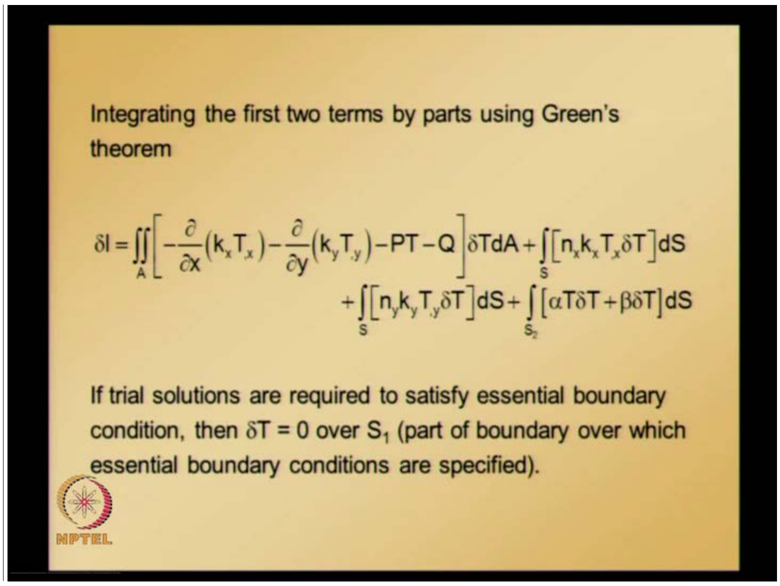

When we apply integration by parts on the first two terms we get additional surface integrals. If trial solutions are required to satisfy essential boundary conditions then variation of T is equal to 0 on the part of boundary over which essential boundary conditions are specified, and this is the one of the fundamental requirements which we already looked at in the earlier classes when we started out learning various steps of Rayleigh-Ritz method.

To simplify this variation of I equation further we apply this condition. Since, boundary S comprises of S 1 plus S 2 and on a S 1 variation of T is equal to 0, because S 1 is the part of boundary over which essential boundary conditions are specified. Based on this requirement this equation of variation of I can be further simplified.

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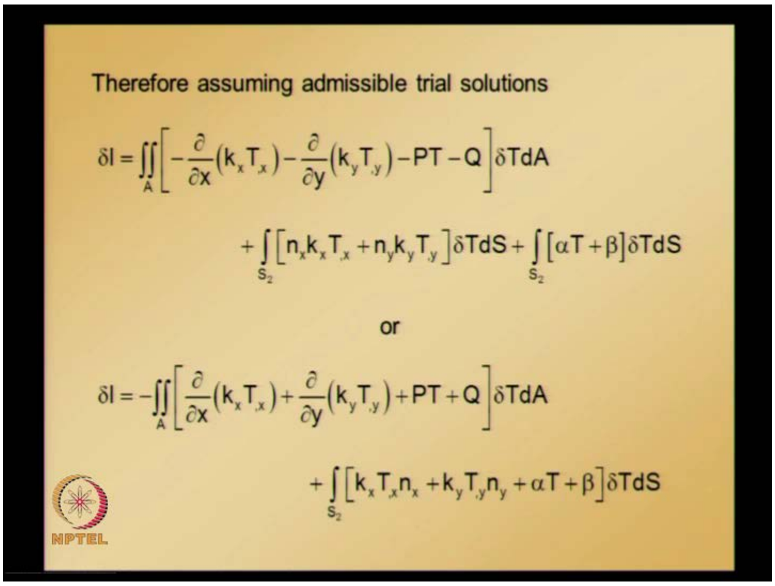

Therefore assuming admissible trial solutions the previous equation of variation of I simplifies to this which can be further rewritten in this manner. If you see the first part of the equation the integrant of the first integral is nothing but the given differential equation would needs to be equal to zero over the domain, and if you see the integrant of the second integral it is nothing but the natural boundary condition that is given which is equal to zero.

The boundary integral is zero for natural boundary condition, the domain integral is zero if the given or the governing differential equation is satisfied. Therefore, using admissible trial solutions variation of I is equal to zero, and which verifies that the given are the equivalent function that is we started out with is the correct functional for this particular two-dimensional general boundary value problem. So, we got equivalent functional for the given general two-dimensional boundary value problem. If you want to derive finite element equations we need to substitute finite element approximations of T, and derivative of T in terms of finite element shape functions of the element that we choose.

Next step if one want to derive the finite element equations is to look at how to derive element shape functions. So, we will be looking at triangular element, simplest two dimensional element, three node triangular element, how to derive shape functions of that element, so that we can express trial solution and derivative of trial solution in terms of finite element shape functions and nodal values before we derive the element equations. Once we have those things that is finite element approximation of trial solution, and derivative of trial solution in terms of nodal values or nodal parameters and finite element shape functions we can get element equations for that particular element.

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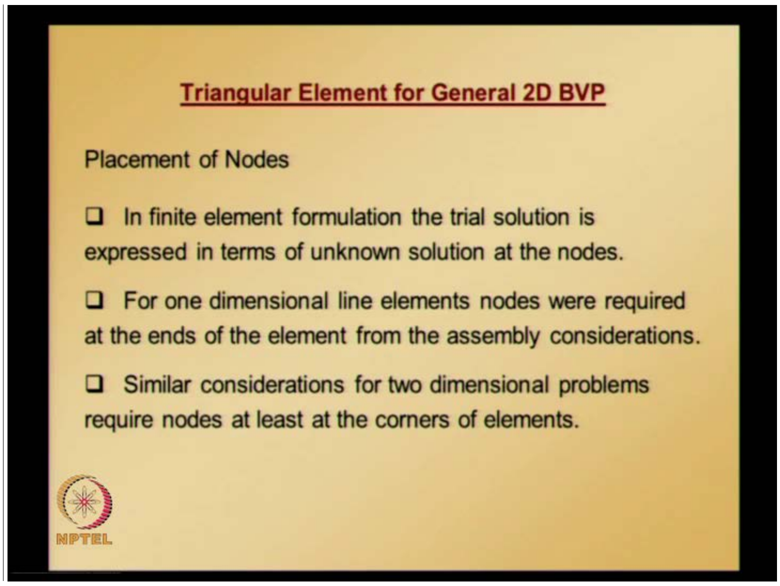

Triangular element for general two-dimensional boundary value problem, placement of nodes in finite element formulation the trial solution is expressed in terms of unknown solutions at the nodes, and this is what we also observed when we are looking at one dimensional problems.

For one dimensional line elements nodes are required at the ends of element for assembly considerations. Similarly or similar considerations for two dimensional problems require nodes at least at the corners of the elements.

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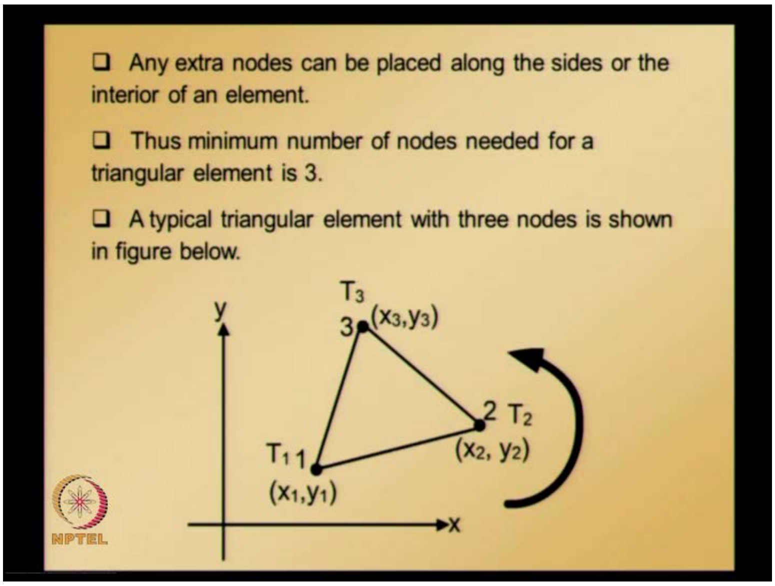

Extra nodes can be placed along sides or the interior of an element. Thus minimum number of nodes required for triangular element is three at the 3 corners a typical triangular element with three nodes is shown. And if you see the local node numbering is in the counter clockwise direction, which is shown or indicated by the arrow and the global coordinates of nodes 1 2 3 are also indicated in the figure, and the unknown solutions at the three nodes t 1 t 2 t 3 is also indicated. Now, we need to derive element shape functions for this three node triangular element. Since, there are three nodes, trial solution must have three parameters. So, we need to start with a polynomial having three coefficients, a linear polynomial in two dimensions satisfies criteria.

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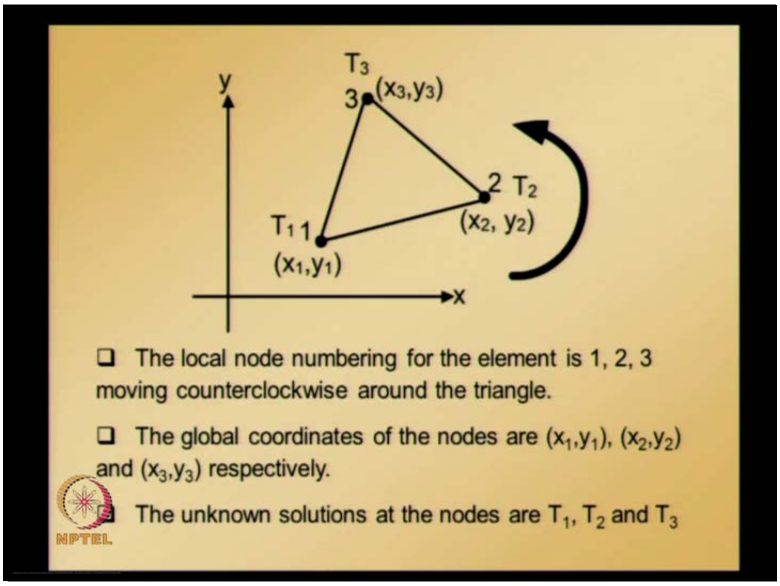

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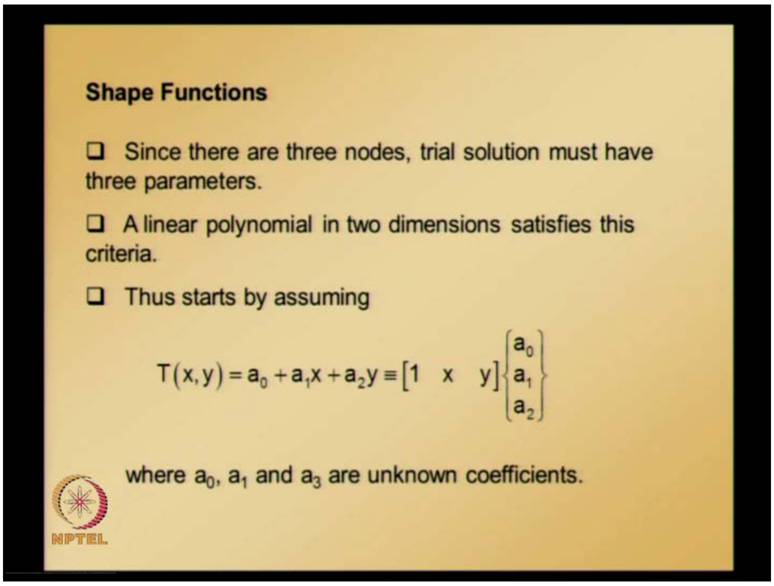

Trial solution T is assumed to be a 0 plus a 1 x plus a 2 y which can be written in matrix in vector form as 1 x y put in a row vector, a 0 a 1 a 2 put in a column vector where a 0 a 1 a 2 are unknown coefficients, and if you recall this is the procedure that we are adopting here is similar to what we did for one-dimensional two node element or three node element is started out with for a two node line element. What we did is we started out with u is equal to a 0 plus a 1 x, and then we obtained two equations to solve for a 0 a 1 by substituting x is equal to x 1, u is equal to u 1 at x is equal to x 1 u is equal to u 2 at x is equal to x 2 by substituting this, we obtain two equations we solved these two equations to express a 0 a 1 in terms of the nodal values. And once we got a 0 a 1 we back substituted a 0 a 1 into u, and group the terms containing u 1 u 2 separately and whatever is coefficient of u 1 and u 2. That is, those two are the shape functions for node one and node two, n 1 and n 2.

The procedure that we adopted when we are deriving shape functions for two node line element, similar procedure we adopted even for three node line element, two node line element is linear element, three node line element is quadratic element in one dimension.

So, similar procedure will be adopting here, to start with to derive the shape functions for three node triangular element, the trial solution is assumed as T is equal to a 0 plus a 1 x plus a 2 y.

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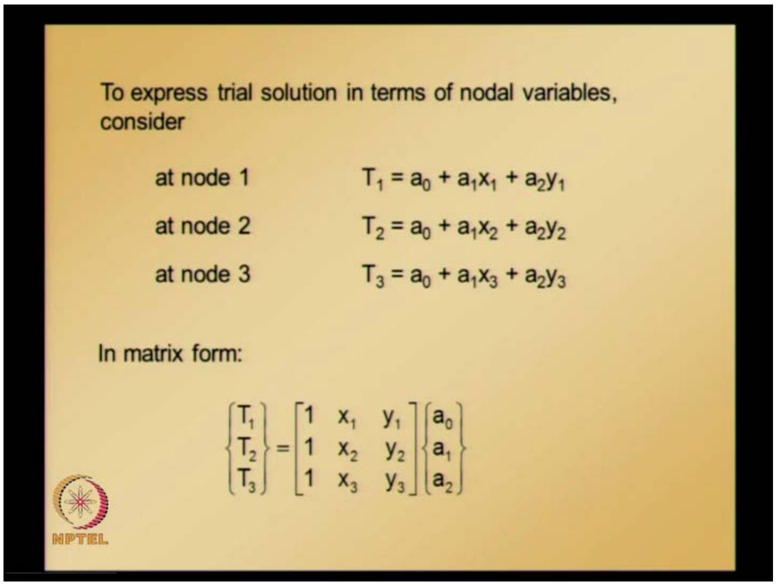

Now T value at node one is T 1 to express trial solution in terms of nodal variables or nodal values. We apply this condition at node one T is equal to T 1 and x is equal to x 1, y is equal to y 1. Similarly, at node two T is equal to T 2, x is equal to x 2, y is equal to y 2 at node three T is equal to T 3, x is equal to x 3 and y is equal to y 3. By using these conditions we get three equations and we can solve these three equations for a 0 a 1 a 2 to solve these three equations we can put them in a matrix form. The coefficients can be expressed in terms of nodal variables by inverting three by three matrix comprising of the coordinates of three nodes of triangular element.

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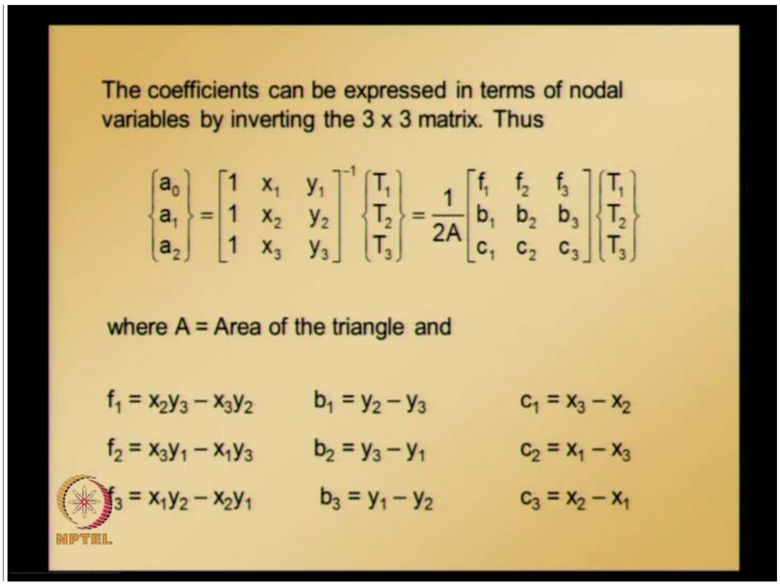

So, a 0 a 1 a 2 can be obtained by inverting a 3 by 3 matrix consisting of coordinates of nodes one two three and if you notice inverse of the matrix 1 x 1 y 1, 1 x 2 y 2, 1 x 3 y 3 is not possible, if two of the nodes are not possible if all the three nodes are lying along the same line. That is one of the requirements that is a three node triangular element or a triangular element in which all the three nodes are almost collinear is very bad shaped element, because in such a case inversion of this matrix is not possible or inversion of this matrix is difficult. That is on the side node, now by doing by carrying out the inversion of the matrix, and multiplying with t 1 t 2 t 3 we can get this equation where a is area of triangle, triangular element and f 1, f 2, f 3, b 1, b 2, b 3, c 1, c 2 and c 3 are defined. We now expressed a 0 a 1 a 2 in terms of t 1 t 2 t 3. Now substituting a 0 a 1 a 2 back into the equation that we started out with t is equal to a 0 plus a 1 x plus a 2 y.

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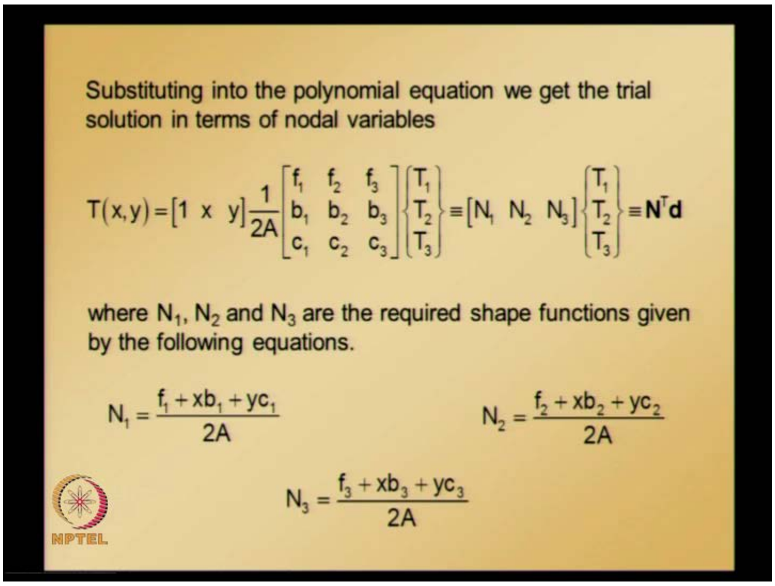

Substituting the polynomial equations equation we get the trail solution in terms of nodal variables or nodal parameters. Substituting a 0 a 1 a 2 back into the equation T is equal to a 0 plus a 1 x plus a 2 y, we get this equation which can be compactly written as N transpose d where N is a vector comprising of shape functions N 1 N 2 N 3, d is a vector comprising of nodal variables or nodal parameters or nodal values T 1 T 2 T 3. Where N 1 N 2 N 3 are the required shape functions and they look like this in terms of f 1 b 1 c 1, f 2 b 2 c 2, f 3 b 3 c 3 where f 1 f 2 f 3, b 1 b 2 b 3, c 1 c 2 c 3 are all defined in the previous slide.

Since all these f‘s b‘s and c’s are dependent on the coordinates of the nodes of three node triangular element. This can be easily computed, once the coordinates of nodes are known and if you see N 1 N 2 N 3, these are linear in terms of x and y. So that is why three node triangular element is also called constants strain triangular element. Since N 1 N 2 N 3 are all linear in x and y, derivative of N with respect x derivative of n, all these n’s with respect x and y is going to be a constant. That is why three node triangular element is also called constant strain triangular elements c s t. So, T can be written as N 1 T 1 plus N 2 T 2 N 3 T 3.

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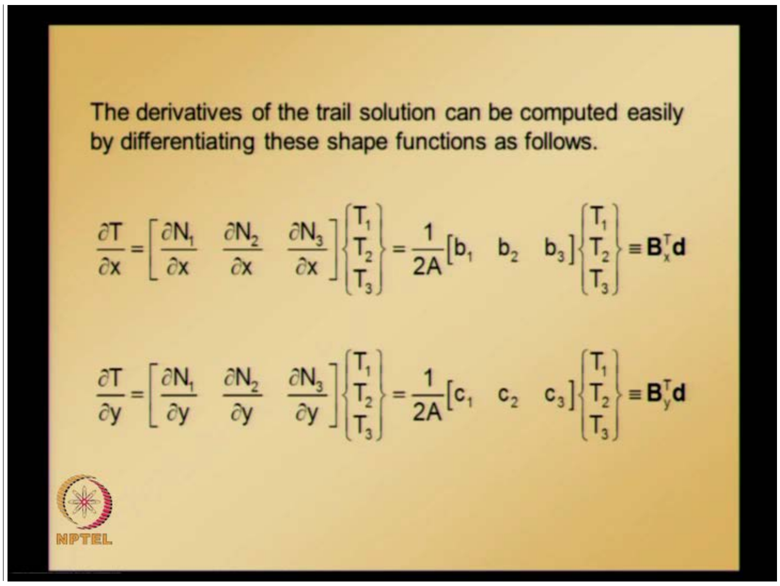

The derivative of trial solution T with respect to x and y can be written like this. You can see derivative of T with respect x is a constant, because b 1 b 2 b 3 are all constant and T 1 T 2 T 3 are the nodal parameters and A is a constant which is area of triangle. Finally, derivative of T with respect to x is constant. That is why this element is called constants strain triangular element and similarly, derivative of T with respect to y is constants and it is going to be function of c 1 c 2 c 3, and each of this can be compactly written as B x transpose d, B y transpose d.

Now we derived element shape functions for three node triangular element, we expressed trial solution, derivative of trial solution in terms of finite element shape functions, derivative of finite element shape functions and the nodal parameters or nodal values. Now, we can plug in all this into the equivalent functional and get the element equations f, we decide to use a Rayleigh-Ritz method and we can also adopt Galerkin criteria to derive the element equations. So, what we will be doing is we will now look at Galerkin criteria to derive element equations for the same general two-dimensional boundary value problem.

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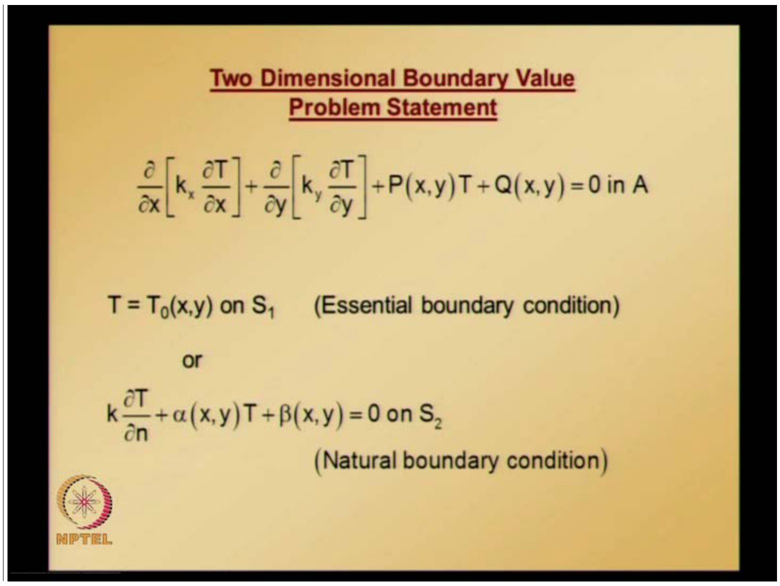

Let us recall the problem that we are looking, this is the differential equation we need to solve over domain A subjected to the boundary conditions, these are the boundary conditions.

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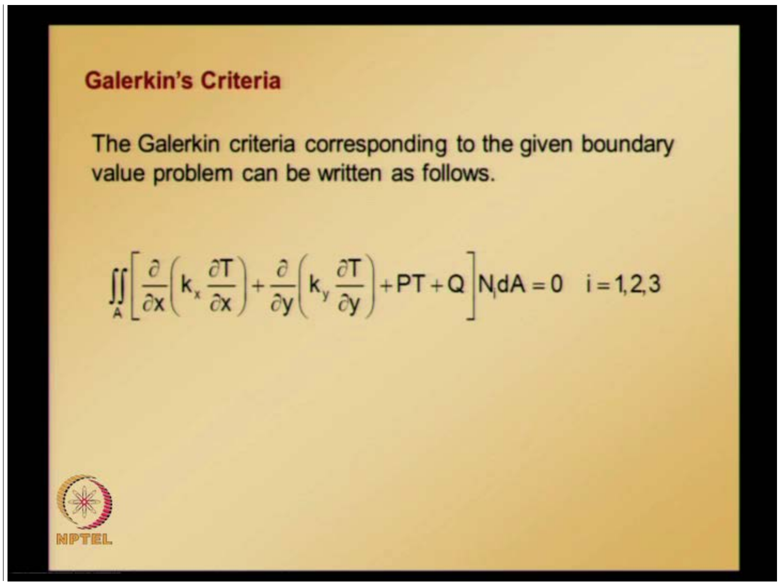

Galerkin criteria corresponding to the given boundary value problem can now be written like this. Galerkin criteria is multiply given differential equation with a weight function integrate over the problem domain equated to 0, and if you decide use finite element method in conjunction with Galerkin criteria, weight function is same as shape function and if it is a three node triangular element there are three shape functions. So, I takes values 1, 2, 3, and the first two terms inside the integral can further be simplified using integration by parts in two dimensions that is the Green's theorem that we already looked at.

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Let us once again look at those identities the formulas. These are the two formulas that we looked at earlier we will be using this two formulas to simplify the previous equation further. So, applying integration by parts.

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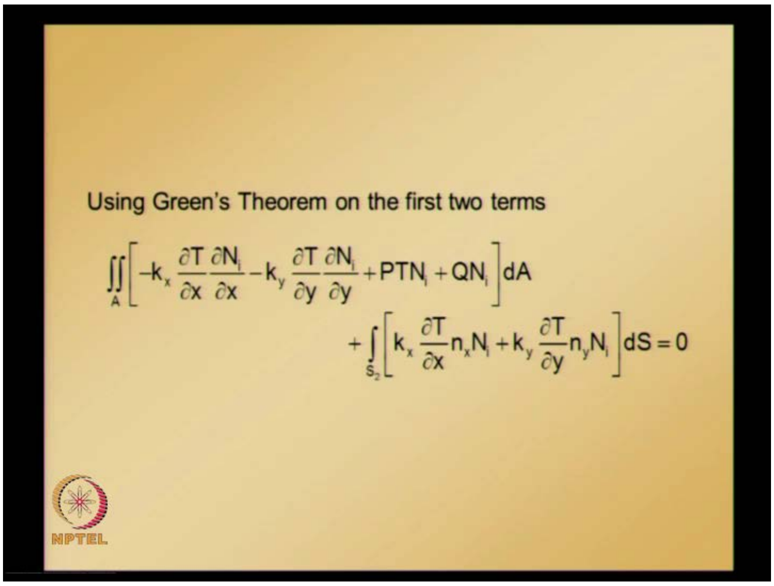

And the first two terms inside the integral of the previous equation we get this equation, and I takes values 1 2 3.

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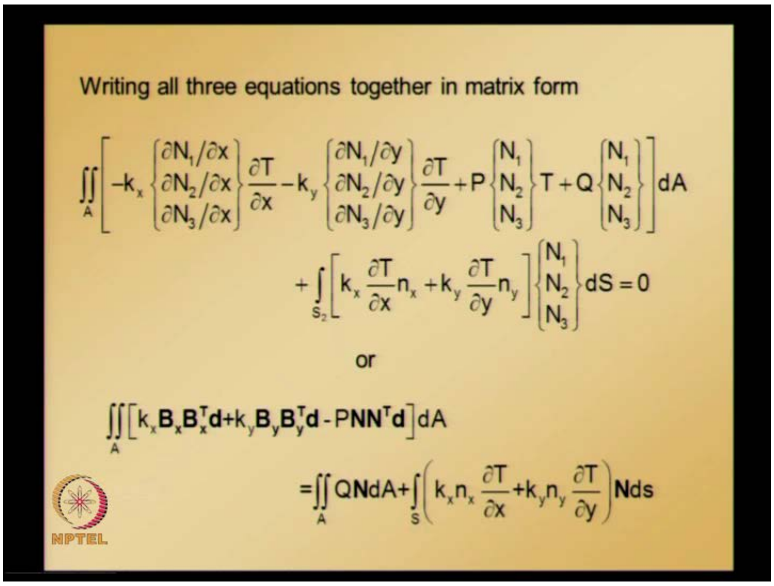

So, further we can write the previous equation like this writing all the three equations together in matrix form we get this form, and if you see the boundary integral it needs to be evaluated over S 2; that is part of the boundary on which natural boundary condition is specified. And this equation can be rewritten in this manner, if you compare these two equations you can see in the first equation the boundary integral needs to be evaluated over S through, whereas in the second equation boundary integral it is written as to be evaluated over the entire boundary.

And why this is written like this is because the boundary S is equal to S 1 the part of the boundary on which essential boundary condition is specified, and part of boundary on which natural boundary condition is specified S 2. And the integral value over S 1 is unknown since along S 1, essential boundary conditions are specified and also this term the integral over S1 as we will see in a while it is going to show up as a reaction term in the element equations, and which we also observed when we are deriving element equations using Galerkin criteria for one dimensional problem. Further explanation on this integral which is over S 2, the first equation is replaced with integral S.

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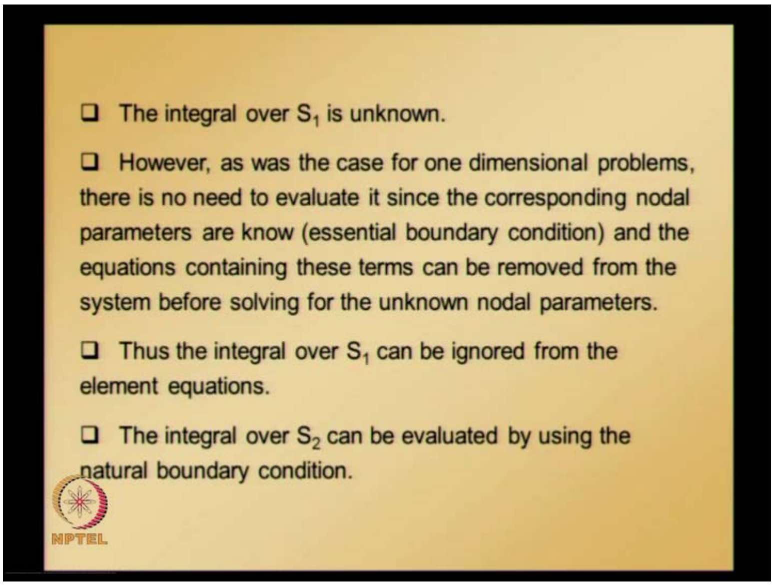

In the second equation further explanation on this can be explained like this, the boundary integral can be split into S 1 and S 2. That is, this integral over the boundary S can be written as integral over S 1 plus integral over S 2, and integral over S 1 is unknown, because essential boundary conditions are specified.

However as was the case for one-dimensional problems, there is no need to evaluate over S 1. Since the corresponding nodal parameters are known, essential boundary conditions are specified and the equation containing these terms can be removed from the system before solving for unknown nodal parameters, and this is what we did even when we are deriving the element equations for one dimensional problems. Thus integral over S 1 can be ignored from the element equations and the integral over S 2 can be evaluated by using natural boundary conditions. So, this is the reason why the integral over S is replaced with integral over S 2 in the second equation.

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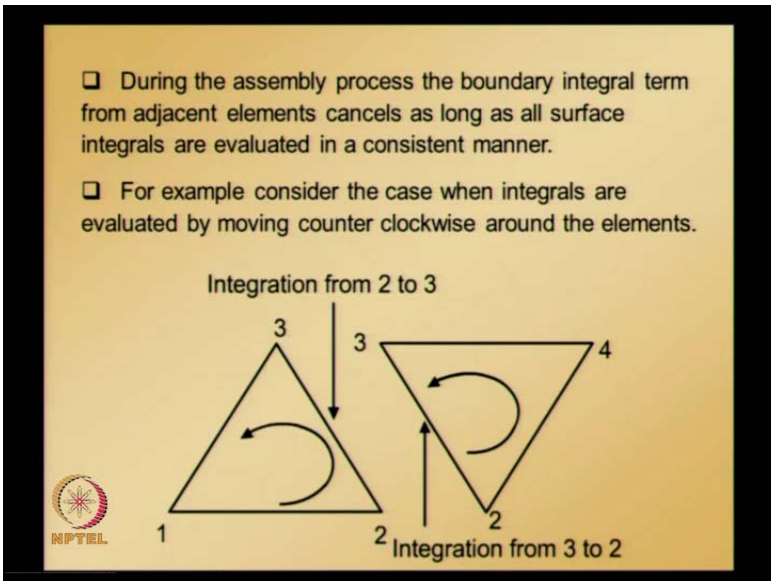

And now integral on the boundary during assembly process the boundary integral term from adjacent elements cancels as long as all surface integrals are evaluated in a consistent manner. To understand this statement, let us consider the case when integrals are evaluated by moving counter clockwise around elements like these two elements. So, what the statement says is during assembly process boundary integral term from adjacent element cancels as long as all surface integrals are evaluated in a consistent manner that is for all element either boundary integrals are evaluated by moving in the counter clockwise direction or clockwise direction. In the figure for demonstration the integrals are evaluated by moving in the counter clockwise direction.

Let us consider evaluation of boundary integral for each of these elements along the edge 2 3, so far one of the element integration is from 2 to 3 for the other element integration is from 3 to 2.

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The common edge between two elements will get integrated from two opposite directions resulting in same integral, but with opposite sign one integral is evaluated from two to three other integral is evaluated from three to two. So, when we add these two integrals they cancel each other, because they have the same value but with opposite sign thus all thus for all interior elements surface integral term can be ignored.

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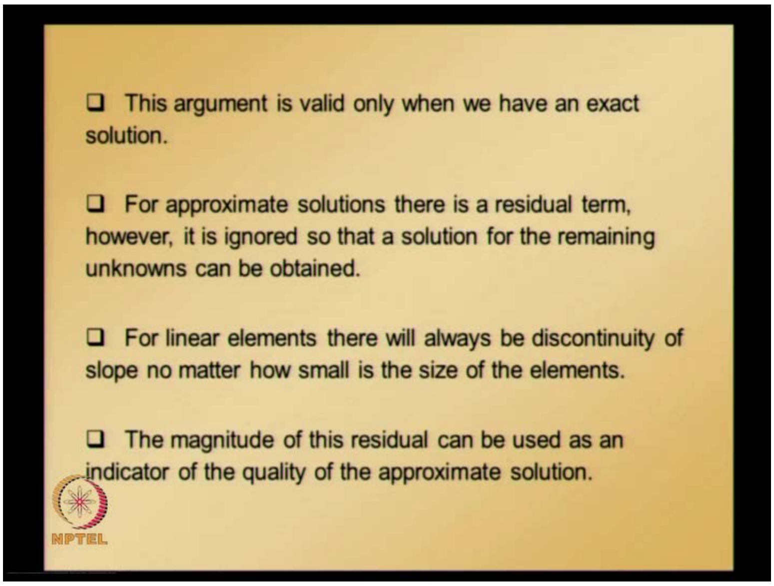

This argument is valid only when we have exact solution that is the integral value coming from each of this element is same, but when we are using finite elements after approximate solution. For approximate solution there is a residual term however it is ignored, so that a solution for the remaining unknowns can be obtained for linear element there is always a discontinuity of slope no matter how small the elements size is. The magnitude of this residual can be used as an indicator of quality of approximate solution. So, whatever is mentioned that is during assembly process boundary term from adjacent elements cancels this is in a way is not going to satisfy hundred percent. Since, we are trying to find approximate solutions, but even though it is not satisfied we are going to ignore that and the magnitude of the residual can be used as an indicator of the quality of approximate solution.

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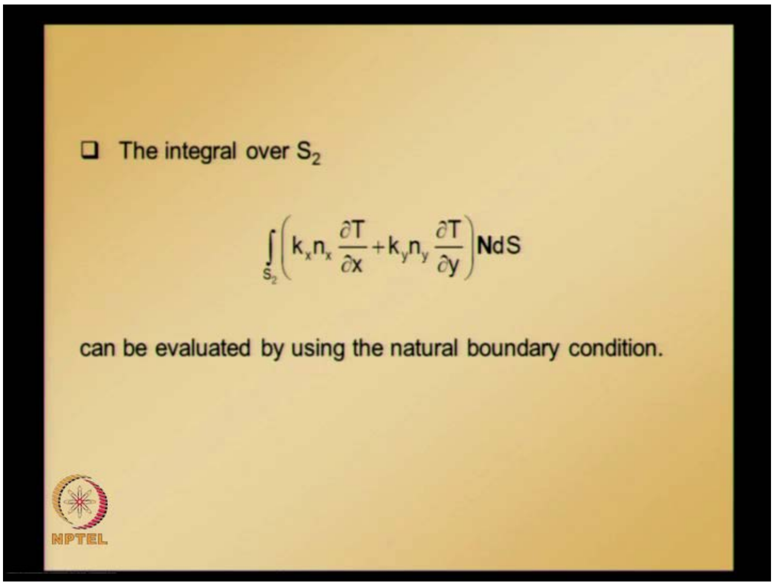

Now, let us look at how simplify further the boundary integral term using the specified natural boundary conditions, the integral over S 2 is this one this can be evaluated using natural boundary condition.

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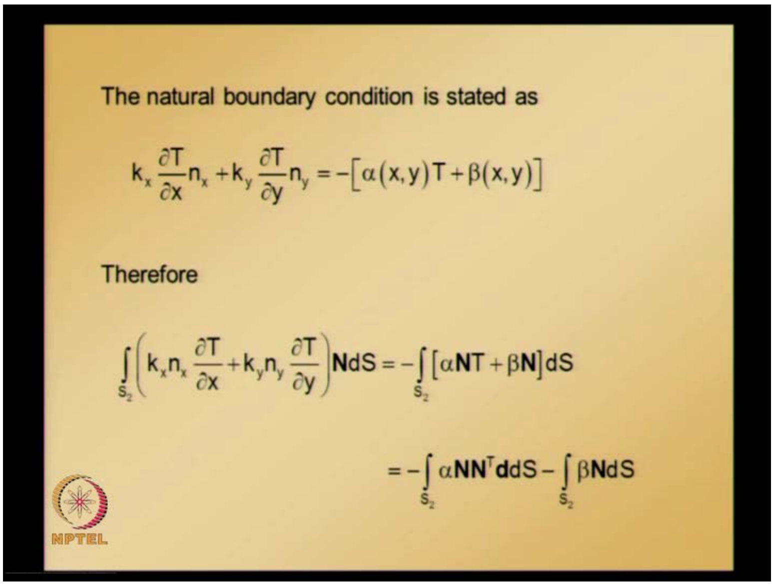

And now let us see what is the natural boundary condition that is given is stated as this one k x times partial derivate of T with respect x times n x plus k y times partial derivate of T with respect n y is equal to minus alpha T plus beta. Substituting this a into the integral it can be simplified in this manner, and substituting the finite element in terms of finite element nodal values, and shape functions it can be further simplify in the manner shown in the equation. Substituting this integral in terms of finite element shape functions and nodal parameters are nodal values into the earlier equation.

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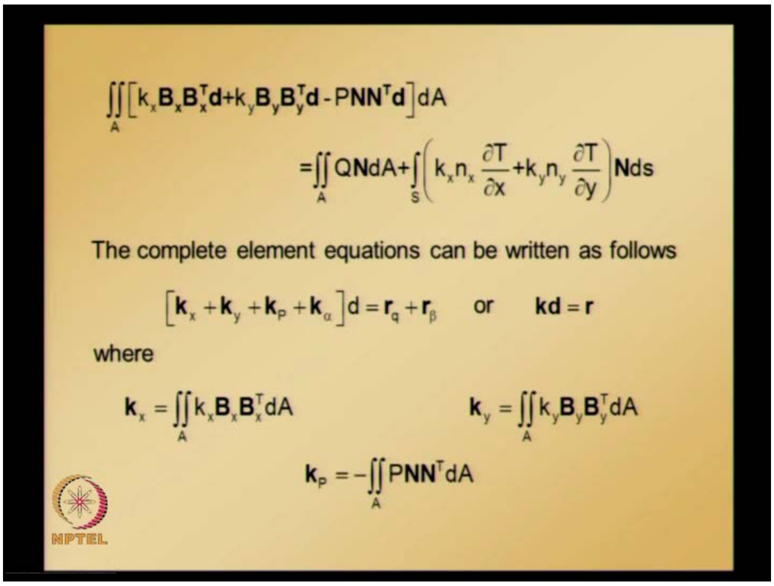

The complete element equations can be written as k x plus k y plus k p plus k alpha times d, where d is the vector comprising of nodal parameters are nodal values is equal to r q plus r beta for it can be compactly written as k d equal to r where k x k y k p are define, and all these integrals can be evaluated using numerical integration or they can be evaluated in closed form, if the integrand is not complicated. So this is how k x k y k p are define and rest of the terms.

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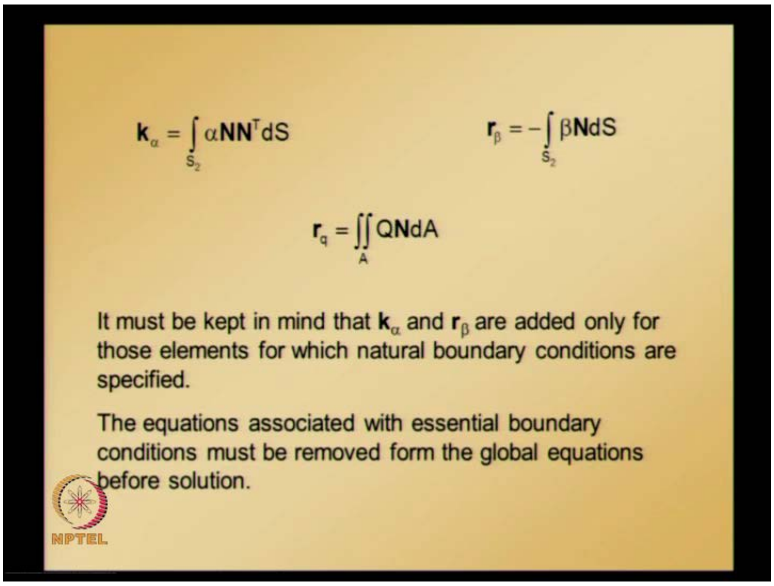

k alpha is defined like this which needs to be evaluated over surface on which essential boundary on which natural boundary condition is specify. r beta again boundary on which natural boundary condition is specify r q is a domain integral, and these evaluating all these quantities we can get element equations for a 3 node a for the two-dimensional boundary value problem adopting. Once we use shape functions corresponding to and trial solution corresponding to three node triangular element we get element equations for three node triangular element.

And please note that k alpha, r beta are added only for those elements for which natural boundary conditions are specified. If there is no natural boundary conditions specified these need not to be evaluated, equations associated with essential boundary conditions must be removed from global equations before the solution. And in these equations we need to substitute triangular element shape functions to further simplify these equations to get element equations for three node triangular element for two-dimensional boundary value problem which we will be seeing in the next class.