Finite Element Analysis

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Lecture No. # 20

Will continue with one-dimensional numerical integration that we are discussed in the last class, and we have seen in the last class that for a 3 node element, if the nodes are uniformly distributed, that is, if node 1 is at x is equal to 0, node 2 is at x is equal to half, node 2 is at x is equal to l over 2 or if the length of the element is 1 half and node 3 is at l, or if the length of unit element is unity, then node 3 is at x is equal to 1.

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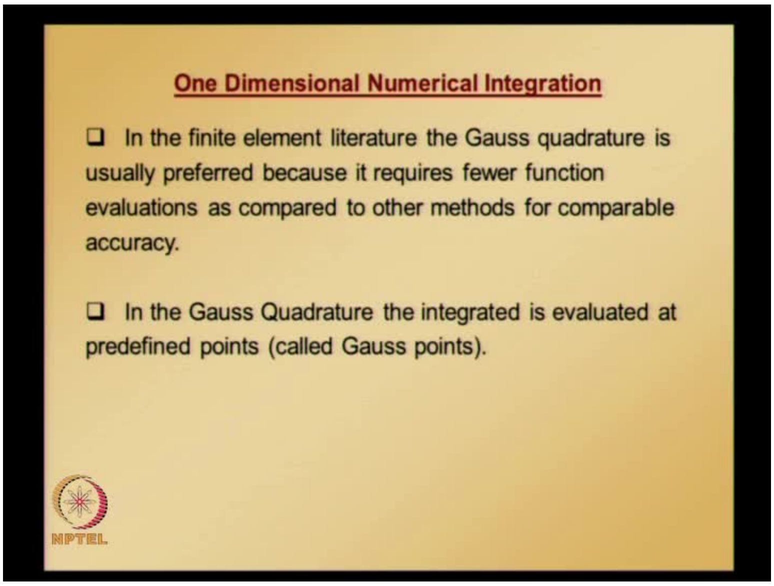

If the nodes are uniformly distributed then, Jacobian is constant. In that case, there is no problem in evaluating the integrals like, what we have seen in the for general one-dimensional boundary value problem like, k k k p k alpha, such kind of integrals and r q r beta, such kind of integrals. Since the integrand is going to be much simpler, we can actually evaluate these integrals using closed form; in a closed form manner, using the integration techniques that we have but when integrand is complicated, then we need to adopt numerical integration.

And also adopting numerical integration is easier, if we are trying to automate this finite element code. So, that is the reason why we are looking at this numerical integration. In finite element literature the Gauss quadrature is usually preferred because it requires fewer function evaluations as compared to other methods for comparable accuracy. This is what we have discussed in the last class and also here. The basically, what I am doing is briefly I am reviewing what we have done in the last class before I proceed further.

So, in Gauss quadrature integrand is evaluated at predefined points called Gauss points and the location of this Gauss points or derived in such a way that with n points a polynomial of degree 2 n minus 1 is integrated exactly and more details, you can find any of this standard books on numerical analysis.

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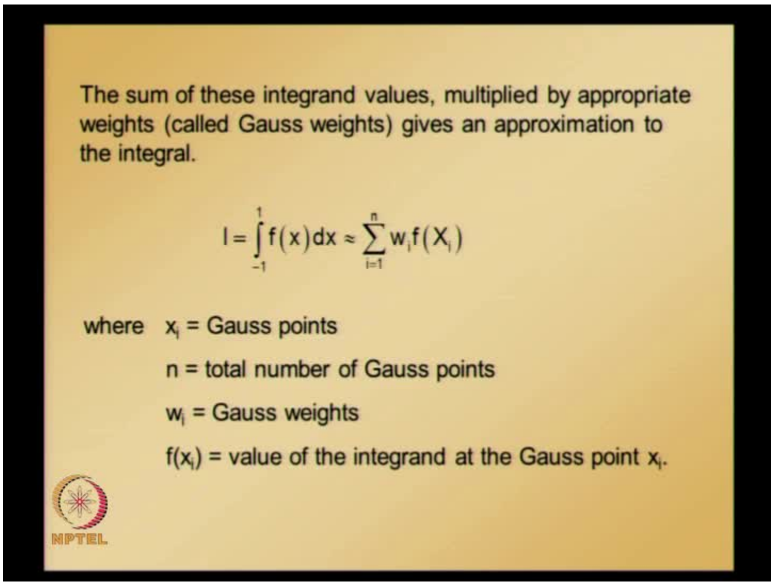

So, once we evaluate integrand at some predefined points multiply the integrand value at those points with some weight. So, the sum of these integrand values, multiplied by appropriate weight called Gauss weight gives an approximation to the integral. So, this is how we try to evaluate integrals using Gauss quadrature, for example, a function which needs to be integrator from minus 1 to 1 can be approximated as function, evaluated at some predefined points n number of small n number of points x. Here, I take values 1 to n. So, at predefined points x 1 to x n evaluate the function and multiply with corresponding weights of those points like w 1 to w n.

So, X i is Gauss point and his total number of Gauss points w i is Gauss weights function or the value of integrand, the Gauss point X i is f X i. Also in the last class, we have seen points in weights up to 10 numbers of points and the tables reproduced here.

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Gauss Quadrature

Gauss Points ($\pm x_i$)	Weights (w_i)
n = 1	
0.00000 00000 00000	2.00000 00000 00000
n = 2	
0.57735 02691 89626	1.00000 00000 00000
n = 3	
0.00000 00000 00000	0.88888 88888 88888
0.77459 66692 41483	0.55555 55555 55555
n = 4	
0.33998 10435 84856	0.65214 51548 62546
0.86113 63115 94053	0.34785 48451 37454
n = 5	
0.00000 00000 00000	0.56888 88888 88889
0.53846 93101 05683	0.47862 86704 99366
0.90617 98459 38664	0.23692 68850 56189

In the last class, the values corresponding to n is equal to 1 are not shown but here it is included. So, how to read this table? Here this table is partial, it shows from n is equal to 1 n is equal to 5. What are the Gauss points and corresponding weights and how to read this table? Please note that Gauss points, you have plus or minus X i, So, corresponding 10 is equal to 1, only one point that is 0 there is no plus or minus 0. So, 1 point and corresponding weight is 2, when you decide to go for two Gauss points and that is, if you want to integrate a polynomial of degree 2 n minus 1, exactly then you need to select n number of Gauss points.

So, if you select n is equal 2, you can integrate a polynomial of degree 3 exactly. So, in similar manner, we can select how many Gauss points, we can use for a particular integral depending on the order of polynomial of integrand. So, when n is equal to 2 is selected, you can see here from the table Gauss point coordinate is given as point 0.57735 and a space is there after that 02691 again space 89 626. So, here depending on the number of significant digits that you require for the accuracy, you can select either all the significant digits or you can chop it off after subtend number of significant digits. So, that is the reason why a pace is there in the table.

So, for n is equal to 2, the Gauss points are minus 0.57735, if I decide to take 5 significant digits and the other Gauss point is 0.57735 and the weight of these two points is equal to 1. So, that is how you can read this table and now when n is equal to 3, there are three Gauss points. First Gauss point is minus 0.77459 and the weight is 0.55555 and the next Gauss point is 0 weight is 0.888 and the third Gauss point is 0.77459 and the corresponding weight is 0.555. So, that is how we can read this table to get integration points and weights. And once you know the integration point, evaluate the integrand at the corresponding point and multiply with the corresponding weight and sum up over all the points then, we get the approximate value of the integral and here n is equal to 1 to n is equal to 5.

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Gauss Points ($\pm x_i$)	Weights (w_i)
n = 6	
0.23861 91860 83197	0.46791 39345 72691
0.66120 93864 66265	0.36076 15730 48139
0.93246 95142 03152	0.17132 44923 79170
n = 7	
0.00000 00000 00000	0.41795 91836 73469
0.40584 51513 77397	0.38183 00505 05119
0.74153 11855 99394	0.27970 53914 89277
0.94910 79123 42759	0.12948 49661 68870
n = 8	
0.18343 46424 95650	0.36268 37833 78362
0.52553 24099 16329	0.31370 66458 77887
0.79666 64774 13627	0.22238 10344 53374
0.96028 98564 97536	0.10122 85362 90376

The integration points coordinates and weights are shown and in the next table n is equal to 6 to n is equal to 8 are shown corresponding points in weights.

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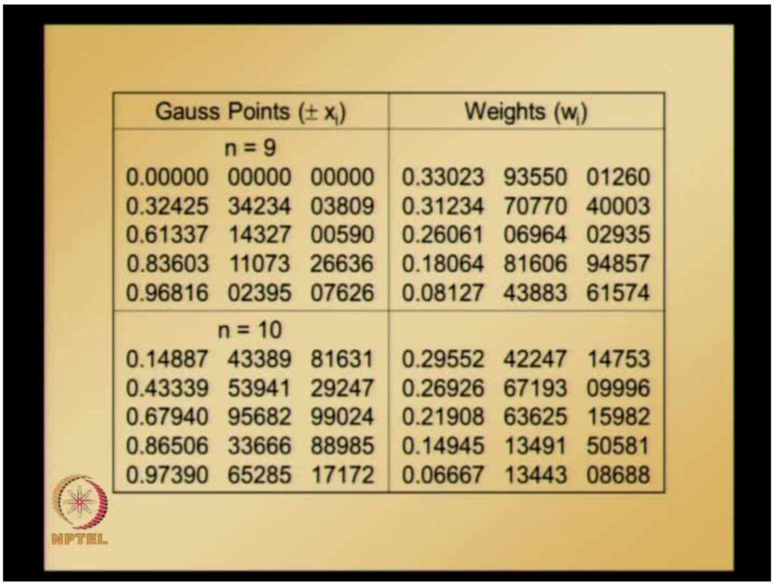

Gauss Points (± x_i)	Weights (w_i)
n = 9	
0.00000 00000 00000	0.33023 93550 01260
0.32425 34234 03809	0.31234 70770 40003
0.61337 14327 00590	0.26061 06964 02935
0.83603 11073 26636	0.18064 81606 94857
0.96816 02395 07626	0.08127 43883 61574
n = 10	
0.14887 43389 81631	0.29552 42247 14753
0.43339 53941 29247	0.26926 67193 09996
0.67940 95682 99024	0.21908 63625 15982
0.86506 33666 88985	0.14945 13491 50581
0.97390 65285 17172	0.06667 13443 08688

The way you have to read this table is similar to what I already explain to you. And n is equal to 9, n is equal to 10, number of Gauss points n is equal to 9 n is equal to 10 are shown here the details of points and weights. And if you require for more number of points and the corresponding weights and coordinates, you can refer any of the commercial software's like Matlab or mathematical and some of the software directly give for any number of points the corresponding coordinates and weights.

So, now let us, try to evaluate in integral numerically using, Gauss quadrature. Let us take an example, evaluate the following integral using Gauss quadrature, integral 8 x power 7 plus 7 x power 6 integrated from minus 1 to 1 and here integrand is 8 x power 7 plus 7 x power 6. We can also integrate this in a closed form manner, because integral x power 7 is x power integral x power 7 dx is x power 8 divided by 8, similarly integral x power 6 dx is x power 7 divided by 7.

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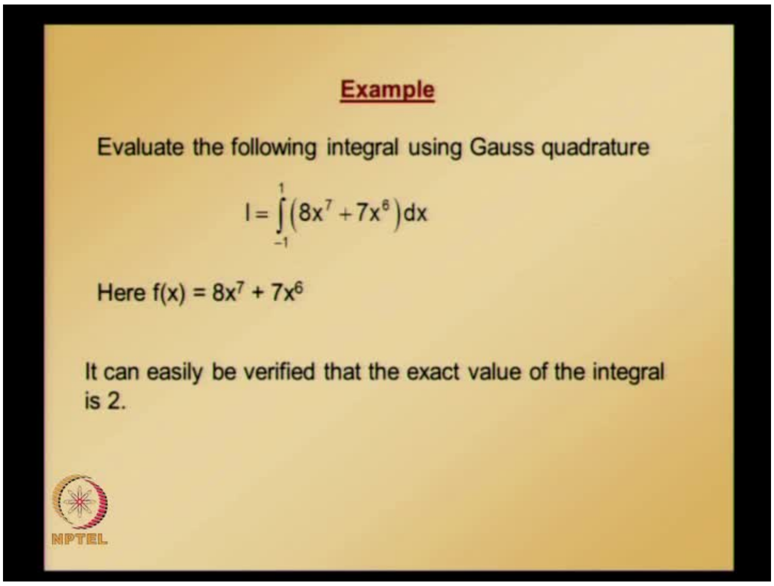

So, we can plug in the limits of integration and evaluate exactly in a closed form manner the value of this integral, it can easily be verified that the exact value of integral is equal to 2, but that is not the purpose of this lecture, we are trying to evaluate this integral using or we are trying to learn to evaluate this integral using Gauss quadrature, and you can see here the integrand order of polynomial is 7. So, you can back calculate how many points of Gauss quadrature exactly gives or how many Gauss points gives exact solution. Since the order of polynomial is 7. So, 2 n minus 1 is equal to 7 so you can back calculate n is equal to 4.

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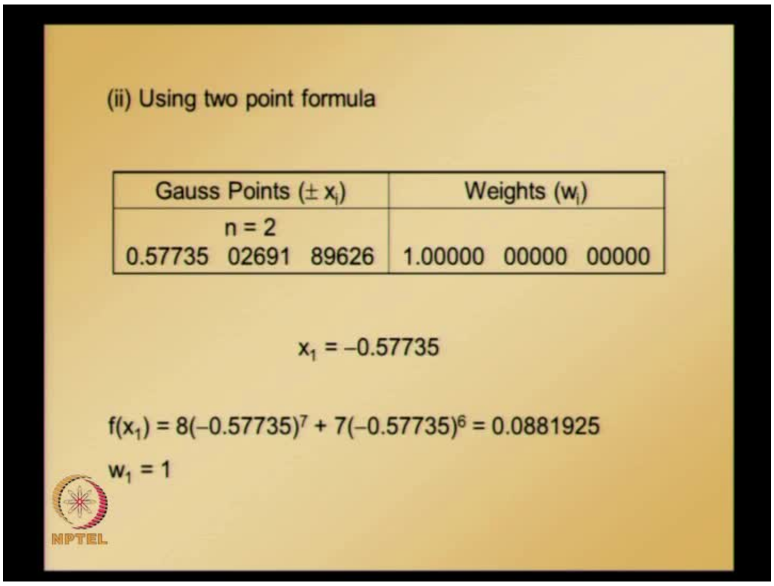

So, when we adopt 4 Gauss points, we are going to get this integral value exactly. Now, let us try using 1 point, 2 point, 3 point and also 4 point and let see, how the solution converges. So, using 1 point formula that means, we decided to use 1 Gauss point and the corresponding weight for n is equal to 1, the coordinate is 0 weight is 2. So, only one integration point is there and corresponding weight is indicated there and also evaluate the integrand at x is equal to 0, that is, f x 1 is equal to 0, if you evaluate that is substituting x is equal to 0, in integrand, that is, 8 times x power 7 plus 7 times x power 6 when we plug in x is equal to 0 in that integrand it terms out that is going to be 0.

And finally function value at the integration point times weight gives us 0. So, 1 point formula, 1 point Gauss quadrature approximates the given integral to be 0, which is not making any sense. So, now, let us go to 2 points; so, when we decide to use 2 points, we need to figure out what are the coordinates and weights. So, the coordinates and weights from the table it can be easily checked these are the values.

So, now, what we need to do is we need to evaluate function at these points, that is, minus 0.57735 multiplied with 1 and add it to function value at 0.57735 multiplied by 1. So, first integration point evaluate the integrand, that is, 8 times x power 7 plus 7 times x power 6 plugging in x is equal to minus 0.57735 and the corresponding weight 1 is also indicated there.

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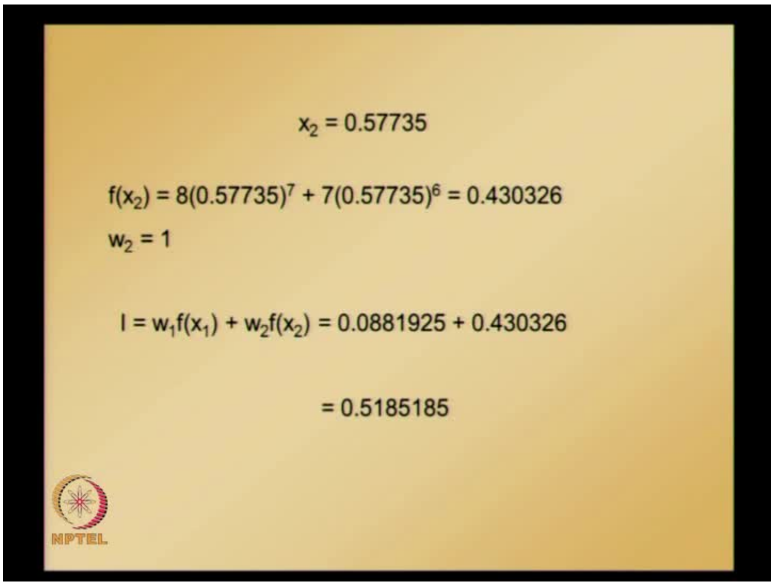

Similarly, evaluate function at the second integration point, which is 0.57735, the corresponding weight is also indicated. So, approximate value of integral is function evaluated at x 1 times w 1 plus function evaluated at x 2 times w 2 and the value is shown there. The approximate value of integral is 0.5185185 for the significant digits that are selected. Here if you want more accurate or more accurate result are the solution to more significant digits accuracy then, we need to select the integration points and weights to more significant digits.

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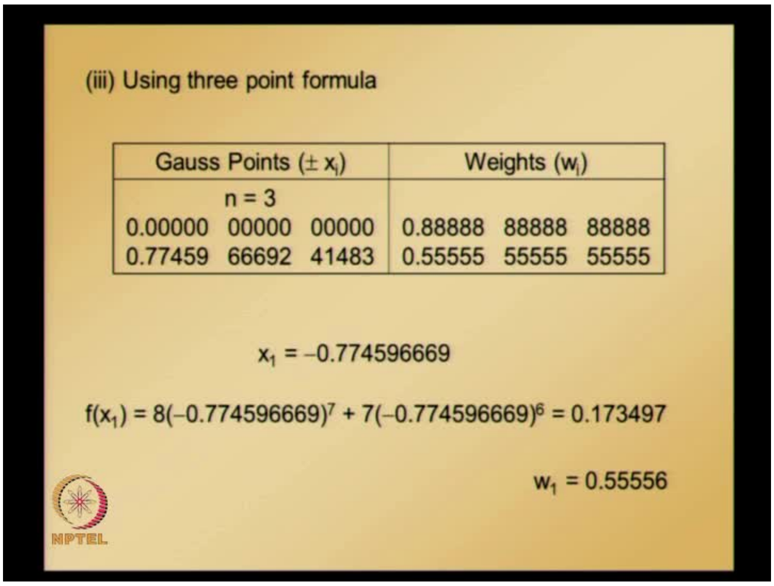

So, now, let us we do this integral evaluation using 3 points. The coordinates and weights from the table it can be easily read, these are the values of points and weights.

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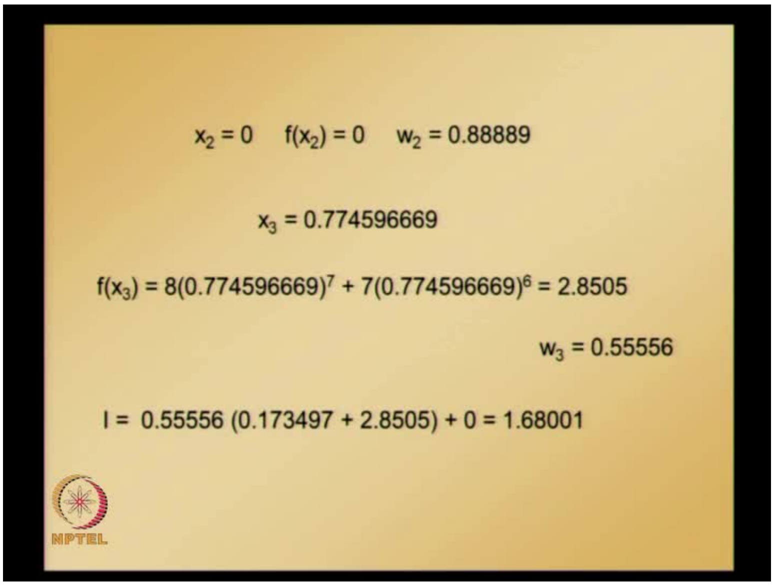

So, we need to evaluate function at 3 points minus 0.77459 0 and multiply with corresponding weights, integrand value at the first integration point and corresponding weight integrand value at the second integration point and corresponding weight, the third integration point integrand value at the third integration point, the corresponding weight. So, we now have at all points, we have the integrand value and corresponding weights and we can approximate integral like this. You can see one integration point when we use n is equal to 1 we got 0, when we adopted n is equal to 2, we got 0.5185 and we and we adopted 3 points we got 1.68.

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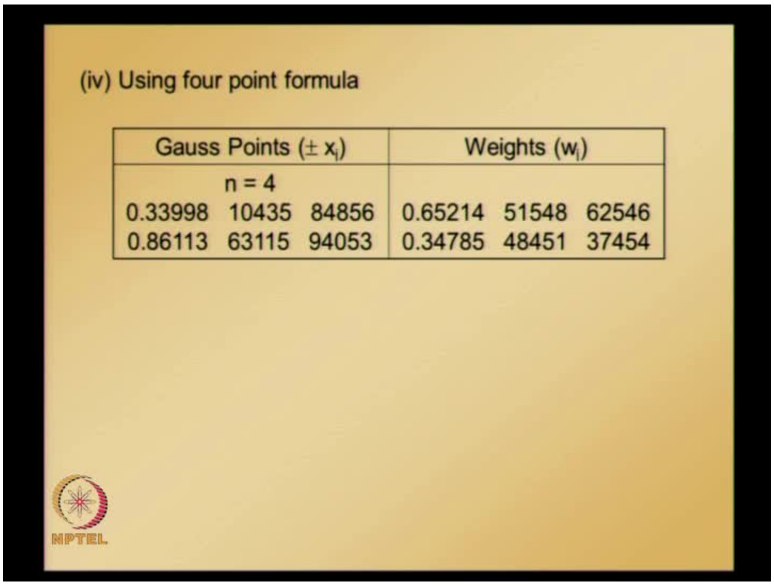

So, let us see, still the solution is not converged. Let see, what we get, when we use 4 points, so for n is equal to 4, the coordinates, the corresponding weights are given here.

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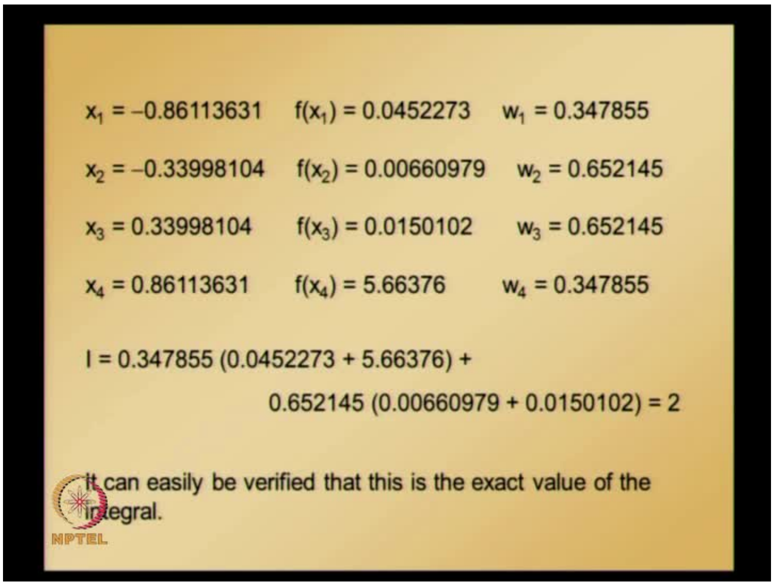

So, we need to evaluate function at the 4 points and multiply with corresponding weights and sum it up value of function and weight at the first integration point, value of function and weight at the second integration point, value of function and weight at the third integration point and similarly at the fourth integration point.

So, now what we need to do is W 1 times f X 1 plus W 2 times f X 2 plus W 3 times f X 3 plus W 4 times f X 4. And we get approximate value using Gauss quadrature, adopting 4 points we get integral value to be 2, and if you check it can easily be verified that this is the exact value of integral, because before we started out using Gauss quadrature to evaluate this integral.

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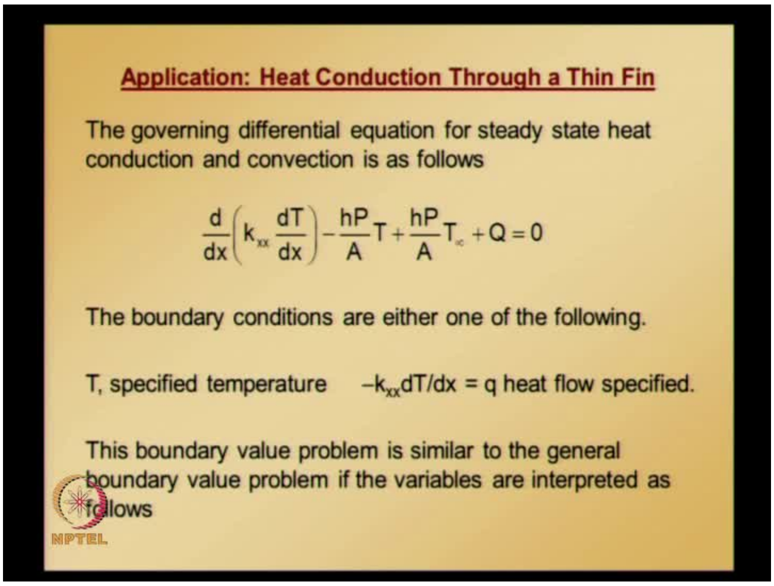

We noted down the exact solution for this problem, which is 2. So, we can use numerical integration it depending on the using the number of Gauss points, depending on the order of integrand keeping in mind n point Gauss quadrature integrates a polynomial of degree 2 n minus 1 exactly. So, now, let us see application of this Gauss quadrature to some general one-dimensional boundary value problem. And let us solve this heat conduction through a thin fin that we already looked at earlier using, when we are actually, when we are actually looking at 2 node linear finite element applied to general one-dimensional boundary value problem and if you recall, the governing differential equation for steady state heat conduction and convection is as given there.

The corresponding boundary conditions are this and to develop the element equations, if you recall what we did is we compared this differential equation corresponding to steady state heat conduction convection with general one-dimensional boundary value problem and identified the corresponding coefficients and once we identified the corresponding coefficients, we already have element equations for general one-dimensional boundary value problem, we can substitute a corresponding coefficients in those element equations to get element equations for this particular case of steady state heat conduction convection. So, we need to make a comparison between this differential equation or this boundary value problem and general one-dimensional boundary value problem. This boundary value problem is similar to general boundary value problem, if variables are interpreted as given in the table here, which shows comparison of corresponding variables.

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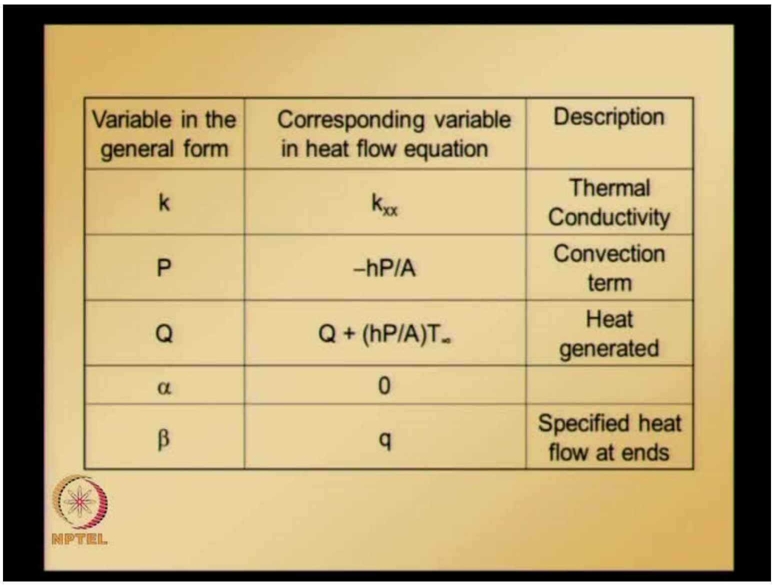

Variable in the general form	Corresponding variable in heat flow equation	Description
k	k_{xx}	Thermal Conductivity
P	−hP/A	Convection term
Q	Q + (hP/A)T_∞	Heat generated
α	0	
β	q	Specified heat flow at ends

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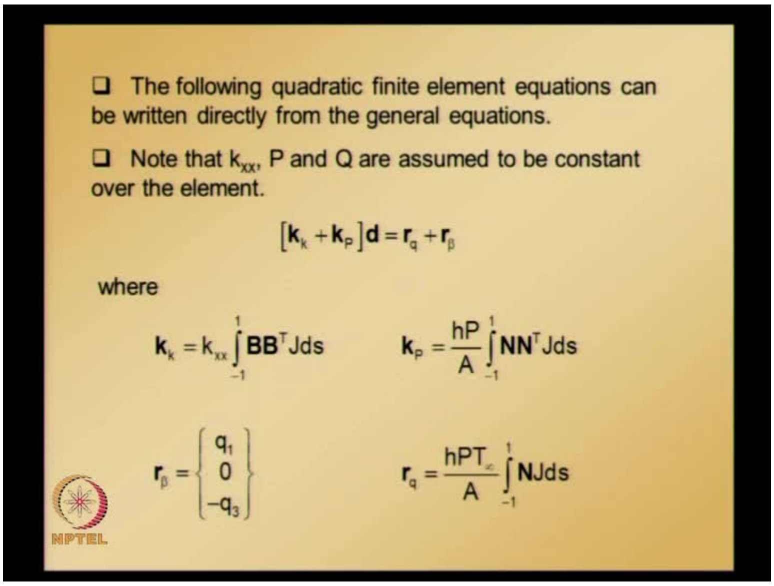

In general form and the specific case of heat conduction and convection once we have this kind of comparison. The following quadratic finite element equations can be written directly from general equations. By replacing the corresponding coefficients in general, one-dimensional boundary value problem with the coefficients corresponding to the specific case note that k x P Q are assumed to be constant over element and the element equations for steady state heat conduction convection are these; where k k k p or beta or q are defined.

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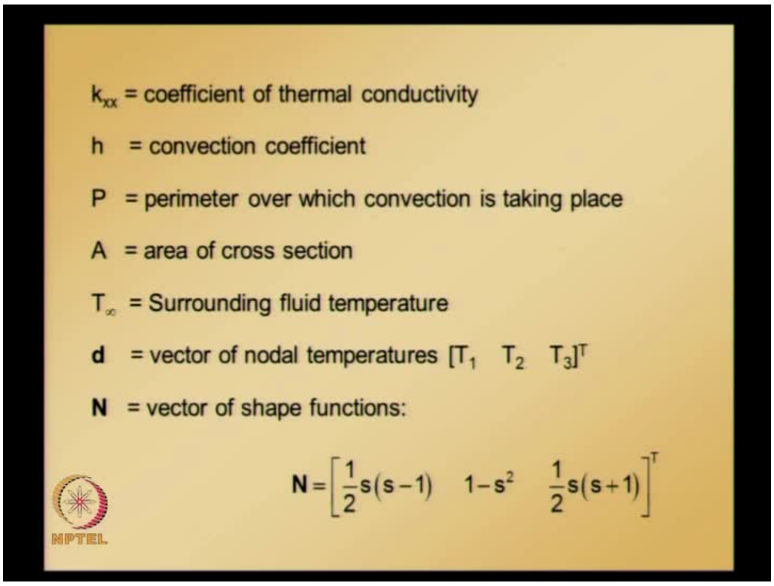

This coefficients, we have already seen the meaning of this coefficients when we are actually solving the same problem using 2 node linear finite element methods. So, k xx is coefficient of thermal conductivity; h is convection coefficient; P is perimeter over which convection is taking place; A is area of cross section; T infinity surrounding fluid temperature; d is vector of nodal temperatures; N is vector of shape functions and this here we are dealing with 3 node quadratic element 1 d element.

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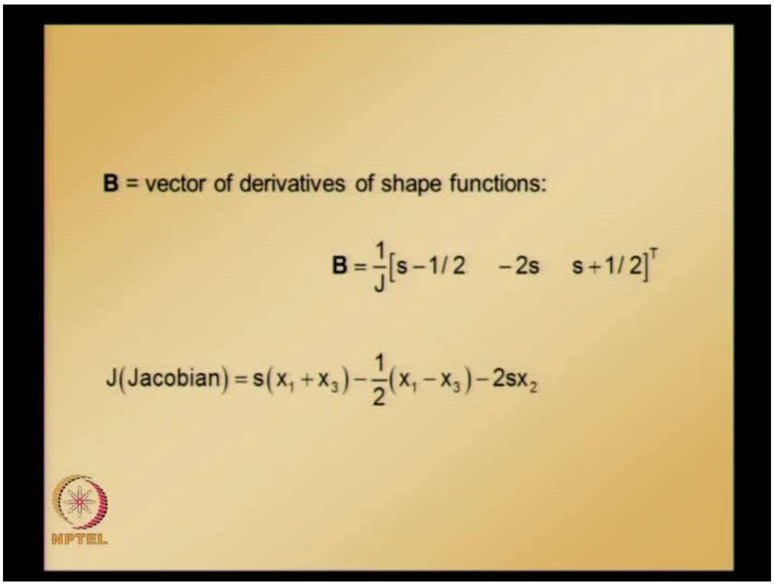

So, the corresponding shape functions in parent element N 1, N 2, N 3 are shown there and the derivatives of shape functions, which is denoted with B bold letter B is obtained by taking derivatives of shape functions N 1, N 2, N 3 with respect to s and multiplying with ds over dx which is 1 over J. J is Jacobian for a 3 node element is defined like this all these things, we have already seen repeatedly many times.

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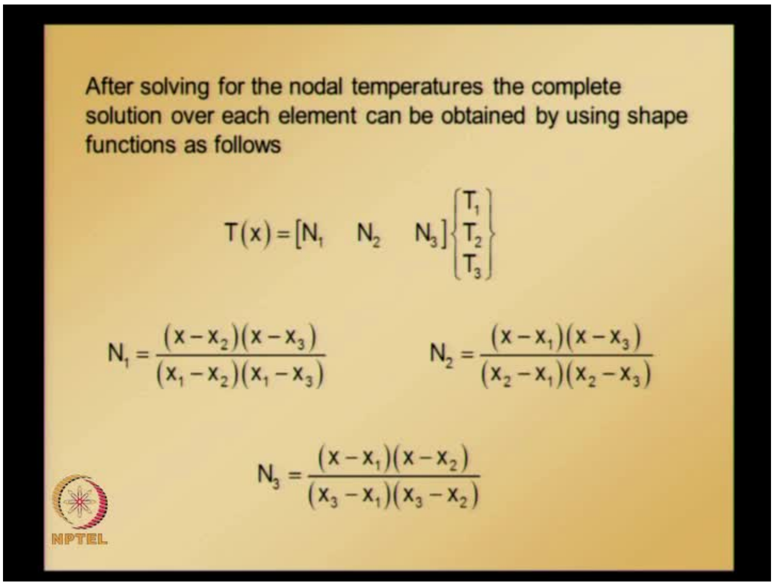

So, now, we have all the quantities. We need to plug in all these into the matrices k k k p r q and evaluate integrals and simplify and get element equations, After solving for nodal temperatures complete solution over each element can be obtained using shape functions just interpolation using shape functions, once we know the nodal values of temperatures and for a 3 node quadratic element. We can derive these shape functions N 1, N 2, N 3 using lagrangian interpolation formula. So, this is required for post processing once we get the nodal temperatures of all temperatures at all nodes to get complete solution over element, we need to do this kind of interpolation.

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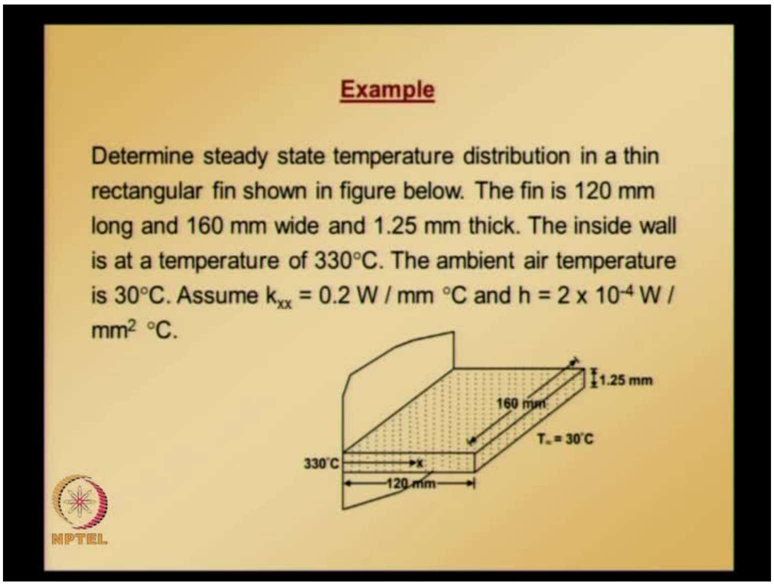

And now, let us look at example. which we already solve using 2 node linear element once again we solve this problem using, 3 node quadratic element and while doing. So, we also learn application of numerical integration; so, the problem statement is here. Determine steady state temperature distribution in a thin rectangular fin shown in figure. The fin dimensions and also temperatures of at the inside wall and also ambient air temperature and also material properties like coefficient of thermal conductivity convection coefficient. All these are given and the schematic shows the rectangular fin

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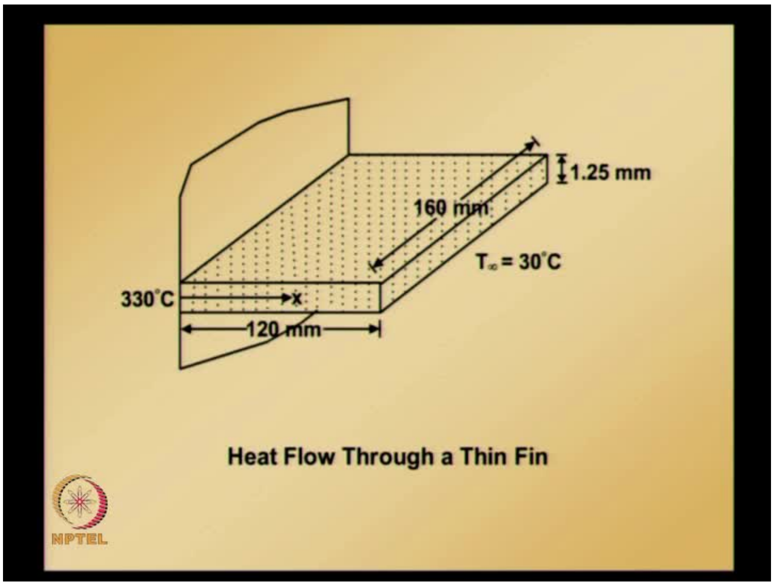

And now, perimeter per unit length is there are the temperature convection takes place from top and bottom surfaces are assumed to be taking place from top and bottom surfaces.

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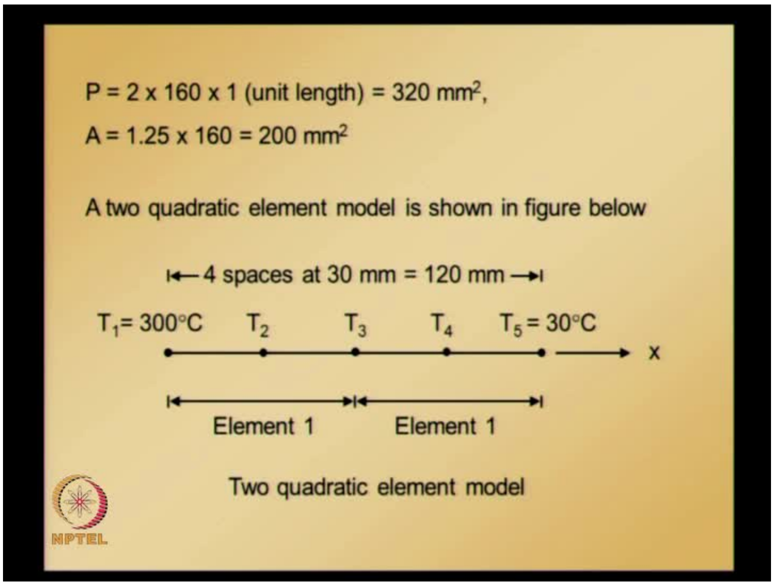

So, perimeter per unit length is obtained using this formula, 2 times 160 times unit length and from the dimensions given, we can easily check area of cross section thickness is given as thickness of rectangular fin is given as 1.25 millimeters and width is given as 160 millimeters.

So, we can find area of cross section entire domain of this rectangular fin is divided using, two quadratic elements with 4, with 5 nodes equally spaced and the nodes are the temperature corresponding to each of this 5 nodes are shown T 1, T 2, T 3, T 4, T 5. T 1 is given from the problem it is given as 330 degree centigrade it in the figure there is a typo it should be 330 degree centigrade and for element 1 the nodes are 1 2 and 3 for element 2 nodes are 3 4 5, again there is a typo it should be element 2. It is printed as element 1 the ambient temperature is given as 30 degrees centigrade. So, T 1 is equal to 330 degree centigrade T 5 is equal to 30 degree centigrade, subjected to these two boundary conditions.

We need to solve this problem of steady state temperature distribution, over this rectangular fin. So, this is the discretization that is adopted for solving this problem and if you notice that the boundary conditions at T 1 or at node 1 and node 5 both of these are essential boundary conditions and 2 elements are there and each of the element all the nodes are spaced in the same manner.

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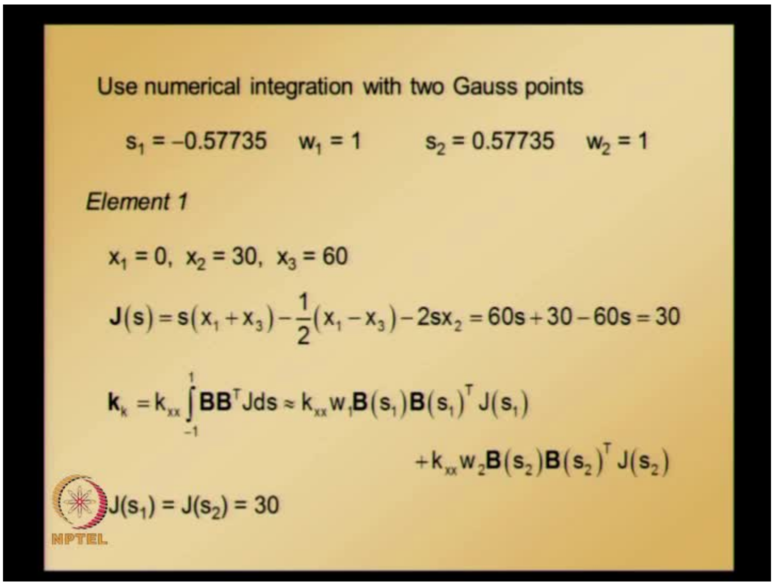

So, element equations for element 1 and element 2 are going to be identical. So, if we derive element equations for element 1, element 2 equations also look similar, so to derive the element equations for element 1, we need to evaluate some of the integrals between minus 1 to 1 k k k p such kind of integrals.

So, we will be using numerical integration to evaluate these integrals with two Gauss points adopting two Gauss points, the coordinates and weights of each of these Gauss points are indicated there. So, for element 1, 3 nodes a corresponding nodal coordinates X 1, X 2, X 3 are shown and once we have X 1, X 2, X 3 values we can easily evaluate what is Jacobian J? We can easily calculate Jacobian value, which turns out to be a constant 30, because all the nodes are equispaced or uniformly distributed.

And now to evaluate the integral k k, please note that k xx thermal conductivity is constant coefficient of thermal conductivity is constant. So, it is taken out of the integral and we need to evaluate integral minus 1 to 1 BB transpose Jds and that can be evaluated using one-dimensional numerical integration by selecting two Gauss points. So, integral k k can we approximated as evaluation of integrand at each of these Gauss points multiply with corresponding weight and sum it up. So, that is what is shown there and J is constant at both integration points which is 30.

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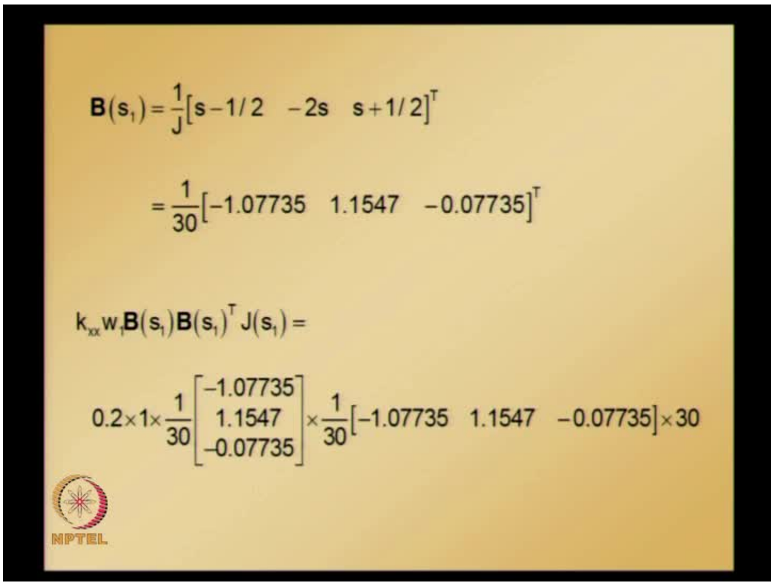

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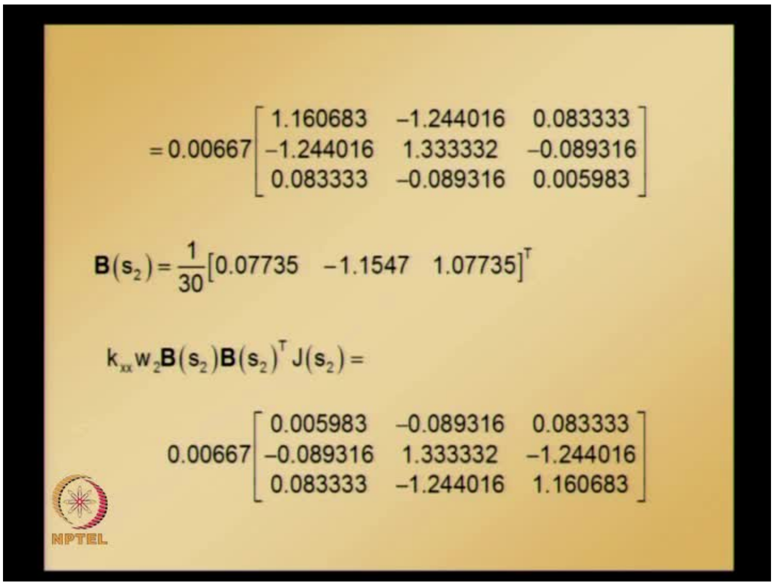

So, substituting all these and also be which is a vector of shape function derivatives. So, once we have this quantities we can plug in and get the integrand value at integration point 1 multiplied by weight simplified form of that and shape function derivatives at the second integration point.

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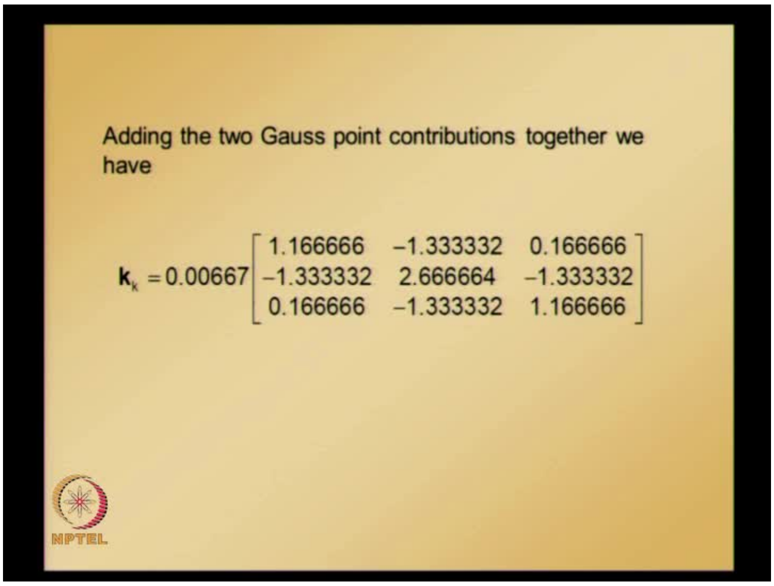

Integrand value multiplied by weight at second integration point and simplification of this gives and also adding the contribution at the two points, we get approximate value of k k using 2 integration points.

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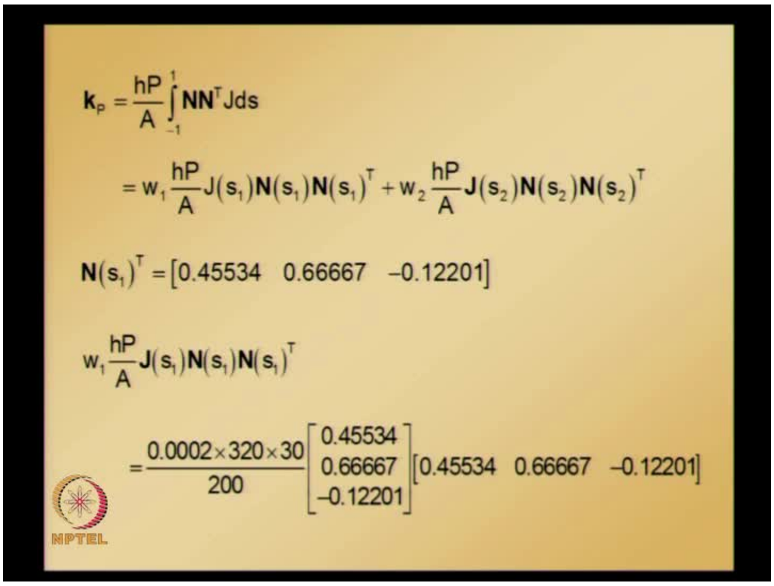

And now, the second integral just k P, this can also be evaluated using two integration points. The details are given here weight times integrand value at integration point one plus weight at second integration point multiplied by integrand value at second integration point.

And similar to the earlier integral, where we evaluated derivatives of shape functions vector at each of the integration points, we need to evaluate here shape function vector at each of the integration points shape function vector at first integration point integrand value multiplied by weight at the first integration point.

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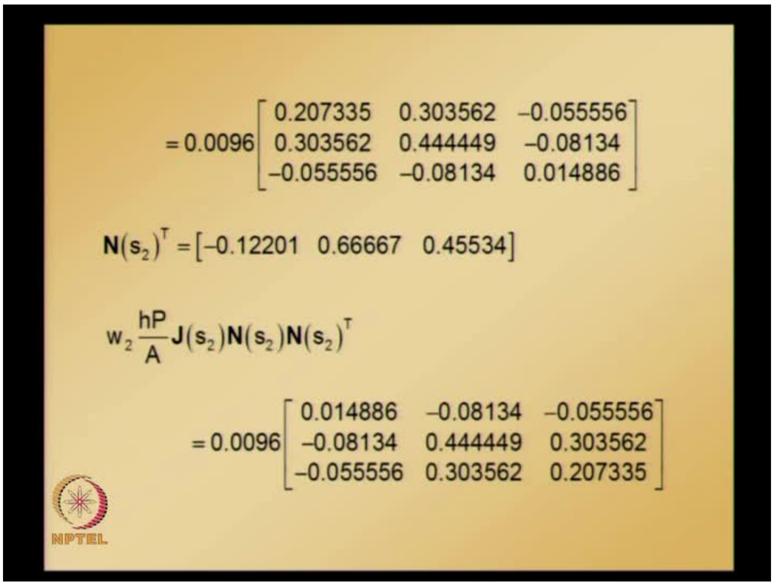

And carrying out vector multiplication, I get this one now shape function vector at second integration point integrand at the second integration point multiplied by weight at second integration point is equal to this one.

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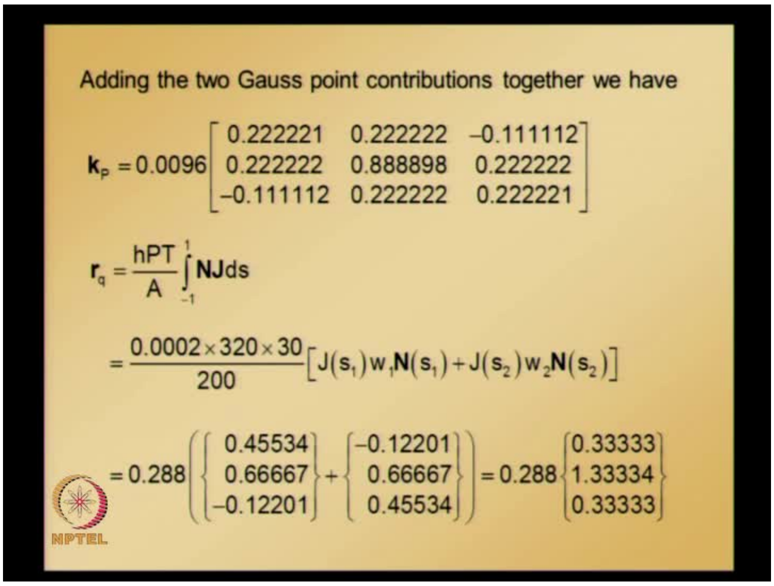

So, now, we have integrand value at the first integration point times W 1 integrand value a second integration point times W 2 add this two together adding the two Gauss point contributions together, we get k P value.

So, this is how we can evaluate integrals using numerical integration. And now, one more integral is left r q r q is defined like this, and using two point integration integrand multiplied by weight at first integration point plus integrand multiplied by weight at second integration, but the details are shown and substituting the corresponding values of shape functions at first integration point, second integration point and summing up we get this one this is r q.

And here, we adopted two point for evaluating all the integrals, we adopted two Gauss points assuming that when we adopt two Gauss points, we can integrate function of order 3 exactly assuming the functions integrate function of in integrand is of order three. We adopted two Gauss points usually in finite element method, two point integration is adopted, but if somebody requires are if they can guess the order of polynomial in integrand is higher than we can go for higher order Gauss quadrature that is using more than 2 number of Gauss points.

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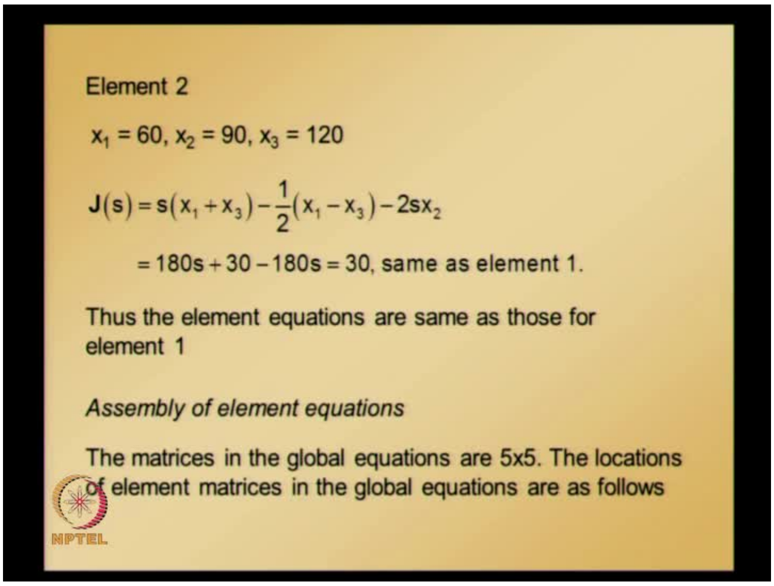

So, now, let us look at these are the matrices and vector corresponding to element 1. What about element 2? Element 2 locally the coordinates of node 1, node 2, node 3 are shown there and all the distance between each of these nodes and the total length of this element 2 are same as that in element 1 and also Jacobian is same as element 1.

So, element equations are same as those for element 1. So, we do not need to go through the entire numerical integration details for element 2, because all those matrices and vector value are same. So, now let us, assemble, so we have element equations for element 1, element equations for element 2. So, we can assemble element quantities only thing is we need to note that at each there are 5 nodes in this problem based on the discretization that we adopted 2 quadratic elements total 5 nodes.

So, global equation system is going to be a 5 by 5 and node one contribution goes into 1 2 3 rows and columns, because the node 1 comprises of nodes 1 2 3 and node 2 element 1 contribution goes into 1 2 3 rows and columns, because element 1 comprises of nodes 1 2 3 and element 2 contribution goes into 3 4 5 rows and columns, because element 2 comprises of nodes 3 4 5.

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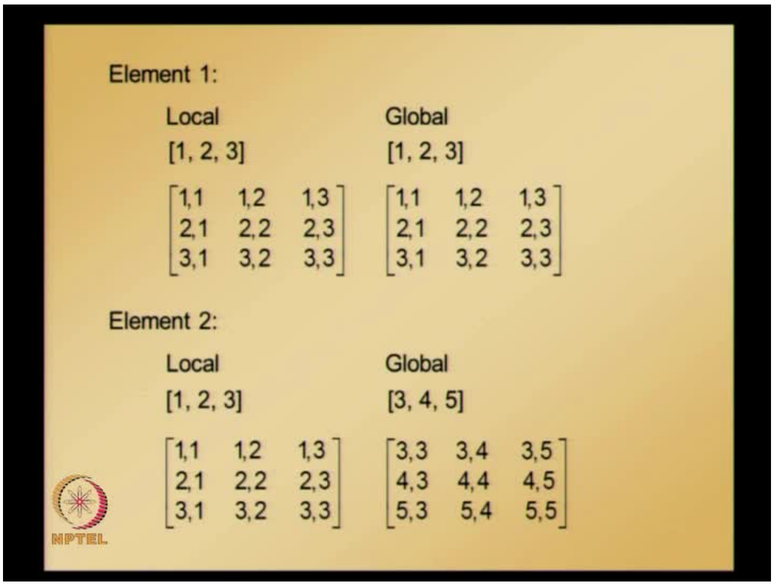

So, the matrices in the global equations are 5 by 5 location locations of element matrices in the global equation are given here. The contribution from element 1, where it goes in with the global equation system and contribution from element 2 where it goes with the global equation system.

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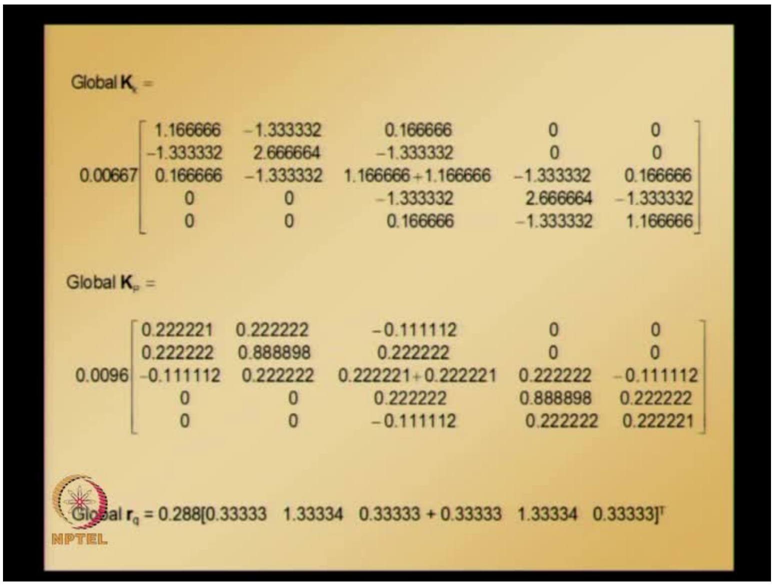

So, local to global relation is given now using this we can directly write global matrices k k k p by plugging in the contribution of element 1 and element 2 into the appropriate locations and global r q.

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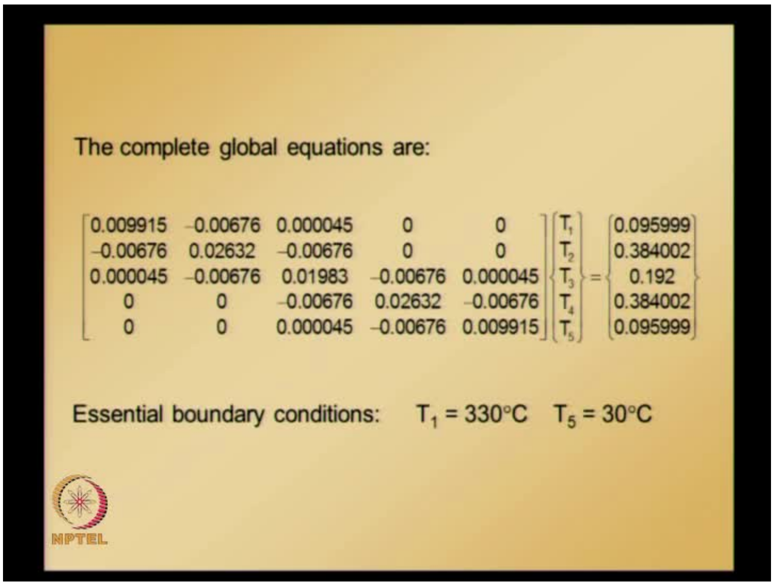

So, once we have all these matrices, complete global equations can be obtain, which is going to be 5 by 5 equation system and now here, before we proceed to solve for nodal temperatures, we need to make substitution of essential boundary conditions that are given T 1 is equal to 330 degree centigrade, T 5 is equal to 30 degree centigrade.

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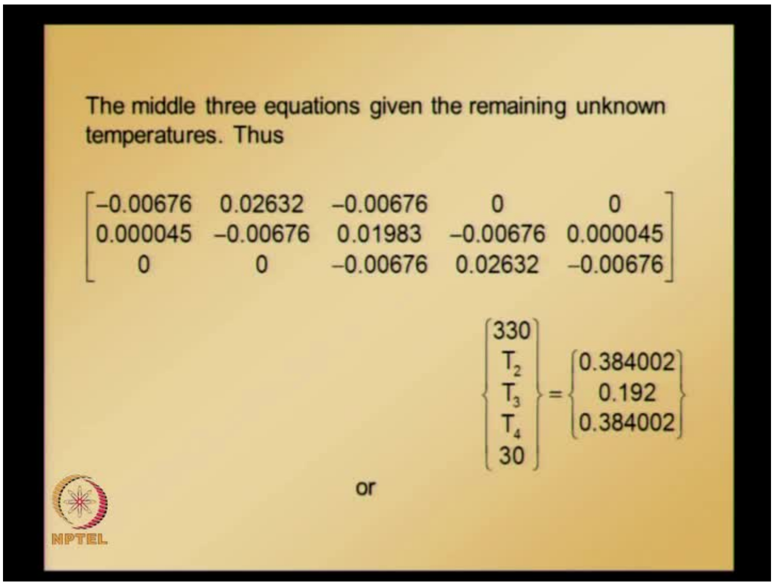

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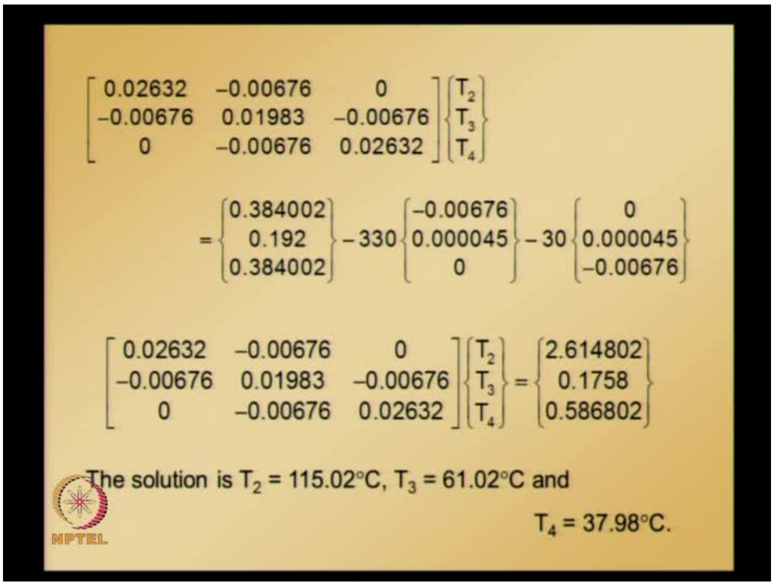

So, substituting essential boundary conditions and deleting or eliminating first and last equation, we get this middle three equations, give the remaining unknown temperatures. So, we get this equation system to solve for T 2, T 3, T 4 we need to rearrange this. And rearranging this we can solve for T 2, T 3 and T 4 solution is given here and T 1 is given T 5 is given.

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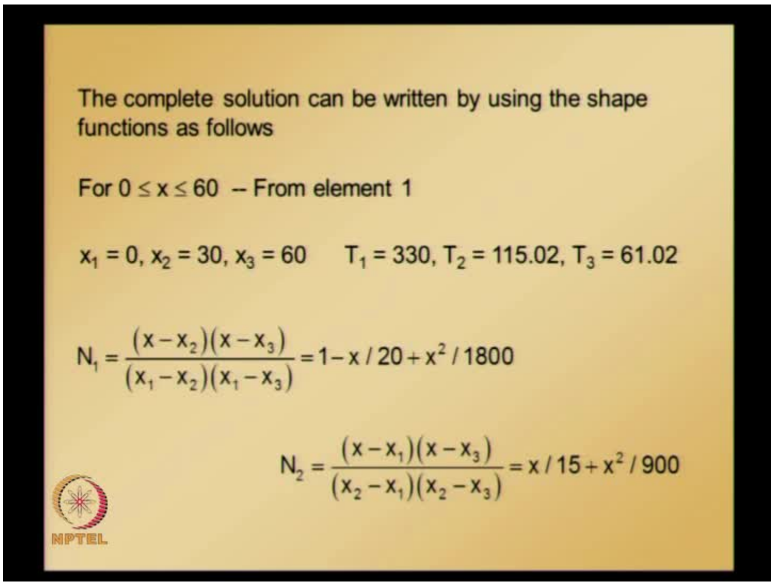

So, now we have all the temperature at all the nodes, now we can go to each element complete solution can be written by using shape functions for element 1, which goes from 0 to 60. The corresponding nodal coordinates and temperatures are given are shown there temperatures, we just obtain after solving the global equation system.

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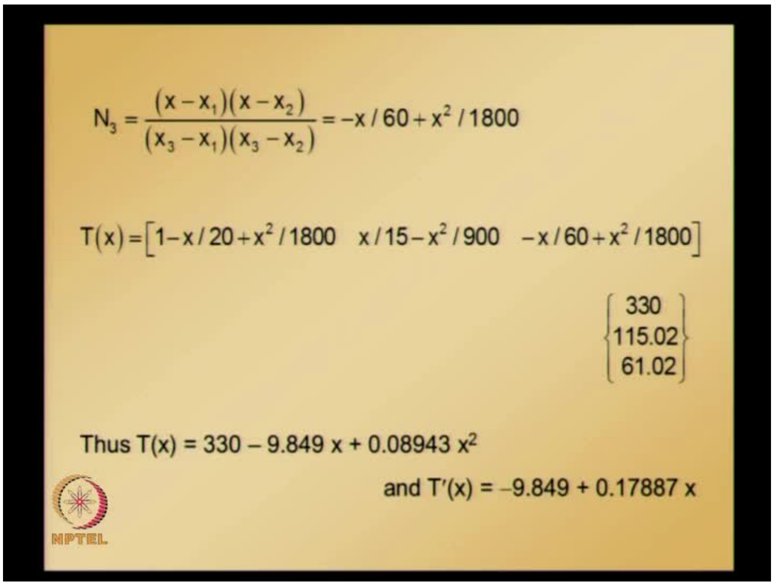

So, using shape functions N 1, N 2, N 3 using the coordinates of nodes we can interpolate temperature inside element 1, using this equation, which can be further simplified and also derivative T is a function of X; so, we can take derivative of T with respect to X that is also given there.

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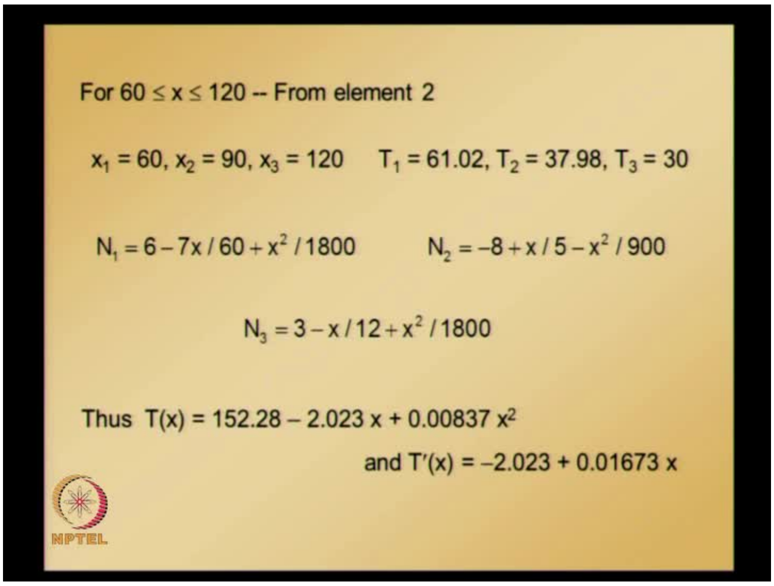

So, this is for element 1, using temperatures at node 1 2 3 and for element 2, which goes from X is equal to 60 to 120 and the nodal. The coordinates of three nodes in the corresponding local temperatures at the nodes, here temperatures are given in terms of local node numbering T 1 corresponds to T 3 T 2 corresponds to T 4 T 3 corresponds to T 5 in the global sense. So, once we have these, we can interpolate using N 1, N 2, N 3 calculated based on the nodal coordinates for this element. So, plugging in these values we can interpolate temperature at any point inside element 2 and also derivative of temperature at any point inside element 2.

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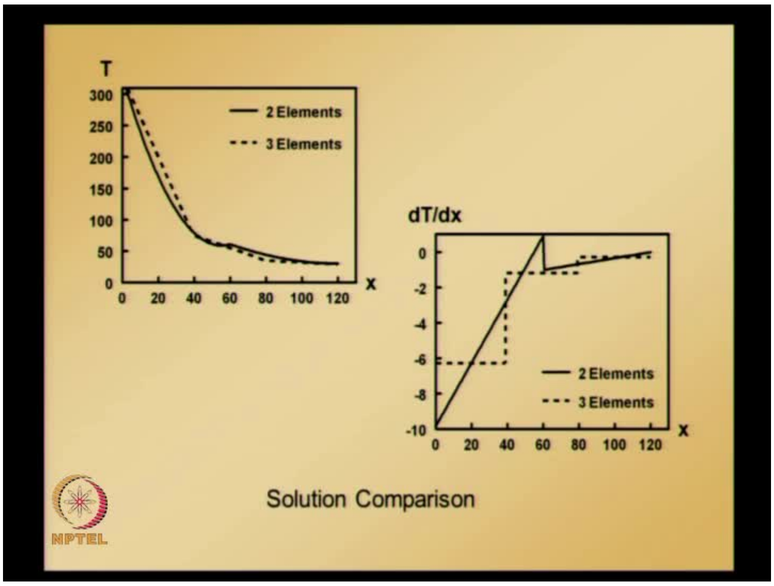

Once we have these relations, we can plot temperature as function of X and derivative of temperature as function of X and here, if you recall, we solve same problem using 3 linear elements and here in this figure solution obtained using 2 quadratic elements is compared with solution obtained using 3 linear elements and you can see solution is fairly close, that is, solution obtained using 2 quadratic elements and 3 linear elements are almost close to each other. But there is a great discrepancy in the derivative of temperature solution that we obtained. So, here derivative of temperature is not we are… What we are calculating? We are actually calculating T and forcefully taking derivative. So, that is expected a large error in derivative of temperature and so this, demonstrates usage of higher order elements and we have also seen, while doing this example, we have seen, how to use numerical integration for evaluating some of the integrals required for assembling the element equations.