Finite Element Analysis

Prof.Dr.B.N.Rao

Department of Civil Engineering

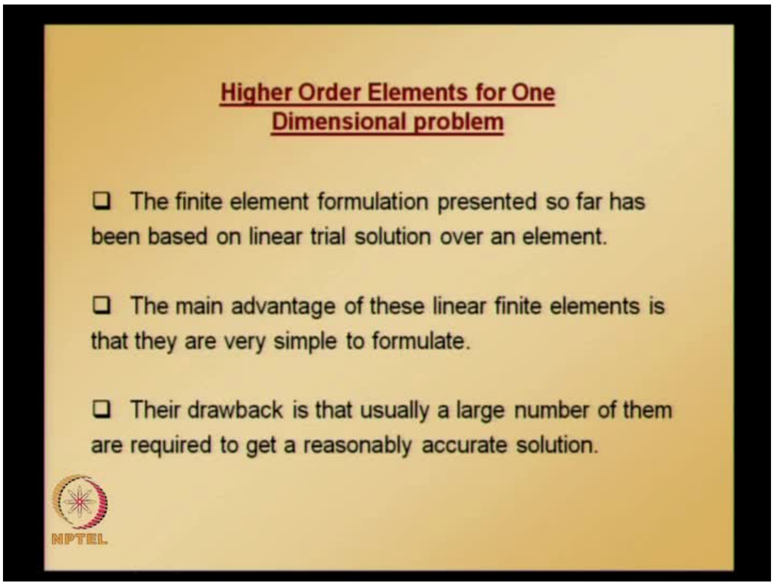
Indian Institute of Technology, Madras

Module No.# 01

Lecture No.# 18

In this lecture, we will be looking at higher order elements for one-dimensional problems. If you recall, in the last lectures, we used two node element, linear element and we also developed the governing… We developed element equations for general one-dimensional problems using two node linear element. And we have also seen applications like heat conduction problems and also a column buckling problem. And in the last class, or in the last lectures, when we have seen this column buckling problem, we solve this problem using four, two node linear elements. And we have seen the critical buckling load, that we obtained has about 5 percent error when we use four linear element.

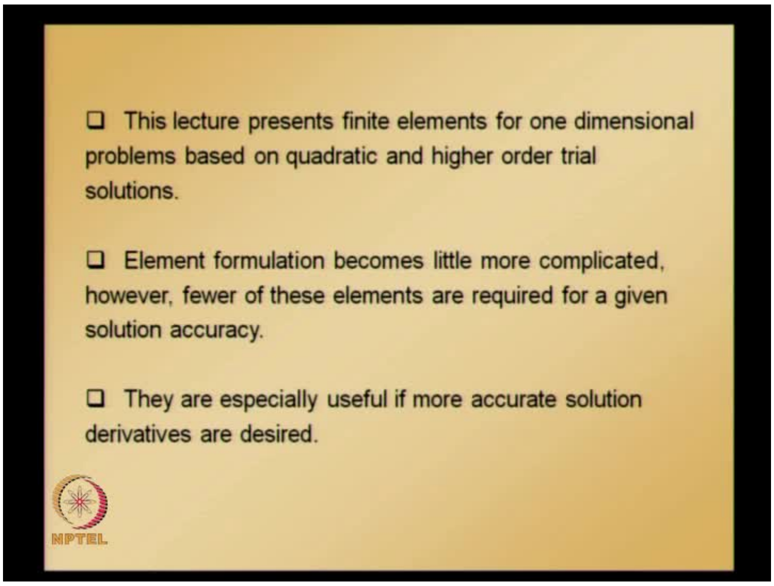
(Refer Slide Time: 01:34)



So, one way to improve the accuracy of the solution is to go for higher order element. So, in today's lecture, or in the next two lectures, we will be looking at higher order elements. So, finite element formulation presented so far has been on linear trial solution over an element, that is, we discussed about two node linear element.

The main advantage of these linear elements is that they are very simple to formulate as you have seen, but they have some drawback. Their drawback is that, usually a large number of linear elements are required to get a reasonably accurate solution. So, one way of improving the solution, which you have seen in the last lecture for column buckling problem, is to use more number of linear elements, but as we increase the number of elements, the equation system also gets increased, because we will be having more number of nodes. So, one way of improving solution is to go for higher order elements, so that is what we will be doing in this class.

(Refer Slide Time: 02:42)

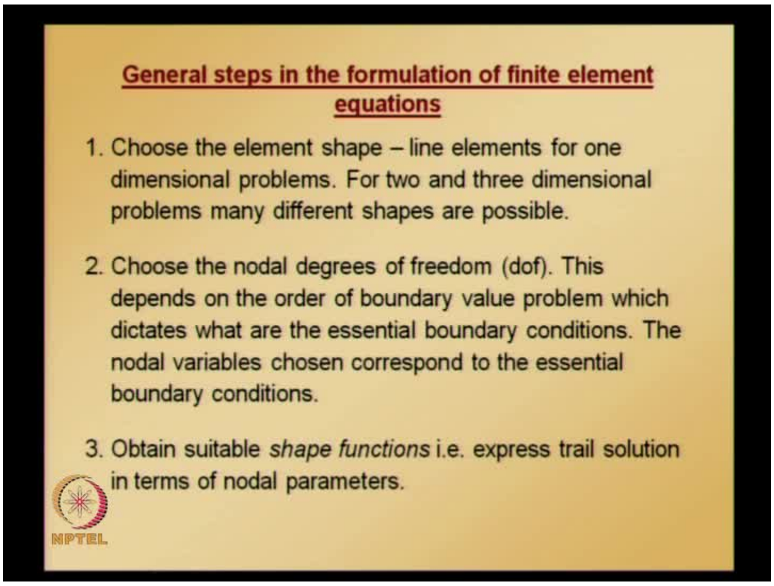


So, this lecture presents finite elements for one-dimensional problems based on quadratic and higher order trial solutions. So, we have seen in the last lectures, we have seen linear, so next one is a quadratic or you can go for cubic and quartic order elements. So, in this lecture, we will see a procedure for doing that. And when we use higher order elements, the element formulation becomes little bit more complicated, however, few of these elements are required for a given solution accuracy.

So, even though so formulation is more complicated, we can use less number of elements, so the size of equation system that we will be solving will be smaller than what we usually get using linear elements. We will be solving same column buckling problem that we are solved using four linear elements at the end or once we complete the element formulation using quadratic elements.

And these higher order elements are especially useful if more accurate solution derivatives are desired. If your recall, we are after the solution, but not after the derivative of solution. But, if you some body is interested in derivative of solution or accurate values of let us say strain, so then higher order elements are very much useful in that case, if more accurate solution derivatives are desired.

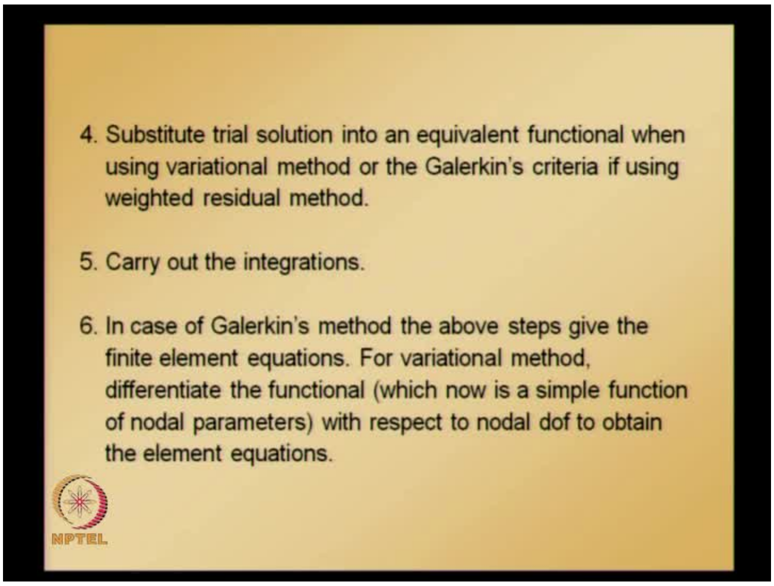
(Refer Slide Time: 04:34)



Now, let us look at what are the various general steps involved in the formulation of finite element equations. So, first step is to choose the element shape, it depends on the kind of problem, line elements for one-dimensional problems. For two and three-dimensional problems, we will be seeing in a little bit later, there are different shapes possible, rectangular or quadrilateral elements, or if this 3D, you can have tetrahedron or a quadrilateral or a cubic element.

And next step is to choose the nodal degrees of freedom. How many nodes you want for an element? This depends on the order of boundary value problem, which dictates what the essential boundary conditions are. And if you recall, if the boundary value problem is of order 2 p, those boundary conditions of order 0 to p minus 1 are essential, and those boundary conditions of order p to 2 p minus 1 are natural boundary condition. So, depending on the order of the boundary value problem which dictates what the essential boundary conditions are, the number of degrees of freedom for each element will be decided based on that. The nodal variable chosen correspond to the essential boundary conditions, because if you see a fourth order differential equation, which corresponds to beam bending problem, there we have zeroth order and first order equations as essential boundary conditions, whereas third and fourth order equations as natural boundary conditions, so we need at least first order continuity.

(Refer Slide Time: 07:13)



And the solution or we need derivatives also for imposing the essential boundary condition, so that is what this step says, choose the nodal degrees of freedom depending on the problem that we are solving. So, after these two steps, we need to obtain suitable shape functions for the chosen element that is express all trial solution in terms of nodal parameters of that particular element. And once we have trial solution in terms of nodal parameters through shape functions, we can substitute a trial solution into an equivalent functional if you are using variational method or if you are using Galerkin criteria, or in Galerkin criteria if you are using weighted residual method.

And once we substitute a trial solution into equivalent functional or Galerkin criteria, then we can do all the manipulations like carrying out integrations. And in case of Galerkin methods, the above steps gives or give the finite element equations, whereas for variational method, we need to apply stationarity condition that is differentiate the functional, which is now, which now is a simple function of nodal parameters with respect to the nodal degrees of freedom, to obtain the element equations.

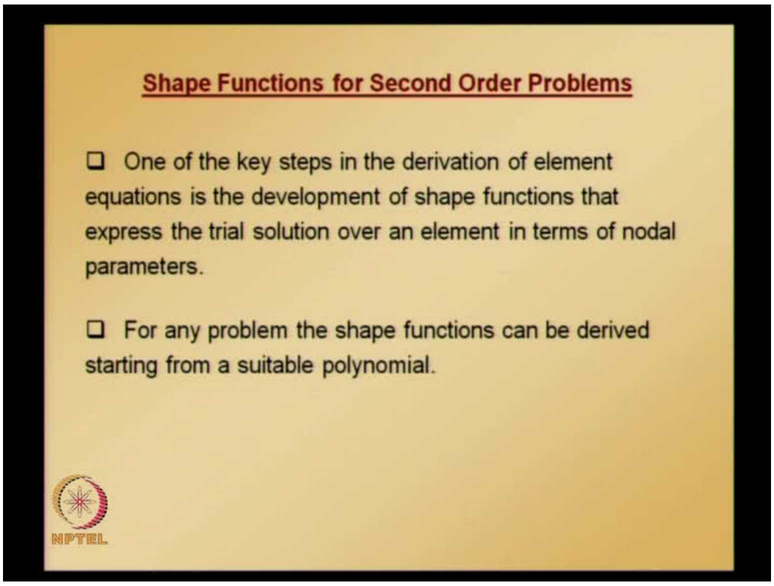
Basically, this all steps you have already experienced, and we have solved problems in earlier lectures. So, in today's lecture, we will be looking at Lagrange interpolation formula for constructing finite element trial solutions that is shape functions for second order boundary value problem. And also, a basic concept of isoparametric mapping we will be looking at it, where shape functions are used for transforming the physical element geometry into a parent element simple geometry, so we will be looking at isoparametric mapping. A quadratic element for general one-dimensional boundary value problem is going to be presented, and we will be developing finite element equations using quadratic element for general one-dimensional boundary value problem using Galerkin method.

So, once we have the element equations for a quadratic element, then we can solve any problem like we did in the earlier lectures, like we can solve heat conduction problem or under various boundary conditions, or we can solve using even the column buckling problem by just comparing or making a comparative table of the corresponding variables in the general one-dimensional boundary value problem and that specific problem.

So, now, for isoparametric element, it is usually difficult to express element equations in an explicit form, because of need to integrate complicated functions, so we will be seeing this. And also for that, because the explicit expressions are possible only if we can carry out integrations accurately, or we can carry out integrations without adopting any numerical techniques, but sometimes the integrant, when it becomes complicated, we need to adapt numerical integration.

So, we will also be looking at numerical integrations schemes: one of the numerical integration schemes is Galerkin's numerical integration procedure, so that also we will be looking as a part of study of these higher order elements.

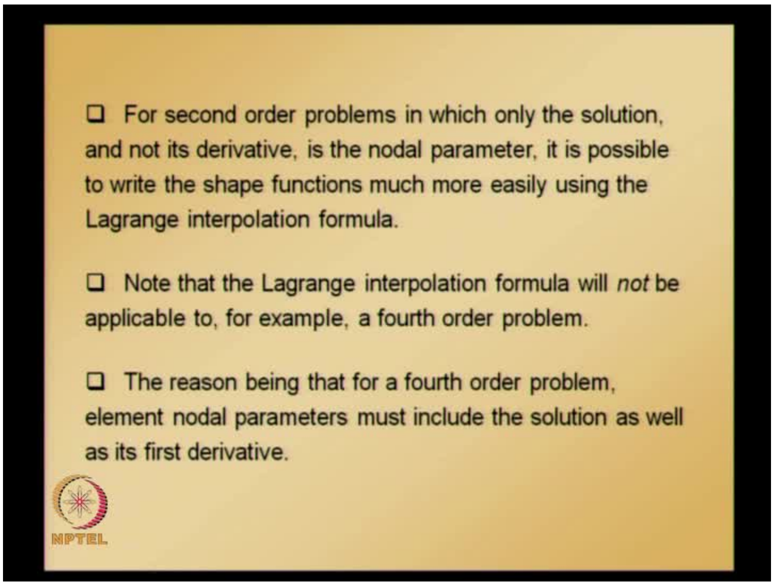
(Refer Slide Time: 11:01)



So, now, let us look at how to derive the shape functions for higher order elements or second order boundary, second order problems. So, one of the key steps in derivation of element equations is development of shape functions that express trial solution over an element in terms of nodal parameters.

For any problem, the shape functions can be derived starting from a suitable polynomial as we did for linear elements. When we are deriving the shape functions for linear elements, what we did is we assume trial solution to be u is equal to a naught plus a 1 x, and we solved for a naught a 1, or we expressed a naught a 1 in terms of nodal parameters, and back substituted at this a naught a 1, and grouped the terms containing nodal values, and then we got the shape function, similar procedure can be adapted.

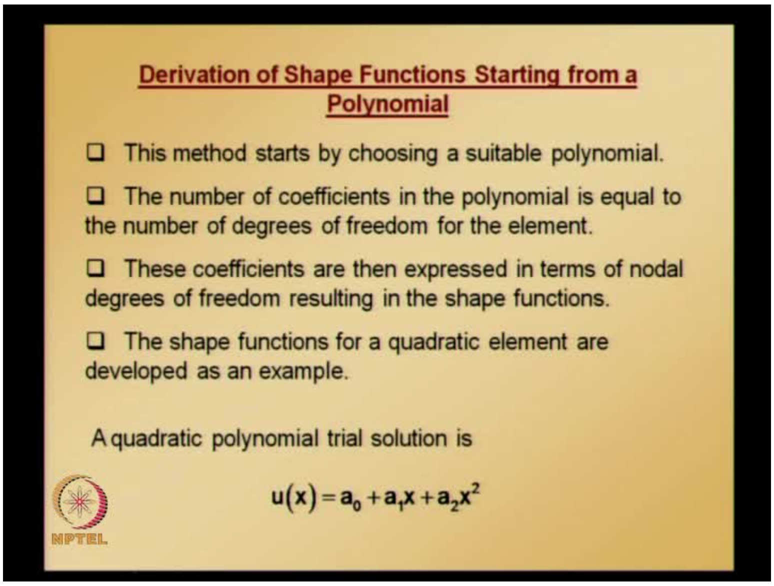
(Refer Slide Time: 12:03)



For second order problems, in which only the solution and not its derivative is nodal parameter, it is possible to write shape functions more easily using Lagrange interpolation formula. So, the procedure which you have seen earlier that is starting with a suitable polynomial, it becomes more cumbersome when you go for deriving for higher order element, so the other alternative is to use Lagrange interpolation formula.

So, one point that you need to keep in mind is Lagrange interpolation formula can all only be used, when the solution, not its derivative, is the nodal parameter. So, note that Lagrange interpolation formula will not be applicable to, for example, a fourth order problem, which is like a beam bending problem, were we have both solution as well as derivative as the nodal parameters. The reason is simple; reason is being that fourth order - for a fourth order problem, nodal parameters must include the solution as well as its first derivative.

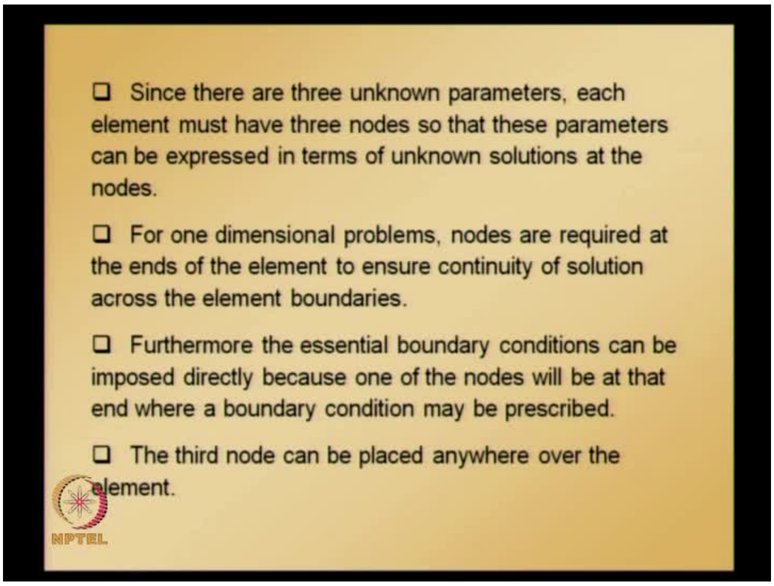
(Refer Slide Time: 13:40)



So, let us look now first starting with a polynomial how to derive shape functions at for a quadratic element and derivation of shape functions starting from a polynomial. And this method starts by choosing a suitable polynomial; how to choose the polynomial? The number of coefficients in the polynomial is equal to the number of degrees of freedoms of element.

If you recall, for linear element - the shape function derivation of linear element, we started with a naught plus a 1 x, because there are only two nodes, so a naught, a 1 are the coefficients - two coefficients. So, here, if we want to derive shape function for quadratic element, we need to start with a polynomial having three coefficients that is a naught plus a 1 x plus a 2 x square. These coefficients are then expressed in terms of nodal degrees of freedom resulting in the shape function.

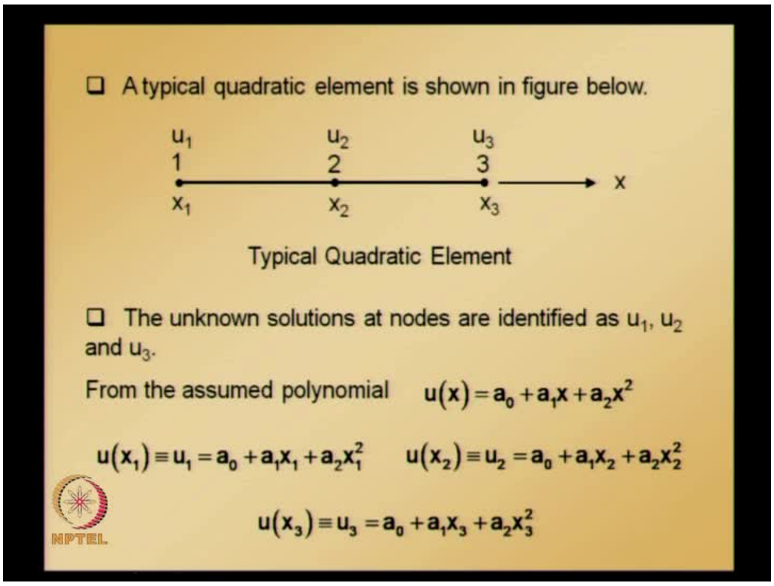
(Refer Slide Time: 15:18)



Shape functions for a quadratic element, the derivation is shown. A quadratic polynomial trial solution is assumed, u is equal to a naught plus a 1 x plus a 2 x square. Why, because since there are three unknown parameters, each element must have three nodes, so that these parameters can be expressed in terms of unknown solutions at the nodes.

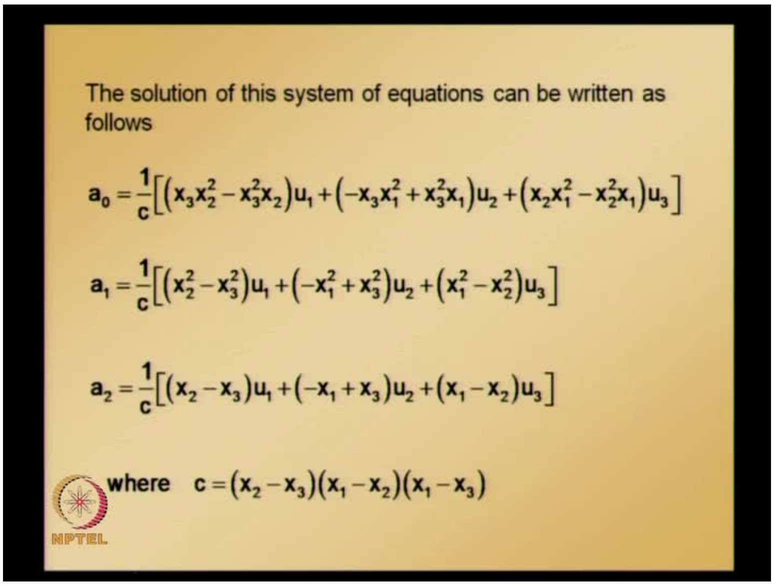
For one-dimensional problem, nodes are required at the ends of the element to ensure continuity of solution across element boundaries. Furthermore essential boundary conditions can be imposed directly, because one of the nodes will be at the end, where boundary conditions may be specified or prescribed. So, for a quadratic element, the third node can be placed anywhere over the element, but the placement of middle node is going to influence the solution that will be seen once we develop the shape functions.

(Refer Slide Time: 16:25)



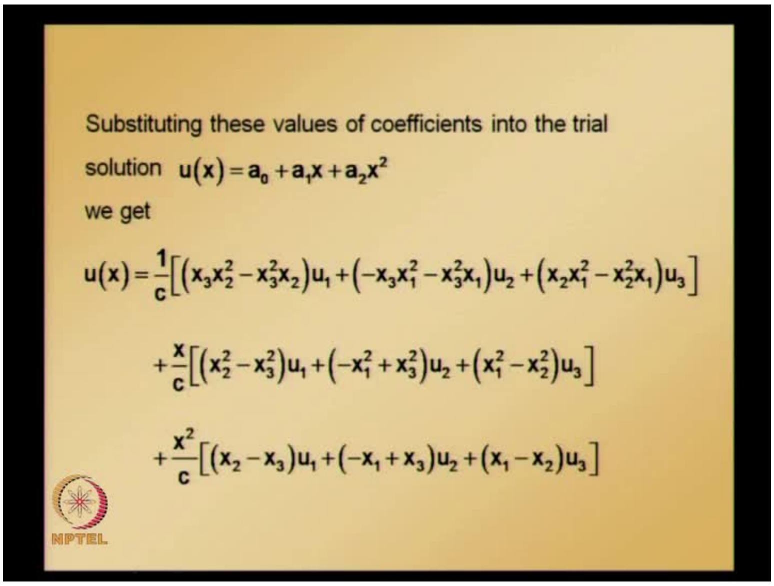
So, a typical quadratic element is shown in the figure here. Having three nodes, node 1, 2, 3; node 1 at x 1, node 2 at x 2, node 3 at x 3 and the corresponding nodal parameters at node 1 u 1, u 2 at node 2 and u 3 at node 3. The unknown solutions at nodes are identified as u 1, u 2 and u 3. From the assumed polynomial, u is equal to a naught plus a 1 x plus a 2 x square. By substituting x is equal to x 1, x is equal to x 2, x is equal to x 3, we are going to get three equations.

(Refer Slide Time: 17:44)

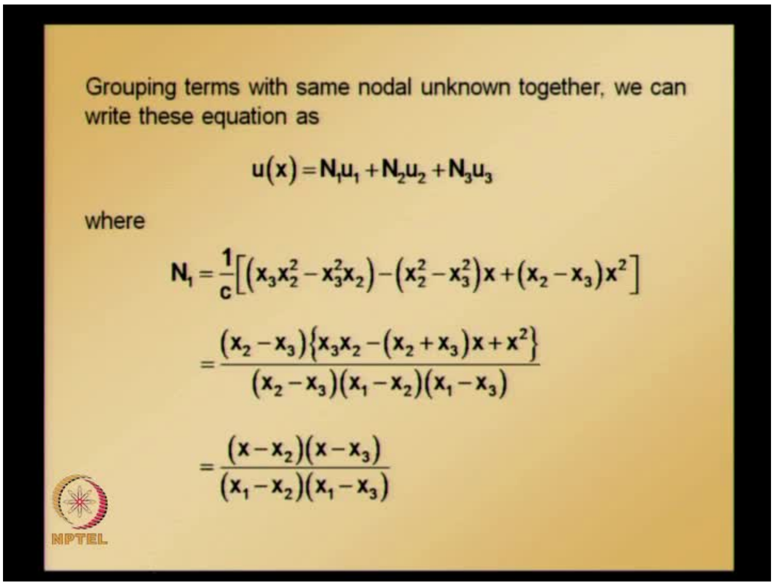


The three equations that we get by substituting x is equal to x 1, x is equal to x 2, x is equal to x 3 are given here. And using these three equations, we can solve for a naught, a 1, a 2. Solution of this system of equations can be written like this. Once we solve for a naught, a 1 and a 2, here the constant or the parameter c is defined like this.

(Refer Slide Time: 18:28)



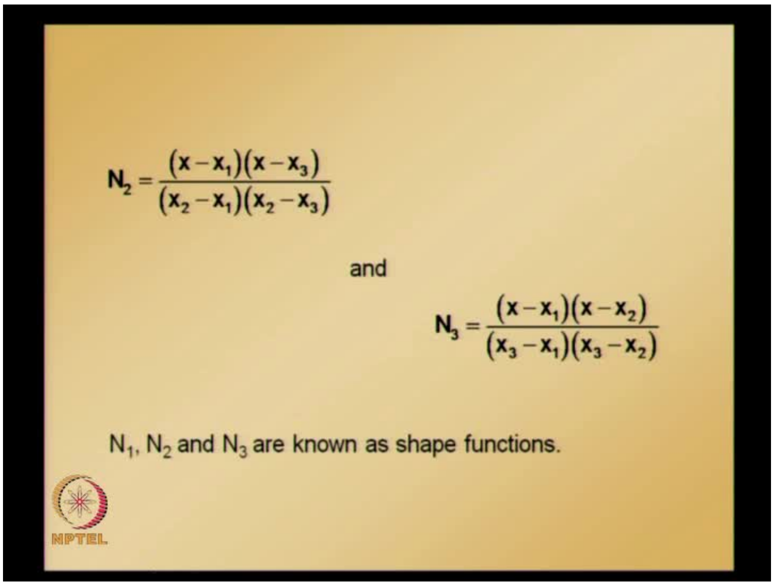
(Refer Slide Time: 18:57)



So, solving three equations for a naught, a 1, a 2, we got the corresponding a naught, a 1, a 2. And now substituting these values of coefficients a naught, a 1, a 2 back into the trial solution, we get this one. And this equation can be rearranged such a way that the terms containing u 1, u 2, u 3 are grouped together. Grouping terms with same nodal unknown together, we can write the previous equation in this form u is equal N1 u1, N2 u2 and N3 u 3.

Where N1 is given by this, substituting the value of c and simplifying, N 1 is equal to x minus x 2 times x minus x 3 divided by x 1 minus x 2 times x 1 minus x 3. And we see this N 1, N 1 by substituting x is equal to x 1, we notice that N 1 is equal to 1 by substituting x is equal to x 2 or x is equal to x 3, N 1 is equal to 0.

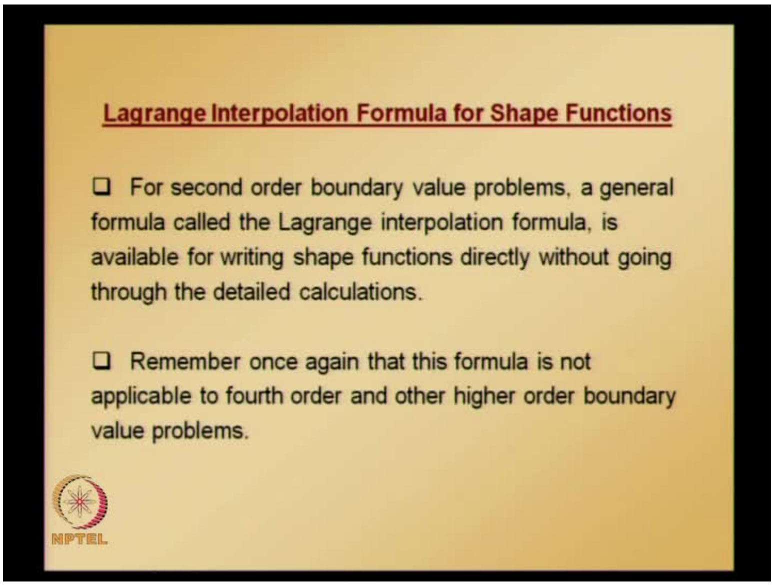
(Refer Slide Time: 20:12)



So, N 1 is going to be 1 at x is equal to x 1; N 1 is going to be 0 when x is not equal to x 1. Similarly, N 2 and N 3 are given by this, so these N 1, N 2, N 3 are called shape functions, shape functions for quadratic element. And instead of going through all this cumbersome procedures starting with a polynomial, as you can notice that when we took this quadratic element and compare to the linear element, the procedure or the effort is very cumbersome.

So, instead of going through this procedure, there is an alternative way, which is Lagrange interpolation formula for shape functions. Using this formula, we can write shape functions for a second order boundary value problem up to any order of trial solution.

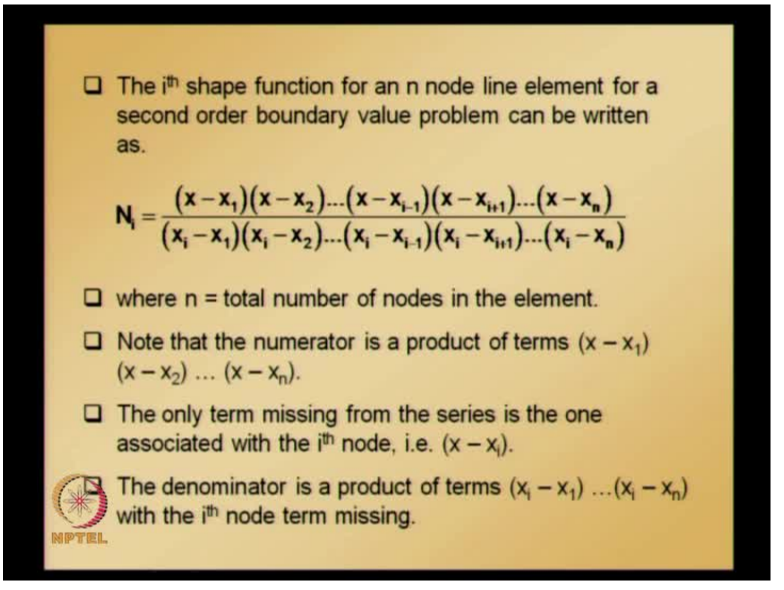
(Refer Slide Time: 21:20)



So, now, let us look at Lagrange interpolation formula for shape functions. For second order boundary value problems, a general formula called Lagrange interpolation formula is available for writing shape functions directly without going through the detailed calculations, which we have seen in the last, in the previous slide or previous slides.

Remember once again that this formula is not, this formula is not applicable for 4th order and higher order boundary value problem, because Lagrange interpolation formula is only applicable for the case when only the solution is nodal parameter, not its derivative; so that you need to keep in mind.

(Refer Slide Time: 22:16)

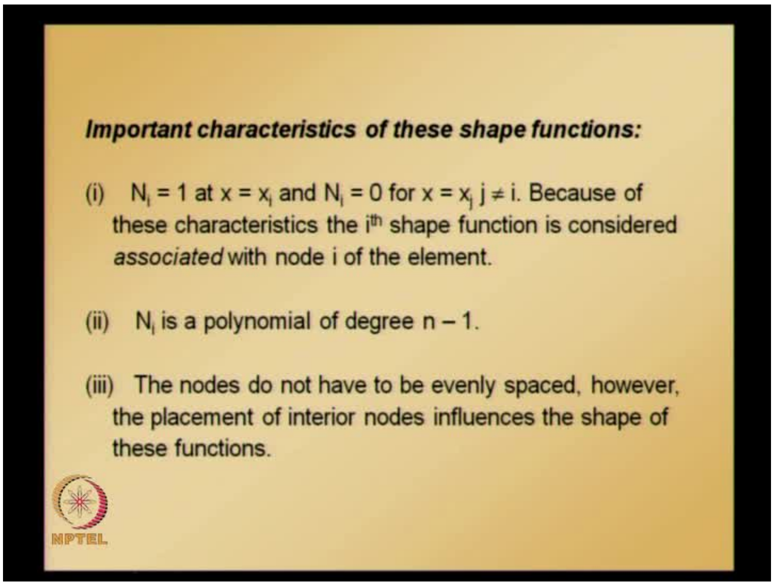


So, according to the Lagrange interpolation formula, the i th shape function for an n node line element, for second order boundary value problem can be written like this. Where n is the total number of nodes in the element, so using this formula we can write shape function expressions for any order element quadratic, cubic, quartic, 5th order, 6th order, or 100th order element.

n is total number of elements; if it is linear element, n is equal to 2; quadratic, n is equal to 3; cubic, n is equal to 4; quartic, n is equal to 5 and so on. And here, in this expression, note that the numerator is a product of terms x minus x 1, x minus x 2, dot, dot, x minus x n. And if you see the numerator carefully, the only missing term from this series is one associated with the i th node, x minus x i is going to be, is not going to be there, that is the term that is missing.

And in the denominator, denominator is a product of terms x i, because we are writing shape functions for i th node, x i minus x 1, x i minus x 2, dot, dot, x i minus x n, with i th node that is x i minus x i is missing that is i th node term is missing.

(Refer Slide Time: 24:31)

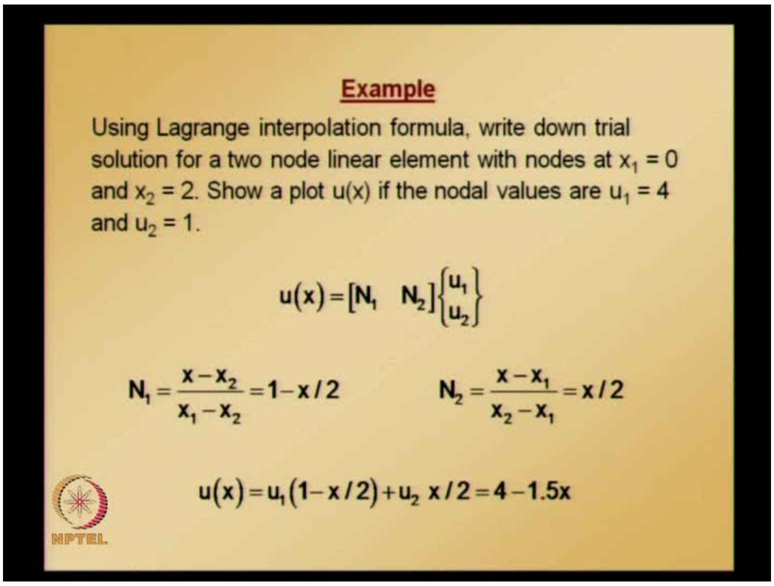


So, using this Lagrange interpolation formula, we can write shape functions for any order element. And some of the important characteristics of this shape functions N i is equal to 1 at x is equal to x i, N i is equal to 0 where x is not equal to x i or x is equal to x j, j is not equal to i, because of these characteristics, the i th shape function is considered to be associated with node i of element. And if you see the shape functions expressions for N i, N i is a polynomial of degree n minus 1.

So, the shape function for a linear element is going to be of order 1, shape function for a quadratic element is going to be order of 2, shape function for a cubic element is going to be order of 3 and so on. And this nodes 1, 2, 3 need not be evenly spaced, the nodes do not have to be evenly spaced, however placement of interior nodes influence the shape functions, which will be seen once we have solved some examples.

So, here we have seen two ways of deriving shape function, one is starting with a polynomial - suitable polynomial, or you can use Lagrange interpolation formula, but using Lagrange interpolation formula, writing shape function expressions for any order element is much much easier.

(Refer Slide Time: 26:22)

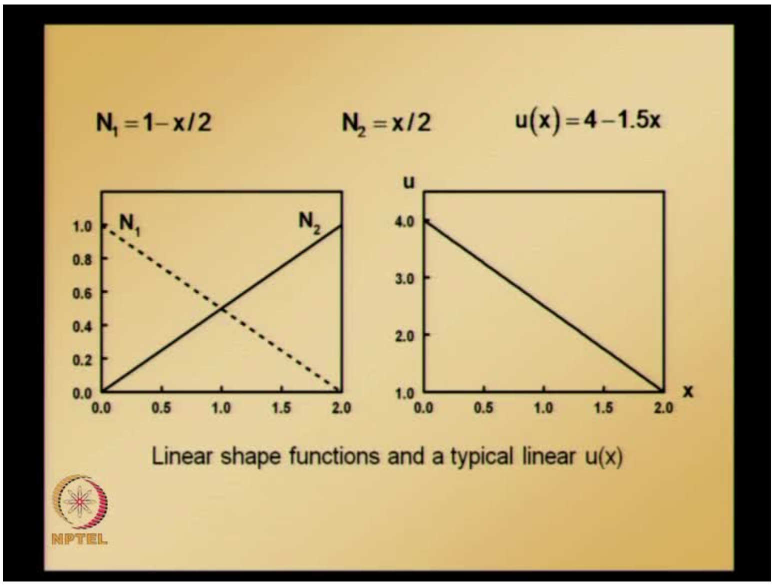


So, now, let us take some examples and apply this formula - Lagrange interpolation formula. Using Lagrange interpolation formula, write down trial solution for a two node linear element with nodes x 1 coordinate is given, x 2 is given for a two node linear element. And also show a plot of trial solution or solution if the nodal values are u 1 is equal to 4 and u 2 is equal to 1.

So, trial solution is given by u is equal to N 1 u 1 plus N 2 u 2, which can be arranged in matrix and vector form like this, where N 1 and N 2 can be obtained using Lagrange interpolation formula. One, you can check these expressions for N 1 N 2 with the expressions that we already looked at, when we derived starting with a polynomial a naught plus a 1 x, we can notice that these N 1 N 2 expressions that we got using Lagrange interpolation formula are identical to that what you already have, using starting with a linear polynomial.

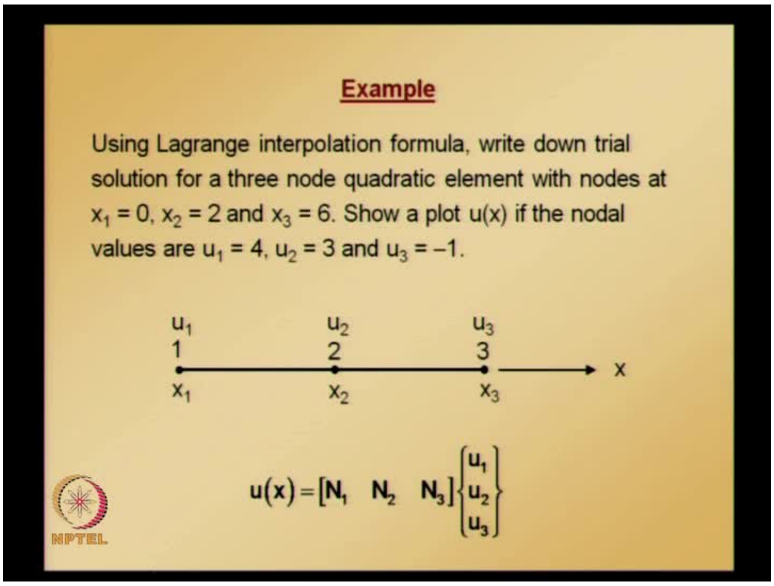
So, these are N 1 N 2 values, so the trial solution is u 1 times N 1; N 1 is substituted u 2 times - plus u 2 times N 2, N 2 is also substituted. And once we simplify this, it turns out that this is equal to 4 minus 1.5 times x. So, u - approximate solution as a function of x is obtained, so now we are ready to plot this as a function of x and we can also plot N 1 N 2 as a function of x.

(Refer Slide Time: 28:32)



So, the plot showing N 1 N 2 and the approximate solution are given in the figure. And note that N 1 N 2 are linear shape functions, and since u is a linear function, with slope - negative slope, so that is how the variation of u with respect x is going to be as shown in the plot.

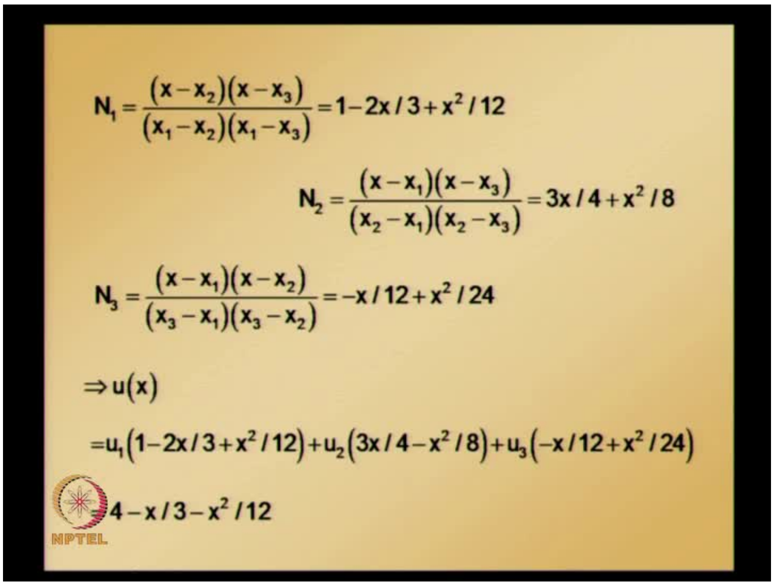
(Refer Slide Time: 29:23)



So, this is how you can apply Lagrange interpolation formula to get the shape functions and also to interpolate the nodal values. Now, let us apply Lagrange interpolation formula for a quadratic element, now let us taken another example. Use Lagrange interpolation formula, write down trial solution for a three node quadratic element with nodal coordinates given x 1 x 2 x 3 and also show a plot if nodal values are given u 1 u 2 u 3.

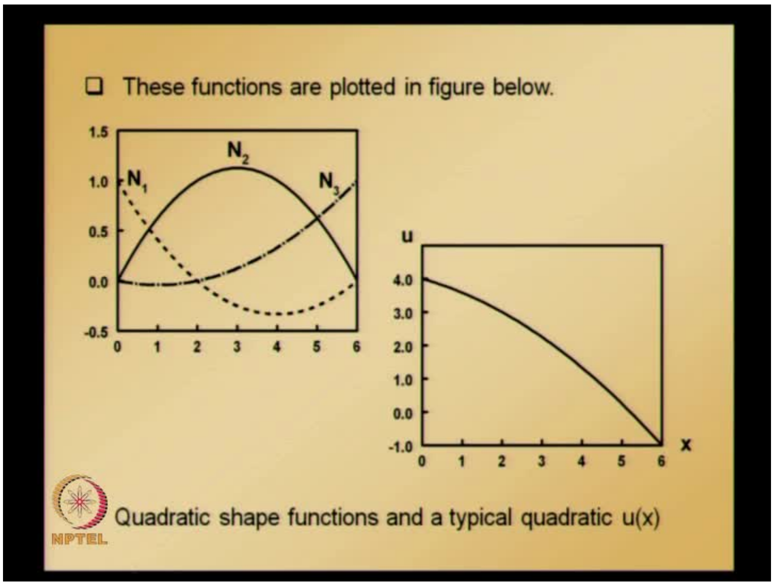
So, first step is we need to write shape function expressions N 1 N 2 N 3 by substituting x 1 x 2 x 3 values, because N 1 N 2 N 3 are required for obtaining the approximate solution, which is u is equal to N 1 u 1 plus N 2 u 2 plus N 3 u 3, which can be put in a matrix and vector form like what is shown there on the slide.

(Refer Slide Time: 30:28)



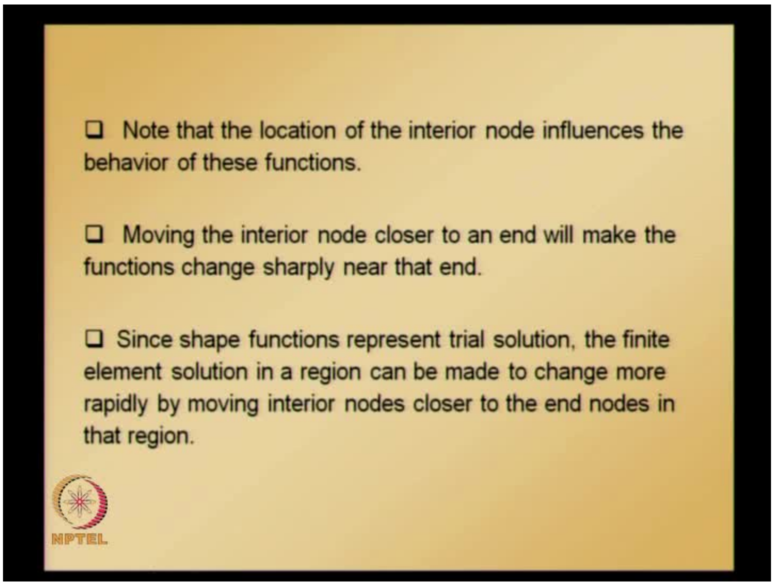
So, now, N 1 N 2 N 3 after substituting x 1 x 2 x 3 values and simplifying, we will obtain what is shown there. Now, using this N 1 N 2 N 3 values, approximate value of u or approximate expression for approximation of u can be obtained by substituting N 1 N 2 N 3 and simplifying. And u 1 u 2 u 3 numerical values are given, so simplifying that we are going to get u is equal to 4 minus x over 3 minus x square over 12. You can see here, this trial solution is quadratic and also each of the shape functions is quadratic.

(Refer Slide Time: 31:40)



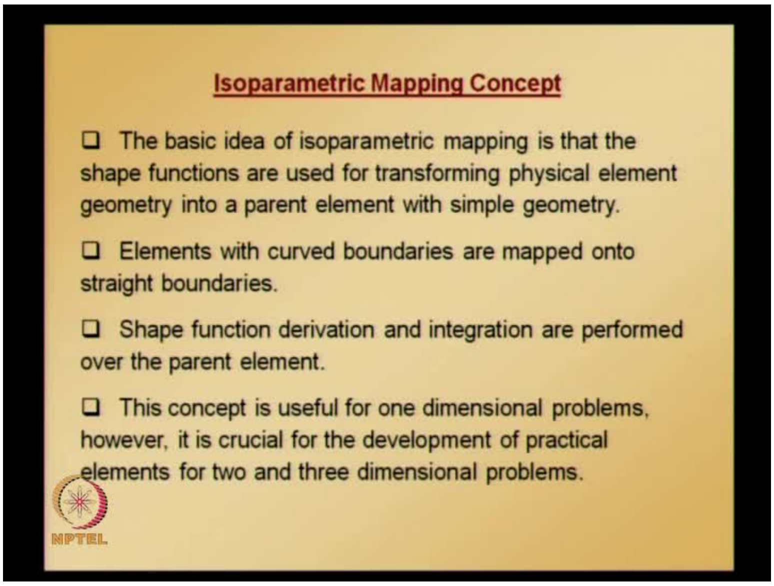
So, now, we can plot this N 1 N 2 N 3 and also the approximate solution. And you can see by looking at this, you can note by looking at these figures or these plots that the location of interior node that is x 2 influences behavior of these functions N 1 N 2 N 3, moving the interior node closer to an end, will make functions change sharp linear at the ends. Since shape functions represent trial solution, finite element solution in a region can be made to change more rapidly by moving interior nodes closer to the end nodes in that region.

(Refer Slide Time: 32:50)



So, by playing with this location of interior node, we can change the way the trial solution is varying. Note the location of interior node influences behavior of these shape functions, and moving to the interior node closer to an end, will make functions change sharp linear that end.

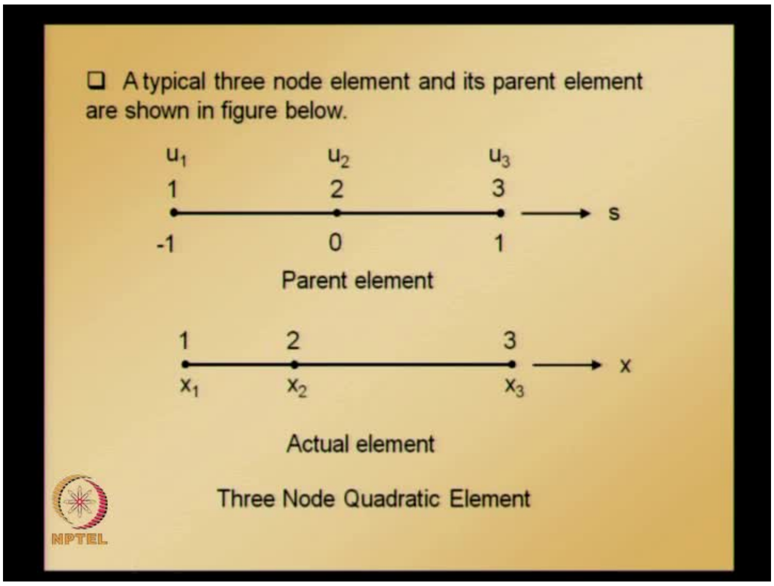
(Refer Slide Time: 33:40)



Since shape functions represent trial solution, finite element solution in a region can be made to change more rapidly by moving interior nodes closer to end nodes in that region. So, depending on the requirement, we can do this. Now, let us go to the next concept - isoparametric mapping. The basic idea of isoparametric mapping is that shape functions are used for transforming physical element geometry into parent element with a simple geometry.

So, the physical element, the actual element, we are going to map on to a parent element. And especially elements with curved boundaries are mapped onto straight boundaries. Shape function derivation and integration are performed over the parent element. And this concept of isoparametric mapping is useful for one-dimensional problems; however, it is crucial for development of practical elements for two and three-dimensional problems, which will be seen in the later lectures.

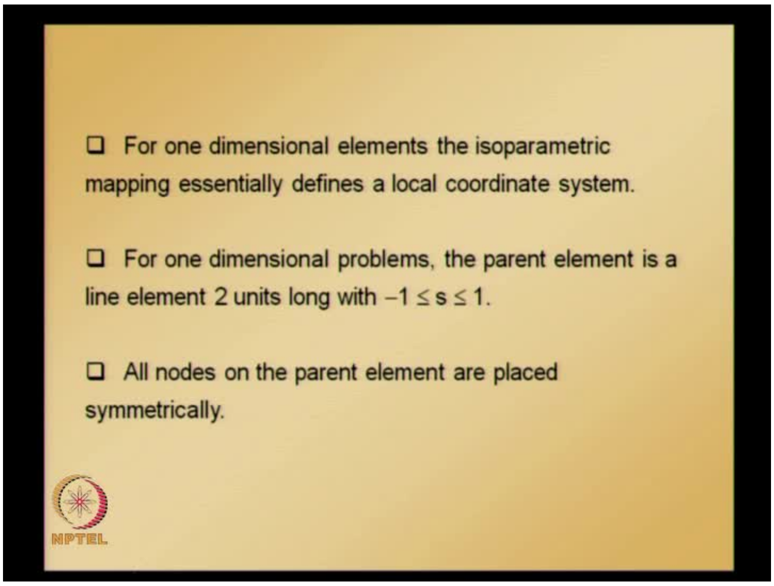
(Refer Slide Time: 34:48)



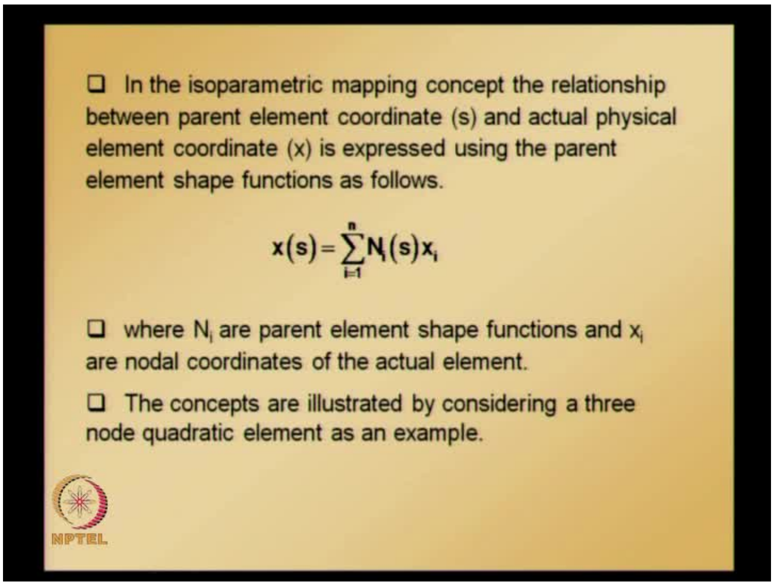
A typical three node element and its parent element are shown here. So, the actual element is having three nodes, located at node 1 at x 1, node 2 at x 2, node 3 at x 3. And this actual element is mapped on to a parent element, in s coordinate system, with node 1 at s is equal to minus 1, node 2 at s is equal to 0, node three at s is equal to 1.

So, the portion x 1 whatever is there between x 1 and x 2, it is mapped to 0 between minus 1 to 0 in s coordinate system. And whatever is there between s 2, x 2 and x 3 that is mapped on to 0 and 1 in s coordinate system.

(Refer Slide Time: 35:36)



(Refer Slide Time: 36:44)

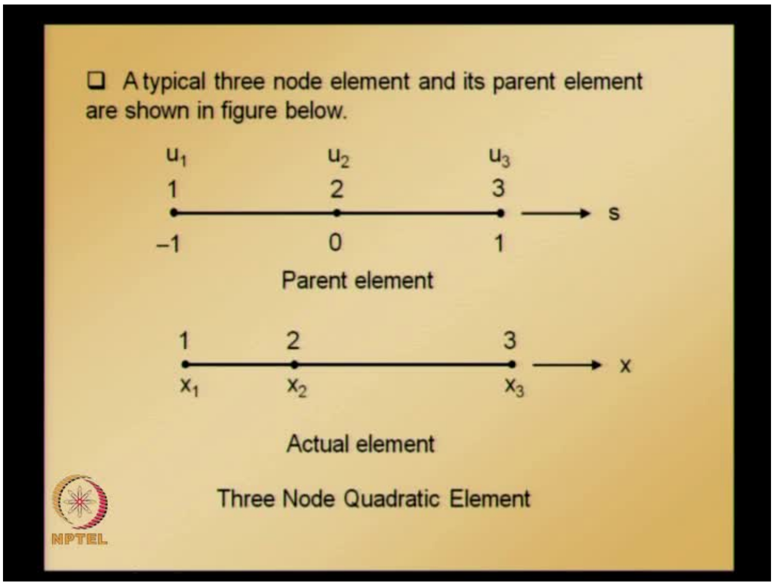


For one-dimensional elements, the isoparametric mapping essentially defines a local coordinate system. For one-dimensional problems, the parent element is a line element of length 2 units with s going from minus 1 to 1. And also, all the nodes on the parent element are place symmetrically. In isoparametric mapping concept, the relationship between the parent element coordinate s and the actual element coordinate x is expressed using parent element shape functions as follows.

If you recall for linear element, for a two node linear element, earlier we have derived the relationship between x coordinate and s coordinate using the formula for linear interpolation, but we can also derive that relation between x coordinate and s coordinate using shape functions of the parent element, like using the expression that is given here, x is equal to small i taking values 1 to n, where small n is the number of nodes, n i is the shape function, x y is the location of these nodes.

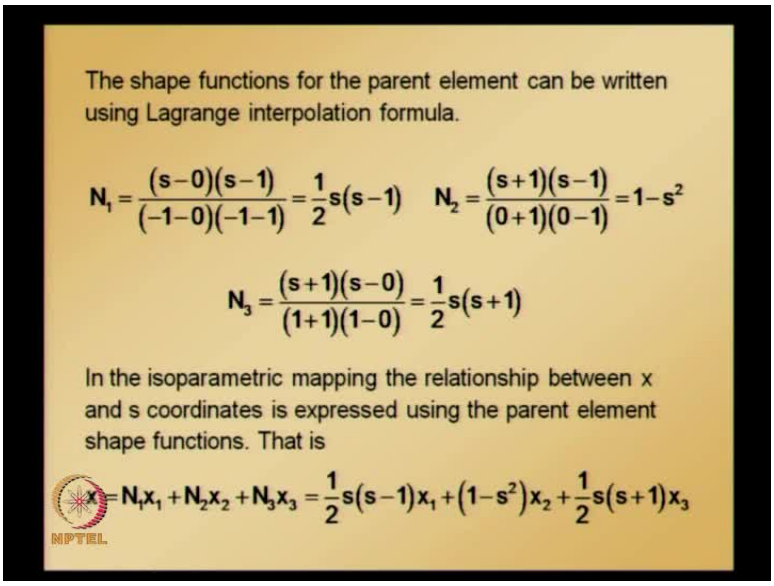
So, this formula is applicable for any order element, where i takes values over the number of nodes, where N i are the parent element shape function, x i are the nodal coordinates of the actual element.

(Refer Slide Time: 38:32)



So, this application of this formula, these concepts are illustrated by considering a three node quadratic element as an example. Now, let us take the three node quadratic element or typical three node element like this. This is same as what you have seen a few minutes back, it is reproduced here.

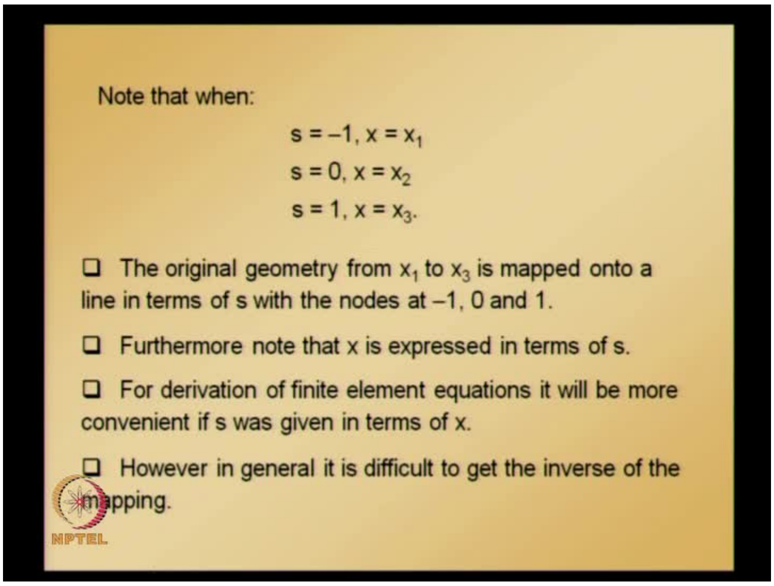
(Refer Slide Time: 39:16)



So, now, we need to write the shape functions of the parent element, because we require the shape functions of the parent element to get the relationship between x coordinate and s coordinate system. And we can get write the or we can get the expressions for shape functions of the parent element using Lagrange interpolation formula. So, the shape functions for the parent element can be written using a Lagrange interpolation formula.

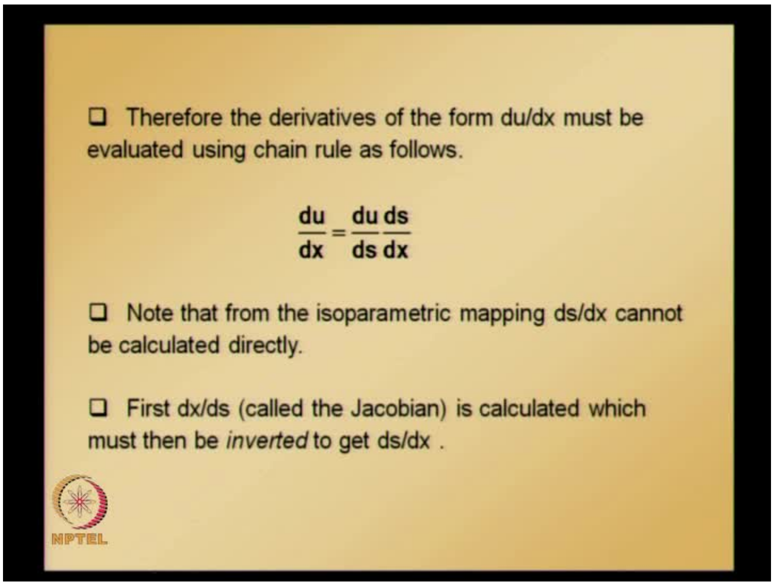
Applying Lagrange interpolation formula, we get N 1 is equal to half s times s minus 1, N 2 is equal to 1 minus s square, N 3 is equal to half s into s plus 1. So, we got the shape functions for parent elements for node 1, 2 and 3. So, in isoparametric mapping, the relation between x and s coordinates is expressed using parent element shape functions, by this relation x is equal to, here there are three nodes, so i takes values 1 2 3, so x is equal to N i or N 1 x 1 plus N 2 x 2 plus N 3 x 3. Substituting N 1 N 2 N 3, the shape functions of the parent element, we get this expression, where x 1 x 2 x 3 are the locations of the nodes in the actual element. And by substituting s is equal to minus 1, you can notice here by substituting s is equal to minus 1 in the relationship between x and s, we get x is equal to x 1, by substituting s is equal to 0, we get x is equal to x 2, by substituting s is equal to 1, we get x is equal to x 3, which you can easily verify.

(Refer Slide Time: 41:04)



And also the original geometry, which is from x 1 to x 3 is mapped onto a line in terms of s with nodes at minus 1, 0 and 1.

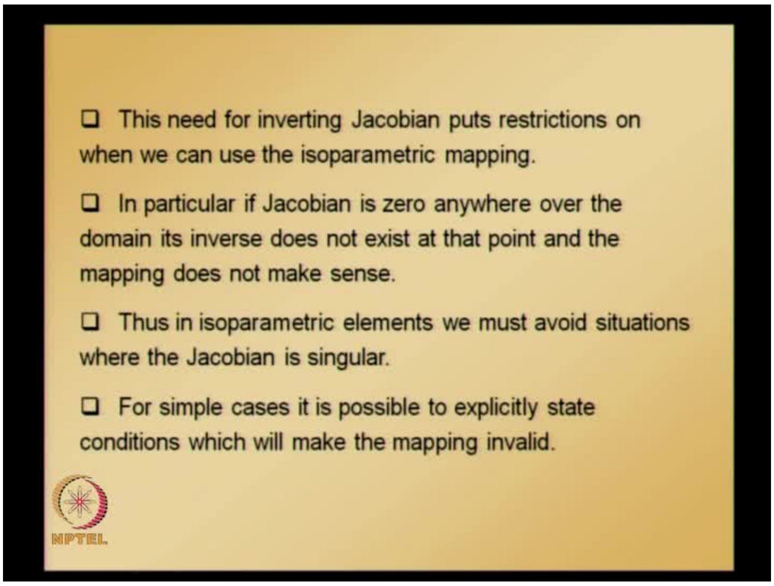
(Refer Slide Time: 41:56)



Furthermore note that x is expressed in terms of s. For derivation of finite element equations, it will be more convenient if we have s in terms of x. However, in general, it is difficult to get the inverse mapping. You will understand in a while why this inverse mapping is required, because once when we have derivatives like this, therefore derivative of form du dx must be evaluated using chain rule as follows, du dx is equal to du ds times ds dx using chain rule.

Note that from the isoparametric mapping, the expression that we have is x in terms of s, so it is easy to calculate dx over ds. So, note that from isoparametric mapping dx ds over dx cannot be calculated directly. So, first, we need to calculate dx over ds, which is called also called Jacobian. And for a linear element, it turns out that this Jacobian is equal to L over 2, L being the length of the element. And for higher order elements, we need to first calculate dx over ds, because we already have expression or the relation between x and s. So, we can differentiate it on both sides, we get dx over ds.

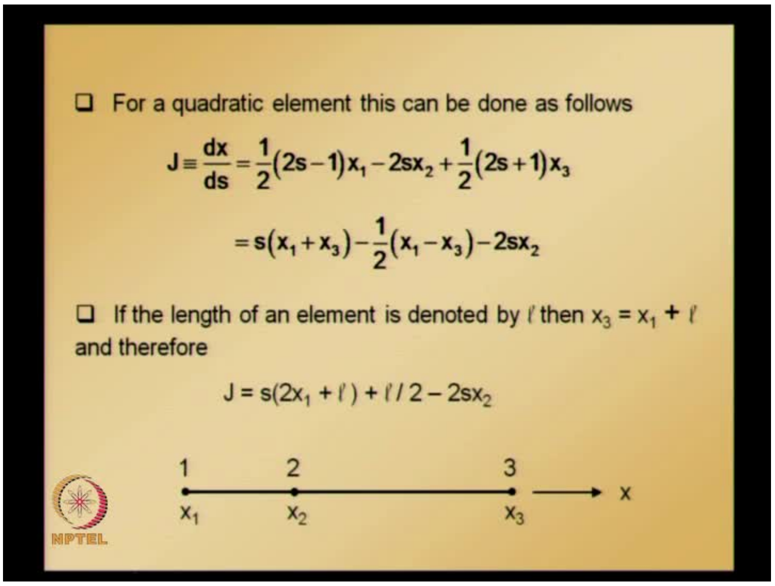
(Refer Slide Time: 43:36)



And once we have this, we can invert this to get ds over dx, but this is possible only if dx over ds is not equal to 0. This is, this need for inverting Jacobian puts restriction on when we can use isoparametric mapping. In particular, if Jacobian is 0 anywhere in the domain or anywhere between minus 1 to 1, its inverse does not exist at that point and the mapping does not make sense.

Because, if Jacobian is 0, inverse of Jacobian goes to infinity, so Jacobian becomes singular. Thus in isoparametric elements, we must avoid situations where Jacobian is singular. It is always not necessary that we need to avoid, but there are circumstances or there are specific cases where we required the singularity. And we will use this property to our advantage when we solve such kind of problems, where singularity is required for regular problems, where singularity is not required, we will try to avoid singularity Jacobian. For simple cases, it is possible to explicitly state conditions, which will make mapping invalid.

(Refer Slide Time: 45:22)

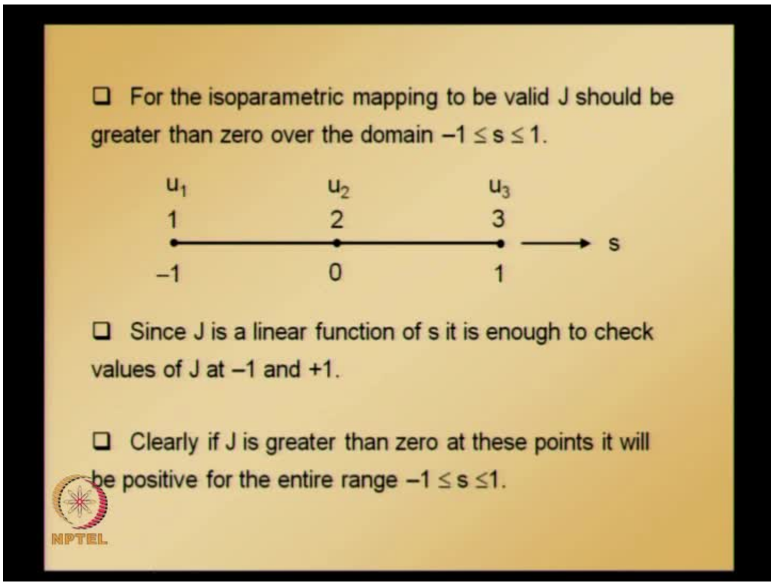


So, here, when we say mapping is invalid, we mean when Jacobian become singular, then mapping is invalid, so we can specify or we can come up with some conditions such a way that Jacobian singularity can be avoided. So, we will be deriving those conditions now. For a quadratic element, we have the relationship between x and s, by taking derivative of that, derivative of x with respect s, we get Jacobian. So, that is what is shown there.

And we note that or this can be rewritten like this, and noting that the length of element is or the x 3 minus x 1 is equal to length of element. If length of element l is denoted using l, then x 3 minus x 1 or x 3 is equal to x 1 plus l, because x 3 minus x 1 is equal to l. So, substituting this, that is x 3 in terms of x 1 and l, we can rewrite that Jacobian in the form shown in the slide.

For isoparametric mapping to be valid J must be greater than 0 over the domain 0 to 1, J must be greater than 0 or the domain s go from minus 1 to 1, not 0 to 1. And you can see here J is the linear function of s.

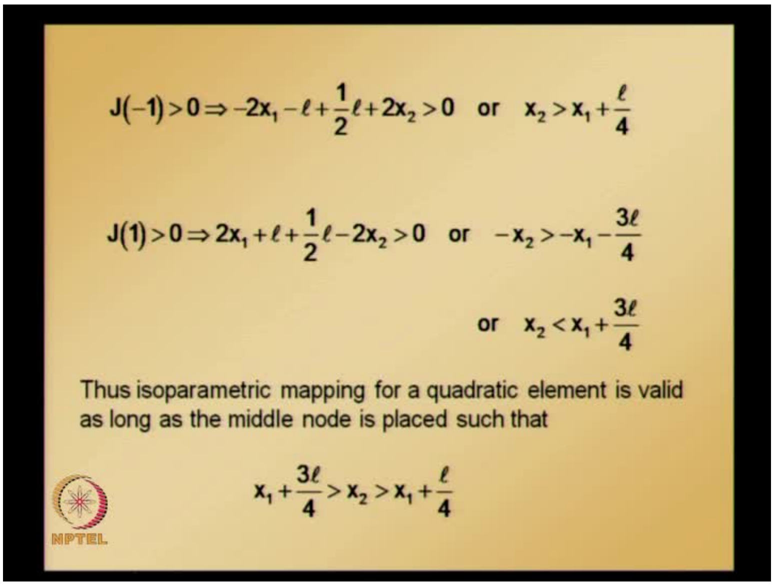
(Refer Slide Time: 47:40)



Since, J is linear function of s, if I make sure that J is positive at s is equal to minus 1, and J is positive at s is equal to 1, then J is going to be positive everywhere between minus 1 and 1. And that kind of check is only possible if J is possible, because here J is linear function of s. For isoparametric mapping to be valid J must be greater than 0 over the domain s going from minus 1 to 1.

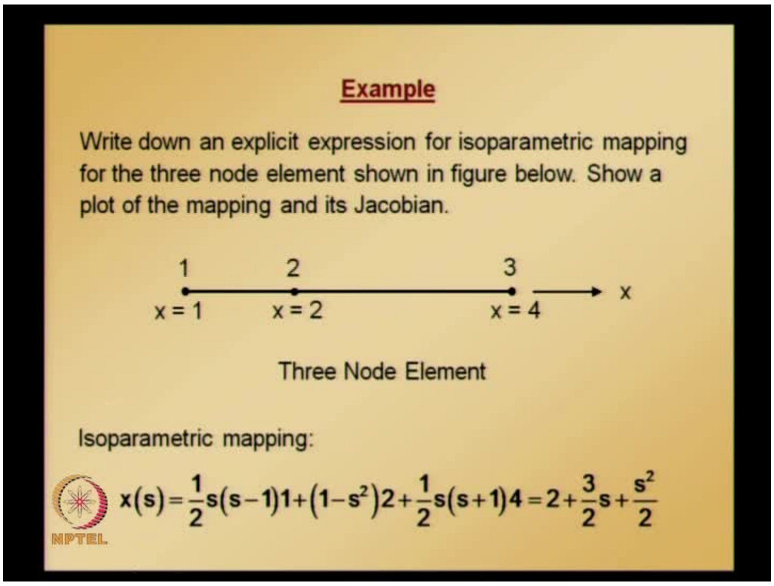
Since, J is linear function of s, it is enough to check the values of J at s is equal to minus 1 and s is equal to 1. Clearly if J is greater than 0 at these points, it will be positive for the entire domain as going from minus 1 to 1.

(Refer Slide Time: 48:18)



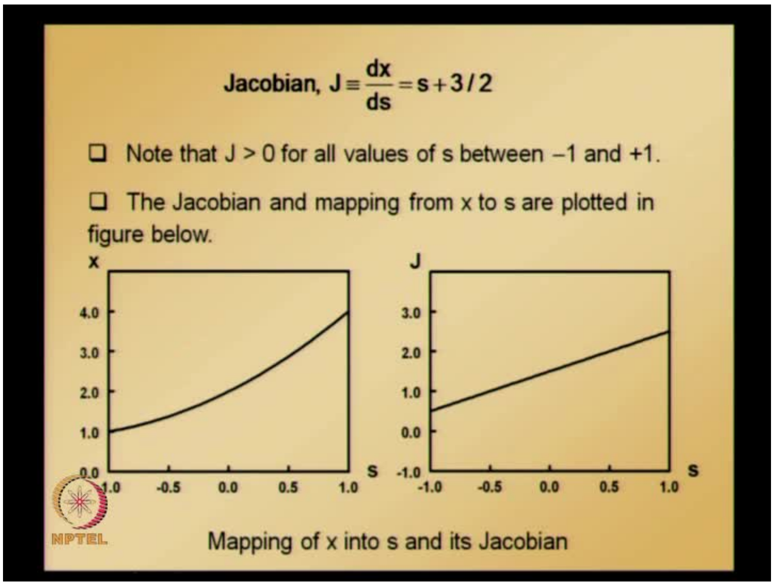
So, that is the condition we are going to apply to get the required conditions, for to avoid singularity in Jacobian. So, substituting the condition that J evaluated at s is equal to minus 1 should be greater than 0, we get that, after simplification we get x 2 should be greater than x 1 plus l over 4. And the other condition is J evaluated at s is equal to 1 should also be greater than 0, and applying this condition and simplifying, we get that x 2 should be less than x 1 plus 3 l over 4. So, the so the middle node that is x 2 should lie or x 2 should satisfy these conditions, thus isoparametric mapping for a quadratic element is valid as long as middle node is placed such that x 1 plus 3 l over 4 or x 2 lies between x 1 plus 3l over 4 and x 1 plus l over 4.

(Refer Slide Time: 49:47)



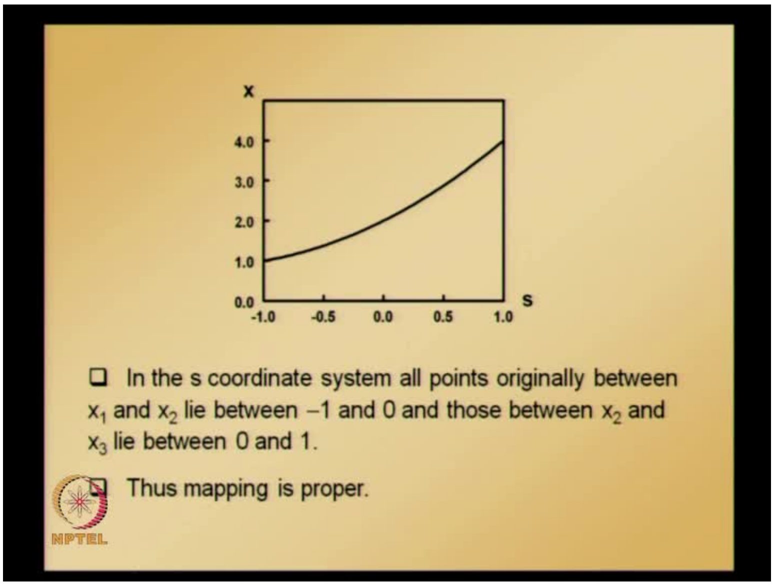
And we can verify this if we evaluate what happens, and if we satisfy this, what happens? We can check that using an example. Write down an expression for isoparametric mapping for three node element shown in figure below and plot the mapping and its Jacobian. The actual element nodal coordinates are given, isoparamatric mapping gives us relationship between x and s, substituting the shape functions of the parent element, and substituting the nodal values of the actual element x 1 x 2 x 3, we obtain this relation.

(Refer Slide Time: 50:30)



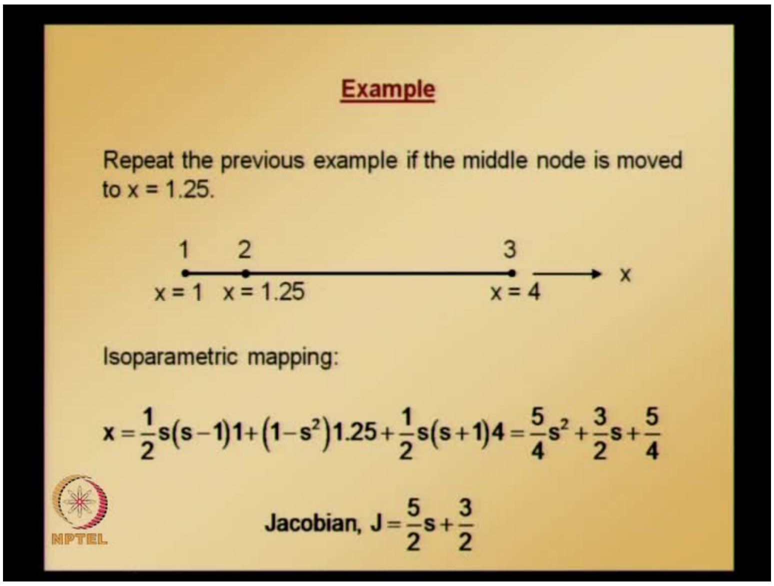
Now, taking derivative of this, we get this relation J is equal to s plus 3 over 2. And it can be easily verified that between s going from minus 1 to 1, J is always positive. Jacobian and mapping from s to s x to s are plotted in the figure below. And you can see, since Jacobian is positive everywhere, it is mapping is valid. In s coordinate system, all points is originally between x 1 and x 2, lie between minus s is equal to minus 1 to 0, and those all and all those points between x 2 and x 3 lie between 0 and 1, and so the mapping is proper or mapping is valid.

(Refer Slide Time: 51:34)



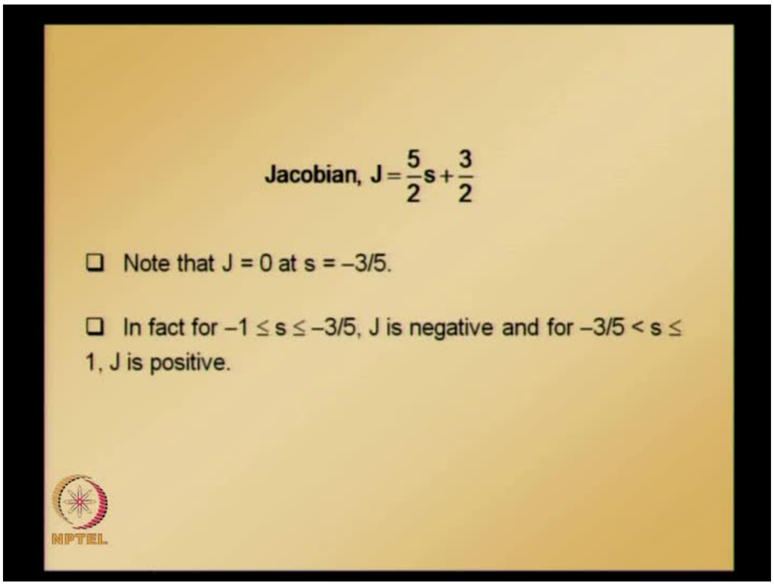
Now, let us take a special case or slightly modify this element such a way that node 2, which is at location x is equal to 2, is moved to x is equal to 1.5, and see how Jacobian and mapping looks like.

(Refer Slide Time: 52:03)



So, now, we will take that example. Repeat the previous example, if middle node is moved to x is equal to 1.25, and lets repeat the entire process, isoparametric mapping becomes this by substituting the parent element shape functions and actual element nodal coordinates, and taking derivative of it, of this, we get Jacobian, Jacobian is equal to 5 over 2 s plus 3 over 2.

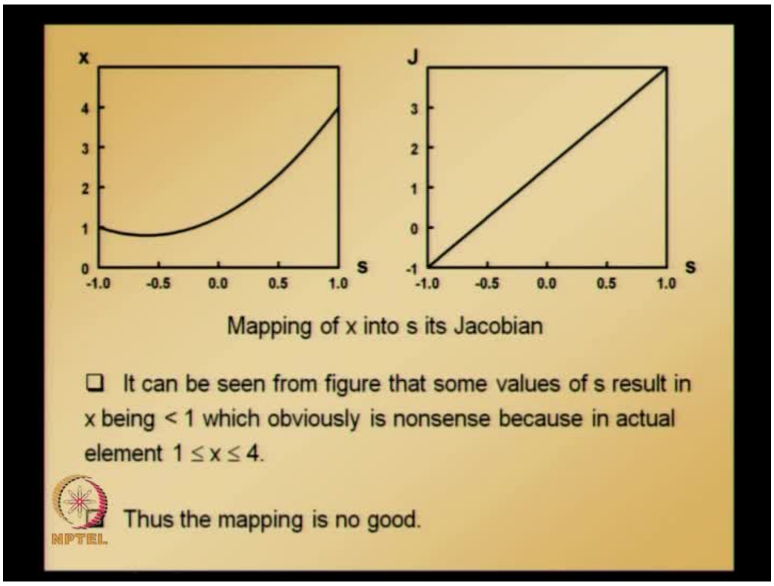
(Refer Slide Time: 52:41)



So, now, we are ready to plot x as a function of s, J as a function of s. And now, let us see those plots, and before doing that note that J is equal to 0 at s is equal to minus 3 over

5. In fact, for s going from minus 1 to minus 3 over 5 J is negative, and for s going from minus 3 over 5 to 1, j is positive, so that you can easily verify by looking at these plots.

(Refer Slide Time: 53:10)



And it can be seen from figure here that some values - that for some values of s, or some values of s result in x being less than 1, which is obviously nonsense, because the actual element is between x equal to 1 and 4. So, the mapping, there is some problem with the mapping, so mapping is not good. So, these two examples give us or illustrate isoparametric mapping. And how to check the mapping and also what are the conditions for placing the middle node in case of a quadratic element or a three node finite element?

So, we will continue in the next class looking at the derivation of the finite element equations, for a quadratic element, for general one-dimensional boundary value problem and its applications.