Finite Element Analysis
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Module No. # 01
Lecture No. # 16

In the last lectures, we have seen one-dimensional boundary value problems like bar under axial deformation; also while we are solving three-dimensional frames, we have seen differential equations, differential equation for torsional problem. So, these two kinds of differential equations for these two problems, that is, bar under axial deformation and also bar under torsion, the governing differential equation falls under the category of one-dimensional boundary value problem.

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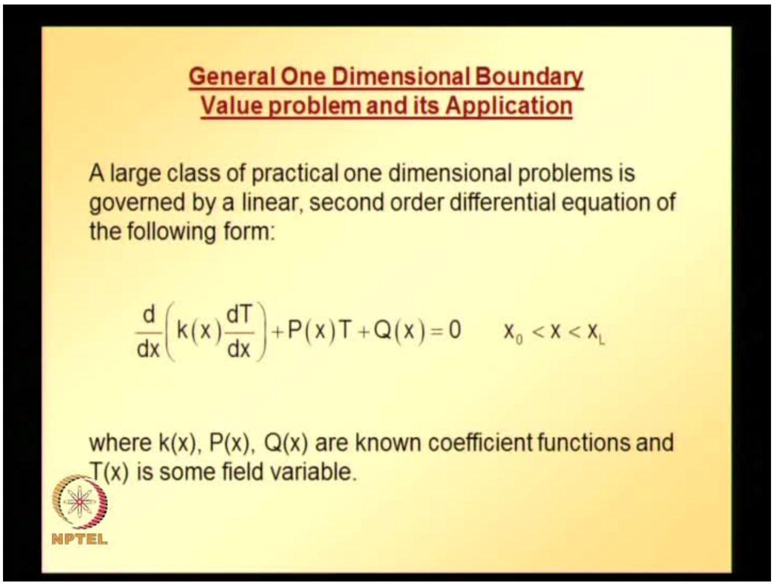

So, in today's class what we will be doing is, we will be actually looking at a more general one-dimensional boundary value problem and subjected to some boundary conditions, which include both essential and natural boundary conditions, and how to derive finite element equations, using linear element. And before doing that, we will first derive equivalent variational functional for this one-dimensional boundary value problem using Rayleigh-Ritz method.

Whatever the procedure that we followed earlier, when we are solving other problems in the earlier lectures, this same procedure we will be following as far as Rayleigh-Ritz method is concerned or Rayleigh-Ritz procedure is concerned. And also, this general one-dimensional boundary value problem can also be solved using Galerkin method, which we will be looking later when we take instead of linear element, when we deal with higher order elements like quadratic elements in the later lectures. But in today's class, we will be using Rayleigh-Ritz method for solving one-dimensional boundary value problem.

Now, let us look at what is this one-dimensional, general one-dimensional boundary value problem. A large class of practical one-dimensional problems is governed by a linear second order differential equation of the form; here this is what we mean by a general one-dimensional boundary value problem (Refer Slide Time: 02:43). So, for a particular or for a specific problem, if you can identify what is this k, coefficient k and P and Q, and so, when we get the equations for this general one-dimensional boundary value problem at those corresponding locations, you can substitute what is this k, P, Q and get the equivalent element equations for that particular problem.

Now, let us look at this problem, which is linear second order differential equations where k x, P x, Q x are coefficients or coefficient functions and T is some field variable and T can be any problem dependent variable. Once we solve this general one-dimensional boundary value problem, you will see application of this and heat flow problems, lubrication problems and structural mechanics problems related to column buckling.

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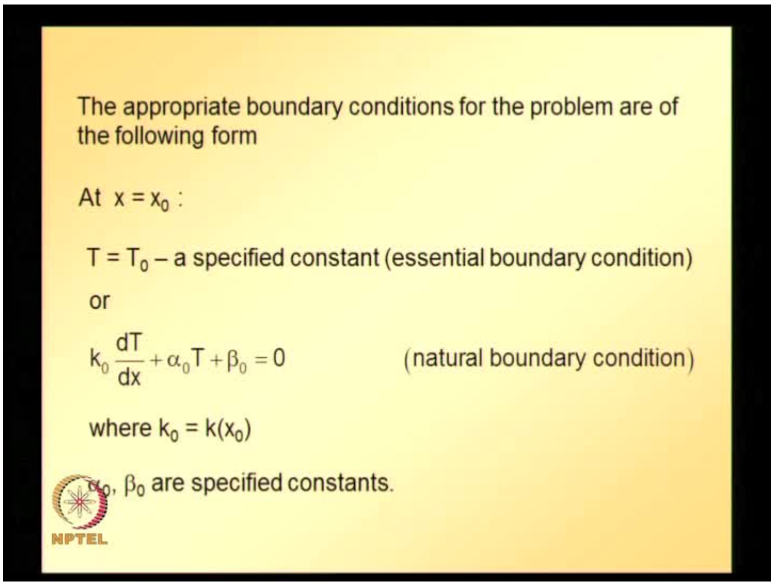

Now, this is the governing differential equation that we are taking. The corresponding boundary conditions for this problem, appropriate boundary conditions for the problem are of the following form; if you see the problem statement, the domain goes from at x naught to x l.

So, the boundary condition at x is equal to x 0 is if it is an essential boundary condition, it is T is equal to T 0, where T is a field variable, a specified constant or you can have natural boundary condition like this (Refer Slide Time: 04:40). Again here, k naught, alpha naught, beta naught are k naught is k value at x naught and alpha naught, b naught are some specified constants.

So, you can have either of these boundary conditions specified at x is equal to x 0 to solve the differential equation or the general one-dimensional governing equation that you have seen. These are the boundary conditions; any of this essential or natural boundary condition can be specified at x equal to x 0.

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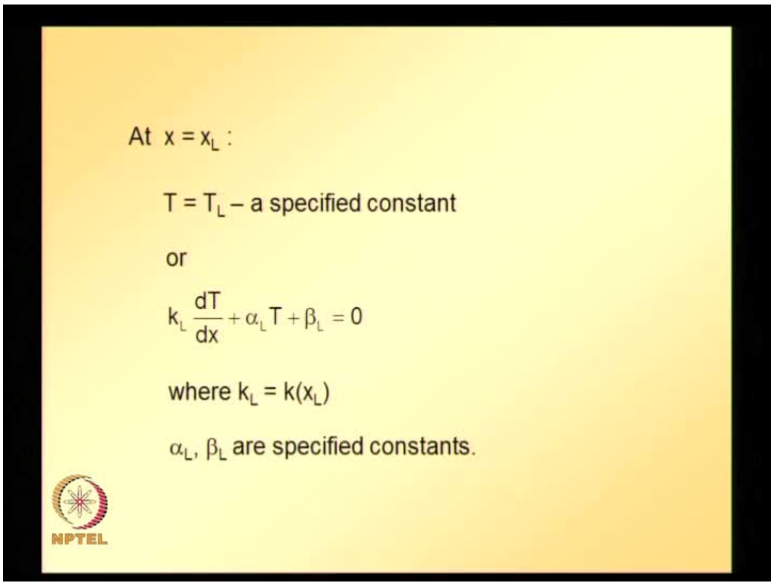

At x is equal to x L, T is equal to T L is specified constant, which is essential boundary condition or natural boundary condition can also be specified, where k L is k evaluated at x is equal to L; alpha L, beta L are some constants.

So, this is the problem definition; the differential equation is given, problem domain is given, x is going from x naught to x L and the two boundary conditions are the boundary conditions at the two ends at x is equal to x 0 and at x is equal to x L, either essential or natural boundary conditions may be specified.

So, this is what we assume and with this we will start. Now, before we derive the element equations using substitute infinite element approximations, we need to arrive at if you are adopting Rayleigh-Ritz method, we need to derive equivalent functional. So, the first step, when deriving equivalent function is the given differential equation is multiplied with the variation of quantity that we are interested, integrated over the problem domain, and equated to 0. Here, the quantity that we are interested is in finding the field variable value.

So, T is what we are looking for so, the given differential equation needs to be multiplied with variation of T, integrated over the problem domain which goes from x naught to x L and d T equated to 0.

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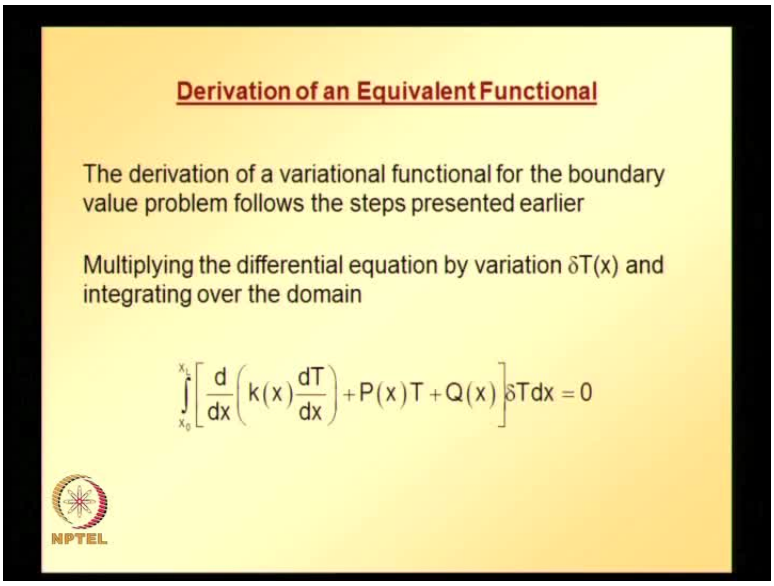

Derivation of a variational functional for the boundary value problem follows the steps presented earlier, in the earlier lectures. So, multiply in the given differential equation by variation of T and integrating over the problem domain and equating it to 0.

We get, this is the first step, and now we need to identify higher order terms and wherever higher order terms appear, we will replace them with lower order possible by integration by parts.

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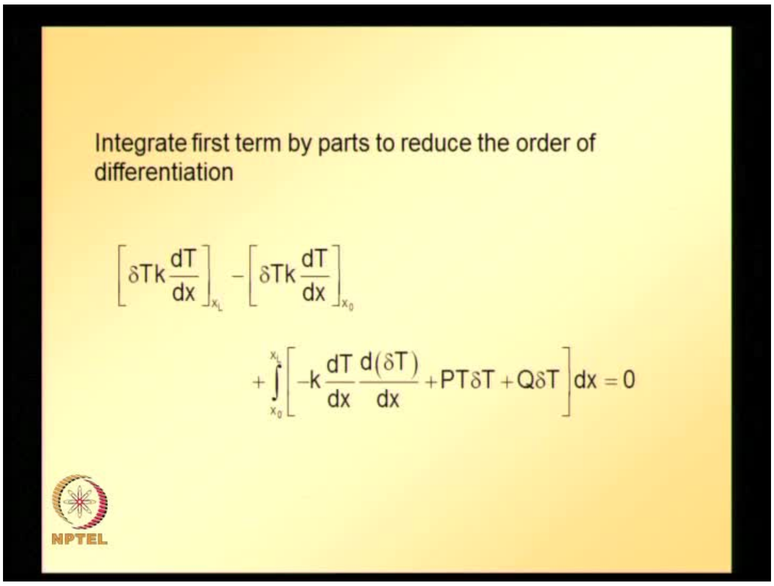

So here, integrate first term by parts to reduce the order of differentiation. Integrate first term by parts to reduce the order of differentiation; so, when you do integration by parts on the first term, we get this equation, and we can see the terms inside the integral, some of these can be further rewritten using the variational identities that we learnt in the earlier lectures.

So, from variational identities k time’s derivative of T with respect to x, derivative of variation of T with respect to x, can be written as variation of half k time’s derivative of P with respect to x square of that. Similarly, P times T times variation of T can also be written as a P times or variation of P over to T square, Q times variation of T can be written as variation of Q T.

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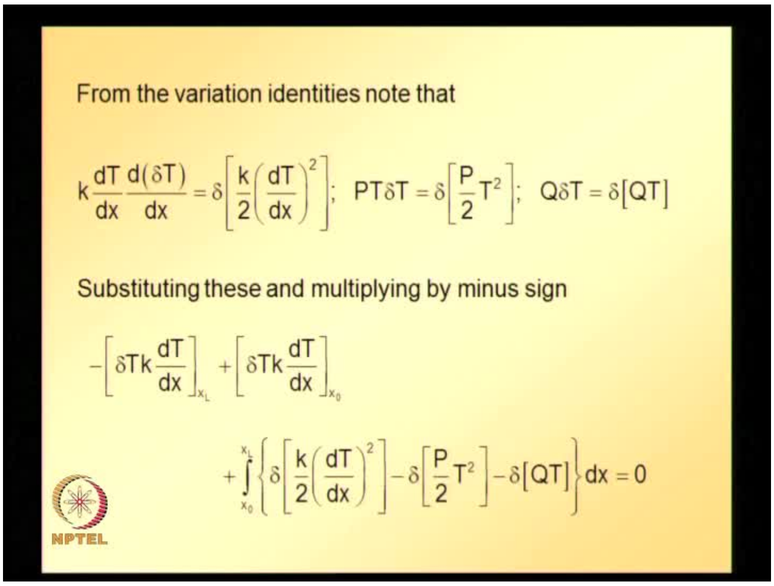

So, from the variation identities we get this and now substituting this and multiplying by minus sign, we get this equation; multiplying the entire equation by minus sign we get this equation and if you see the first two terms, they are related to the boundary conditions that are given, because k times derivative of T with respect x at x is equal to x L is given and also k times derivative of T with respect to x at x naught is also given.

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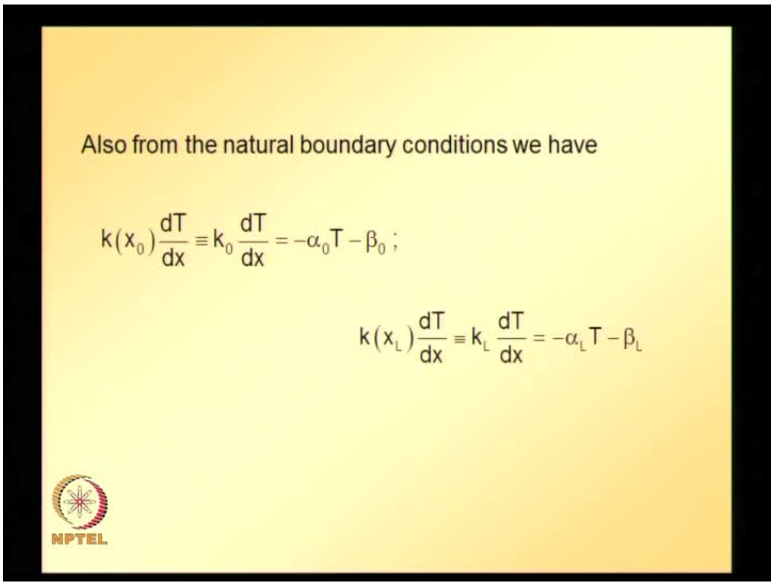

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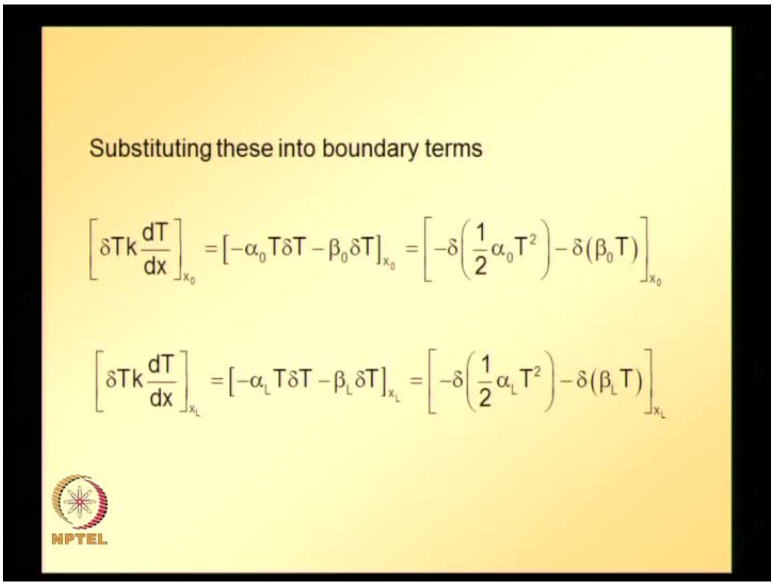

So, from natural boundary condition that are given, we have this equation (Refer Slide Time: 10:36) and also we have this equation from natural boundary condition that is specified at x is equal to x L. Substituting these boundary terms, substituting these into the boundary terms we get, variation of T times k times derivative of T with respect x, evaluated at x is equal to x naught.

We can write like this, and also which can be further rewritten using variational identities. We have the other boundary term, which can also be written in a similar manner, and also which can be rewritten using variational identities.

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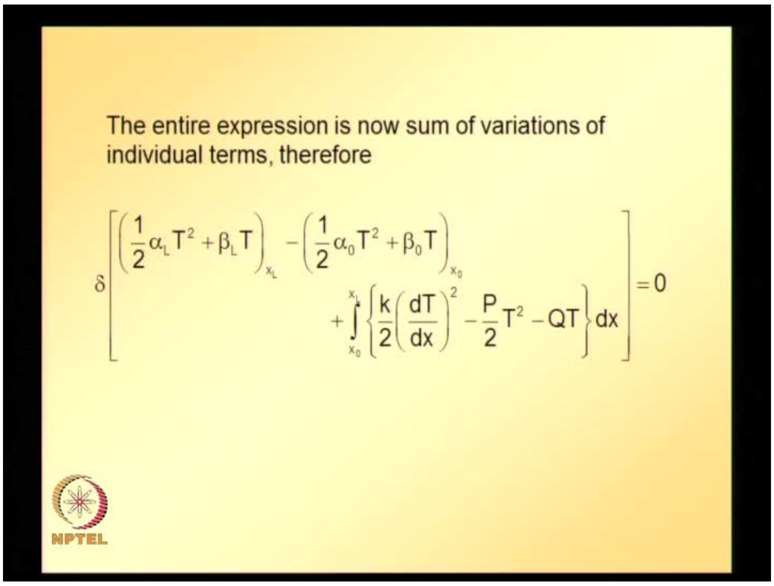

Now, substituting all these back into the equation, we can pull variational operator out; so, the entire expression now is sum of variation of individual terms.

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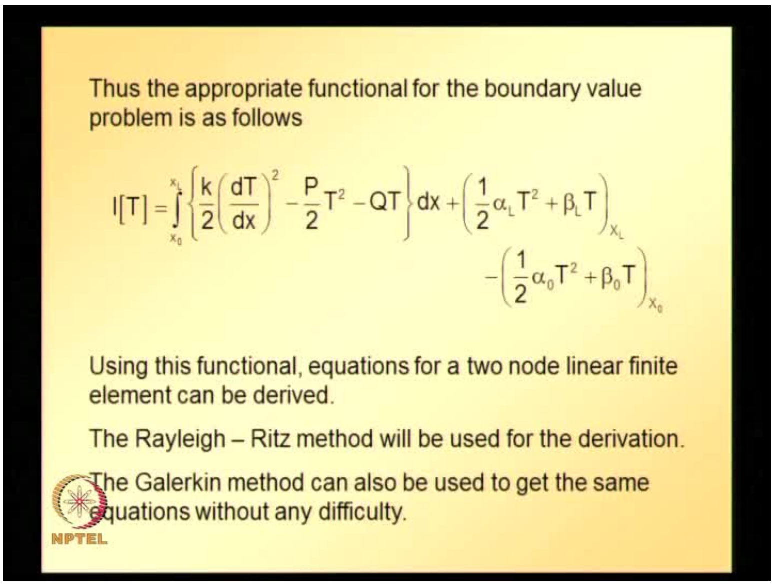

So, whatever is there inside the square bracket that is equivalent variational functional, the appropriate functional for the general one-dimensional boundary value problem is given by this (Refer Slide Time: 12:40).

So, using this functional and equations for a two node linear finite element, if a two node linear finite element is chosen for discretization, using that equations for two node finite element, a linear infinite element can be derived. Here, the rest of the derivation will be using Rayleigh-Ritz method; Rayleigh-Ritz method will be used for the derivation, Galerkin method can also be used to get the same equations without whatever equations that we are going to get using the Rayleigh-Ritz method; these similar equations can also be obtained using Galerkin method.

Now, we have the equivalent functional; so, before we proceed further, we need to look at the finite element approximations of each of the terms that are appearing in the equivalent functional. How to express their approximation in terms of finite element shear functions and nodal values, we need to look into it.

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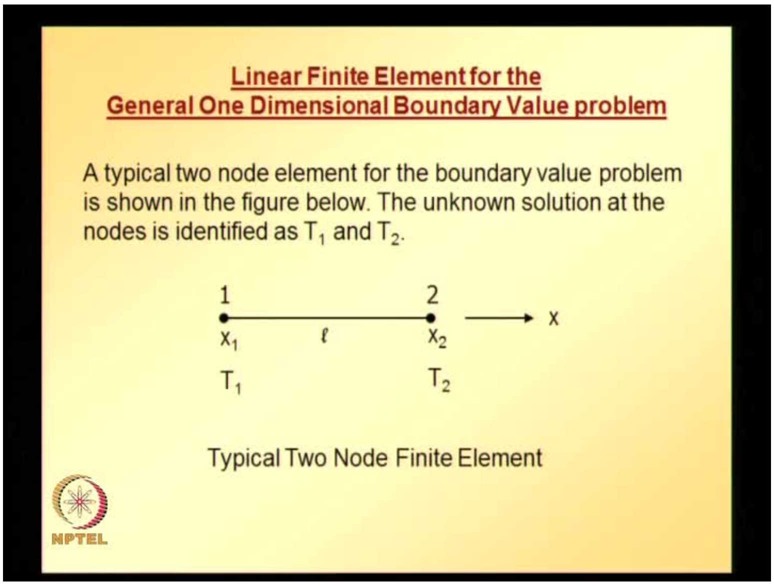

So, linear finite element for general one-dimensional boundary value problem, the procedure will be same even if you use higher order elements which will be looking at the later lectures. A typical two node element for the boundary value problem is shown in figure below (Refer Slide Time: 14:43). Having two nodes, node 1 at x 1, node 2 at x

2, length of the element being L, field variable value at node 1 is T 1, field variable value at node 2 is T 2, and the positive direction of a coordinate system x is also indicated in the figure.

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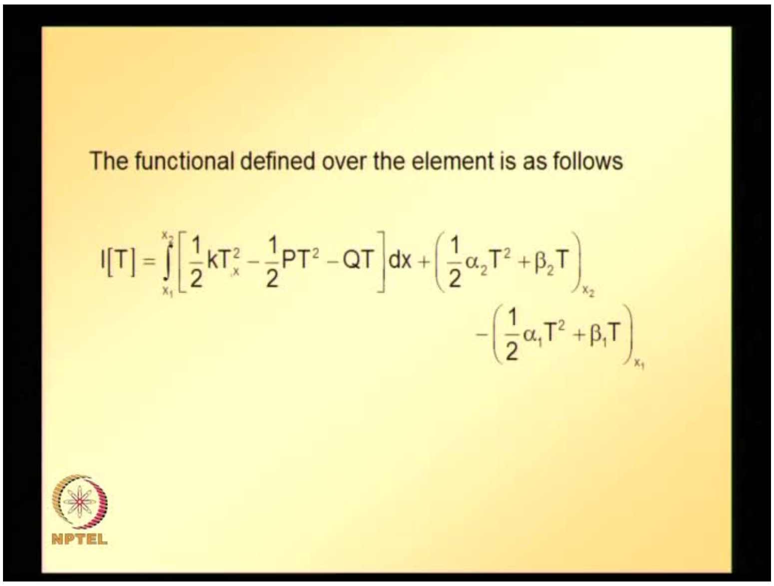

So this is the typical two node element for the boundary value problem that we are going to solve; the unknown solution at the nodes is identified as T 1 and T 2. So, since we are taking a two node finite element having node 1 at x 1 and node 2 at x 2, the functional that we derive, can be defined over this element is like this. Functional defined over the element is as follows: so, x naught is replaced with x 1, x L is replaced with x 2, and also alpha naught, beta naught are replaced with alpha 1, beta 1; alpha L, beta L are replaced with alpha 2, beta 2.

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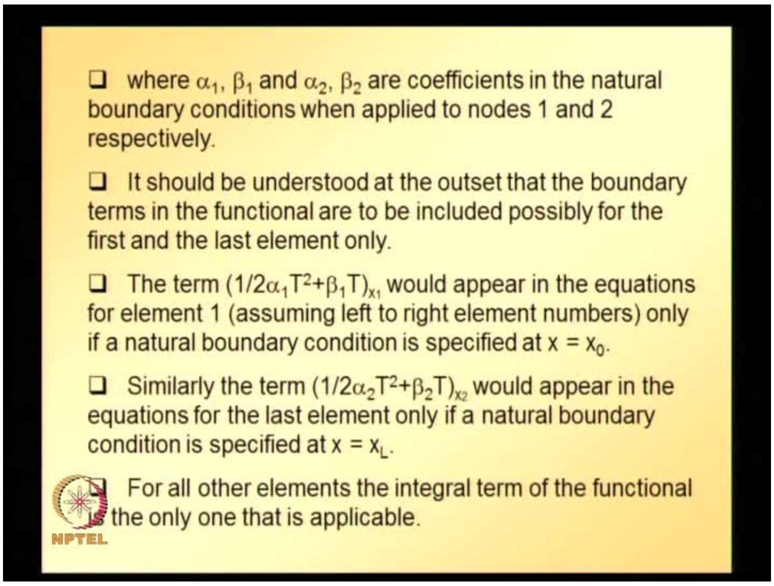

So, here in this equation which you have seen alpha 1, beta 1 and alpha 2, beta 2 are coefficients in the natural boundary conditions, when applied to nodes 1 and 2 respectively. Natural boundary conditions are applied to nodes 1 and 2 respectively. It should be understood that, the boundary terms in the functional are to be included possibly for the first and last element only that will be clearer, when we are actually solving a problem.

So, the term half alpha 1 T square plus beta 1 T evaluated at x 1 would appear in the equations for element 1, assuming left to right element numbers, only if a natural boundary condition is specified at x is equal to x naught.

Similarly, the term half alpha 2 T square beta 2 T, evaluated at x 2 would appear in the equation for the last element only if natural boundary condition is specified at x is equal to x L. These points will be clearer once we solve a problem, for all other elements which are in between first element and last element; for all other elements, integral term of the functional is the only one that is applicable.

So, whatever the functional that we are defining over the element is more general; so, for some elements, only the first boundary term and the integral may be possible. For some cases, for some elements, second boundary term and integral term are only possible whereas, for some other elements which are neither first element nor last element, for such kind of elements only integral term of the functional is applicable.

Now, we have functional defined over element; we need to substitute finite element approximation. So, we need to make the field variable approximation in terms of finite element, two node finite element shear function in nodal values.

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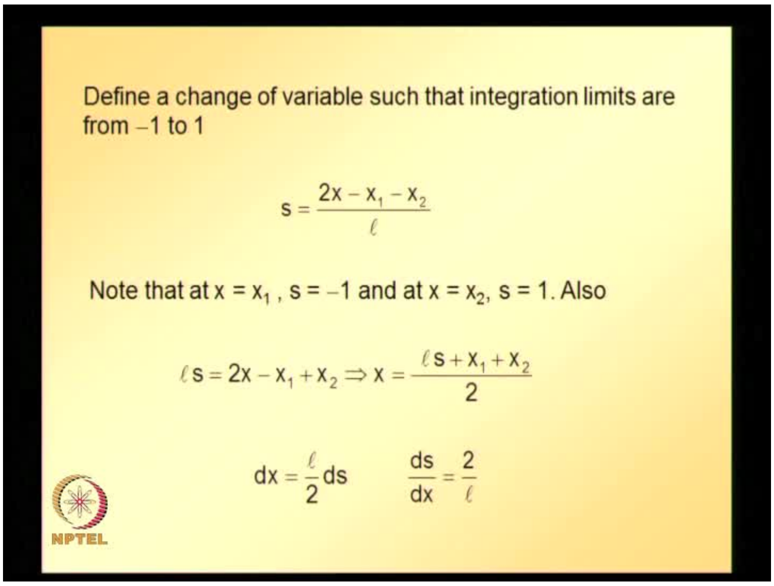

So, the element x 1 to x 2, two node element x 1 to x 2 will be mapping on to S coordinate system, where S goes from minus 1 to 1 S is equal to minus 1 corresponds to x 1, S is equal to 1 corresponds to x 2. So, define a change of variable such that integration limits are from minus 1 to 1 instead of x 1 to x 2.

So, the relation between the x coordinate system and S coordinate system is given by this equation, which can be easily derived using linear interpolation formula, which we already did in earlier lectures.

So, in this relation note that x is equal to x 1 gives S is equal to minus 1 and x is equal to x 2 gives S is equal to 1; the equation can be rearranged as shown in the slide, we get the inverse relation between x and S. From the relation, first equation you have the relation between s and x and rearranging it, we get relationship between x and s; after taking derivative on both sides, we obtain this which can be rearranged such a way that d S over d x is equal to 2 over l.

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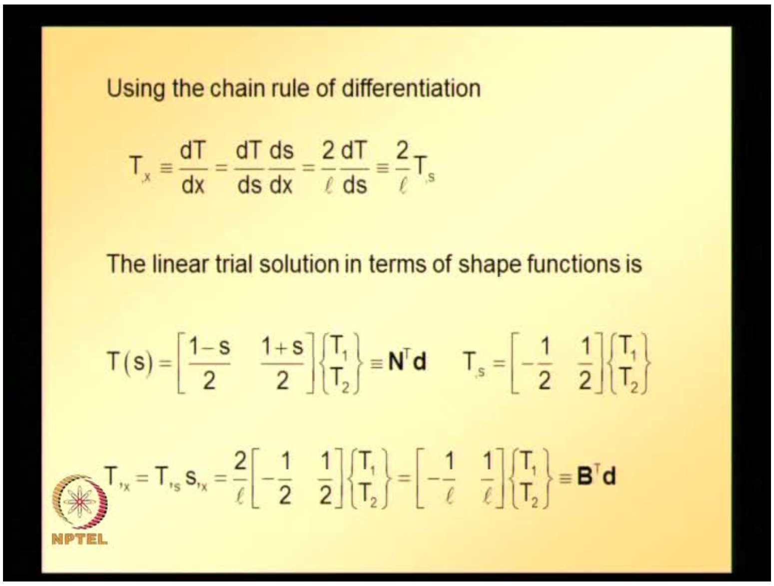

If you see the functional, we have derivative of T with respect x so, there will be using this relation d S over d x. Using chain rule of differentiation, derivative of T with respect to x can be written as d T over d S d S over d x, just now we have seen d S over d x as 2 over l, where l is length of element. So, the previous substituting whatever we have in the previous equation, that is say d s over d x is equal to 2 over l here, we get what is shown there.

So, derivative of T with respect to x is 2 over l times derivative of T with respect to S. Also we know that for 2 node element, the linear trial solution in terms of finite element shear functions is given by this one (Refer Slide Time: 22:12); p as a function of S is n 1 T 1 times n 2 T 2 which can be written in a matrix and vector form in this manner and which can be compactly written as N transpose d. From this we get, what is derivative of T with respect to S; it can be easily verified, it is given by minus half half put in a matrix form, multiplied by T 1 T 2 put in a vector form.

So using this, if you multiply derivative of T with respect S with 2 over length of element we get derivative T with respect to x, but that is what is shown here; derivative of T with respect x is derivative of T with respect S times derivative of S with respect x and which can be compactly written as B transpose d whereas, B transpose d b transpose sorry whereas, b transpose is defined as minus 1 over L, 1 over L put in a matrix form or it is a row matrix.

So, we have the field variable approximation in terms of nodal values and shear functions and also derivative of T values, derivative of field variable with respect x in terms of derivatives of finite element shear functions and the nodal values.

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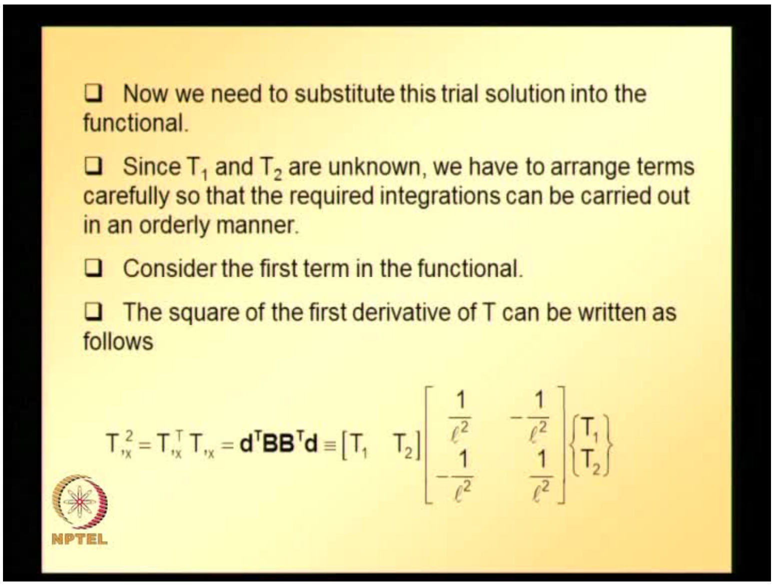

So, we need to substitute this trial solution and it is derivative into the functional. Since T 1 T 2 are unknown, we have to arrange the terms carefully so, that required integrations can be carried out in an orderly manner. Now, we will take each term in the equivalent functional.

So, let us consider the first term in the functional and if you see the first term in the functional, square of first derivative of T with respect to x is appearing. So, derivative of T with respect to x square of that can be written as derivative of T with respect x transpose times derivative of T with respect x because derivative of T with respect x is a scalar quantity, square of S scalar can be written as scalar transpose scalar.

So using that logic, we get this, we can write derivative of T with respect x square in this manner and we know that derivative of T with respect x is B transpose d. So, substituting that, we obtain this equation after carrying out the multiplications of B B transpose.

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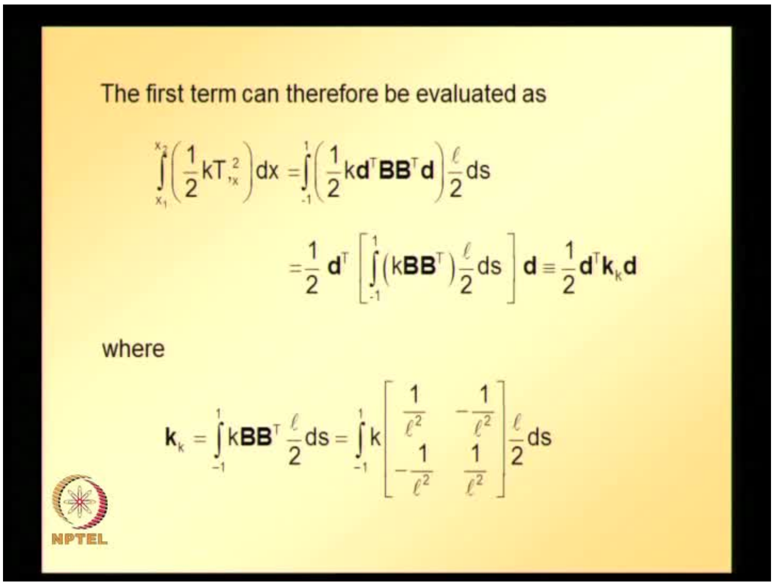

So, substituting this square of the first derivative of T with respect x into the first term, the first term can be therefore evaluated as, this is the first term appearing in the equivalent functional.

Now, substituting derivative of T with respect x square in terms of finite element, finite element shear functions or derivate of finite element shear functions, which is d transpose B B transpose d and k is the coefficient. We know that d which is a T 1, T 2 which are the nodal values and they are not functions of special coordinates.

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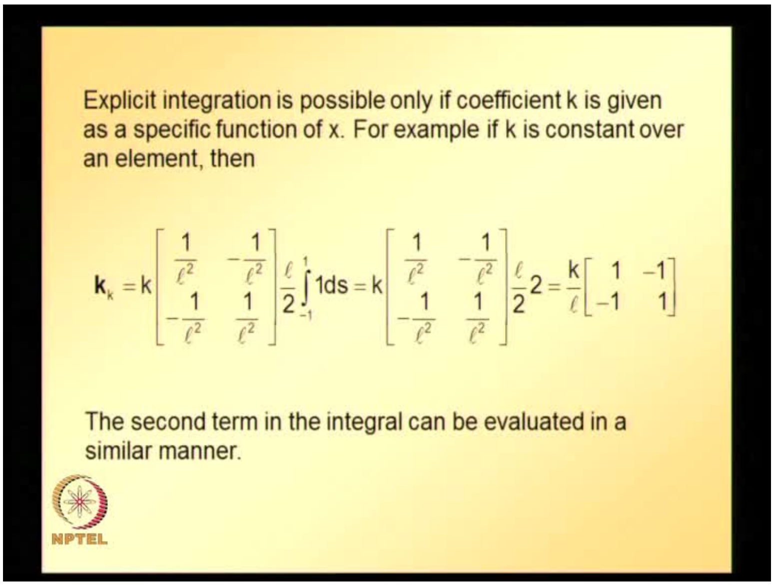

So, d transpose and d can be taken out of the integral and defining k k as integral k trans B B transpose l over to d S integrated between minus 1 to 1 and carrying out the multiplication of B B transpose, this can be further written and the weight is shown (Refer Slide Time: 27:32). Here, the explicit integration is possible only if coefficient k is given as a function of x and if k is constant over the element, then k can be taken out of the integral and rest of the integration or rest of the terms can be carried out and we get, k divided by l times 1 minus 1, minus 1 1 as shown in the slide.

So, the first term appearing in the equivalent functional, after making substitution in terms of finite element shear functions and derivative of finite element of shear functions, we get this and here, this equation or this is valid only if k is constant over element. So, this is the first term simplification, now let us look at second term. Second term integral can be evaluated in a similar manner, except that in the second term we have T square.

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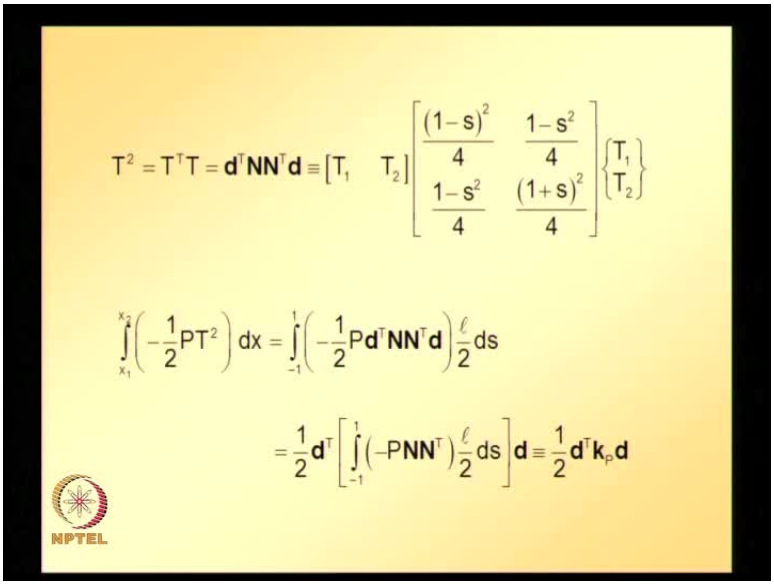

So, T square T is again being field variable, it is a scalar quantity T square can be written as T transpose T, T is N transpose d. So, T transpose is d d transpose N so, T transpose T is T transpose N and transpose d, and after making substitution of d and N values and simplifying, we get this equation.

Making substitution of this T square into the second term appearing in the equivalent functional, we get this equation (Refer Slide Time: 29:59). Here defining k P as integral minus 1 to 1 minus P N, N transpose time's l over to d S defining that as k P, we can compactly write in this form; again here d is being vector of nodal values, it can be taken out of the integral.

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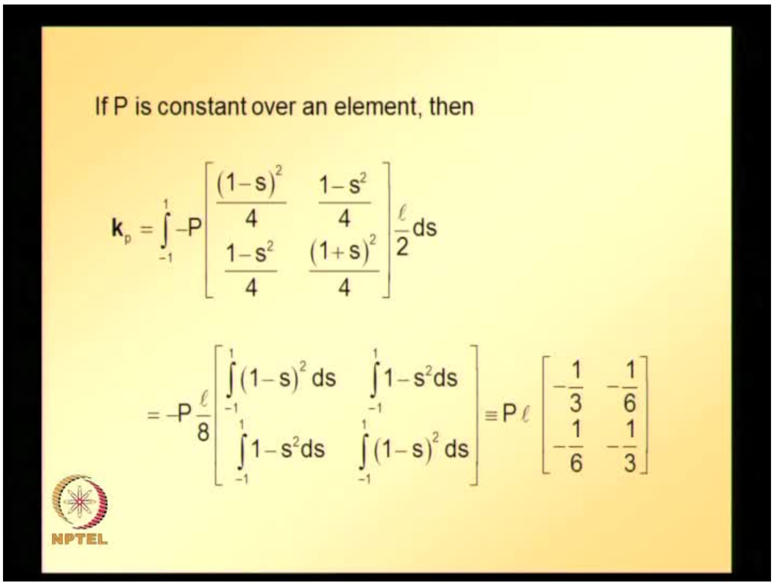

So, this simplification is possible because of that and now looking at k P, where k P is defined like this (Refer Slide Time: 30:53). Here also unless explicitly P is given as a function of x, integration is not possible. But for a specific case, where P is constant over an element, we can take pay out of the integral and we can simplify each of the terms appearing in the matrix that is carrying out integration of each of the terms between minus 1 to 1.

If P is constant over an element, then k P can be simplified to this (Refer Slide Time: 31:40). So, we have simplified first term and the second term appearing in the equivalent functional, well now we need to look at third term.

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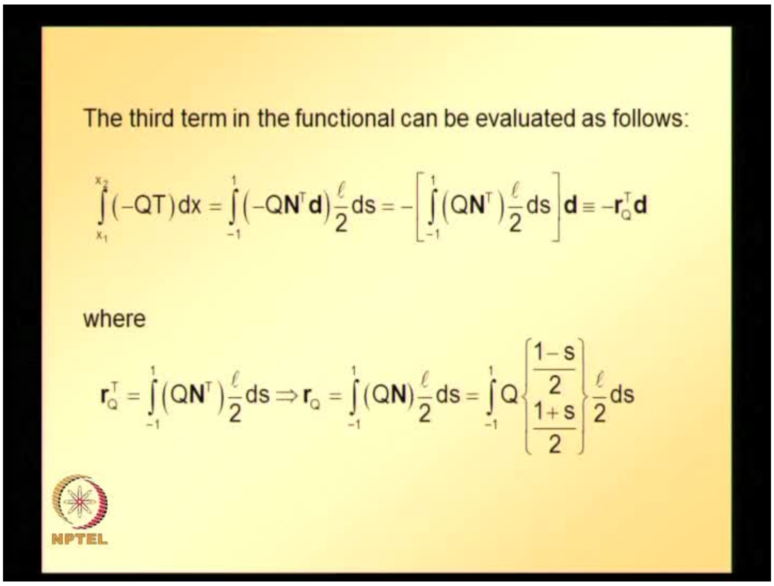

Third term in the functional can be evaluated as follows: substituting T value in terms of finite element shear functions, which is N times (( )) and rearranging it, since d is not a function of spatial coordinate, we can rearrange it as r Q transpose d, where r Q is defined as Q integral minus 1 to 1 Q N transpose l over 2 d s. If r Q transpose is defined in the manner, r Q can be written as integral minus 1 to 1 minus 1 to 1 Q N 2 times N l over 2 d S, substituting finite element shear functions values, N transpose is defined as 1 minus S over 2, 1 plus S over 2.

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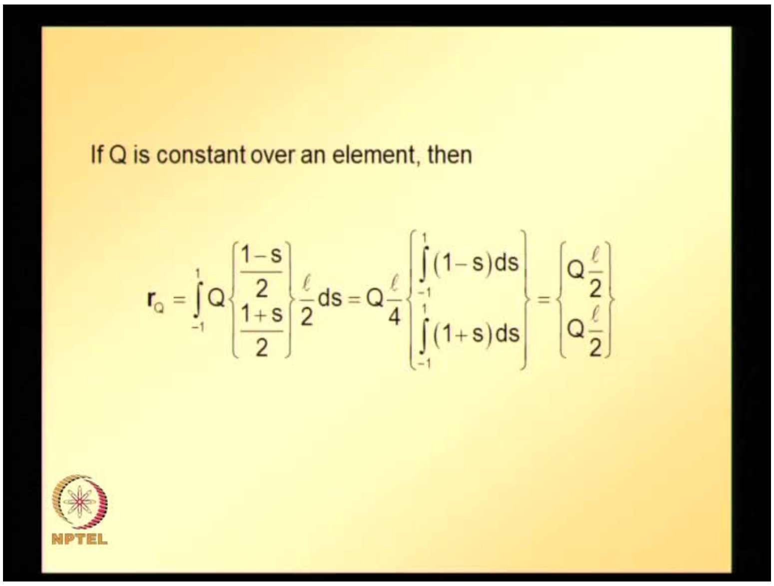

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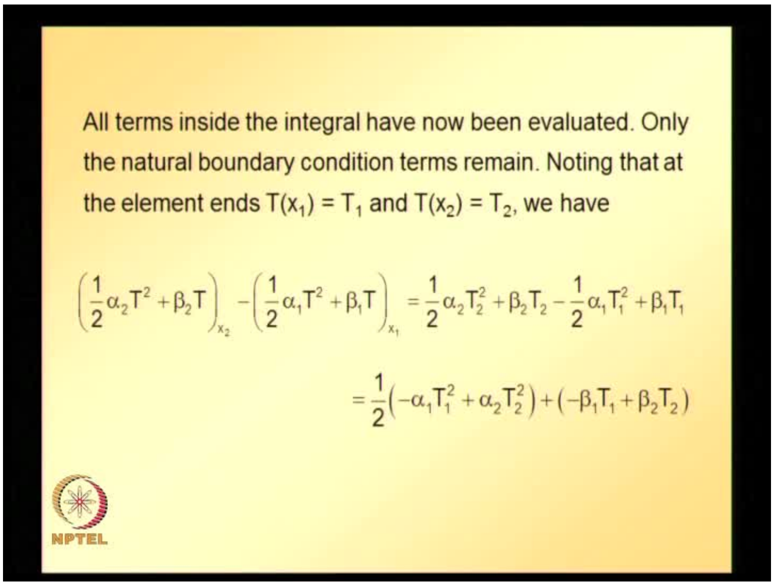

So, N is defined as put in the same thing, put in a column vector form and again here if Q is constant, you can take Q out of the integral, carry out the integration and we get this, r Q r Q is equal to Q l over 2, Q l over 2 put in a column vector form (Refer Slide Time: 33:36). Now, all the terms inside the integral, inside the equivalent functional are all the terms appearing inside the integral of equivalent functional are evaluated and only the natural boundary condition terms are remaining.

We will look into those things, noting that T evaluated at x 1 is equal to T 1; T evaluated at x 2 is equal to T 2. The boundary terms can be written in this manner and in order to combine these boundary terms in the rest of the matrices, they are rearranged or they are arranged in a matrix form.

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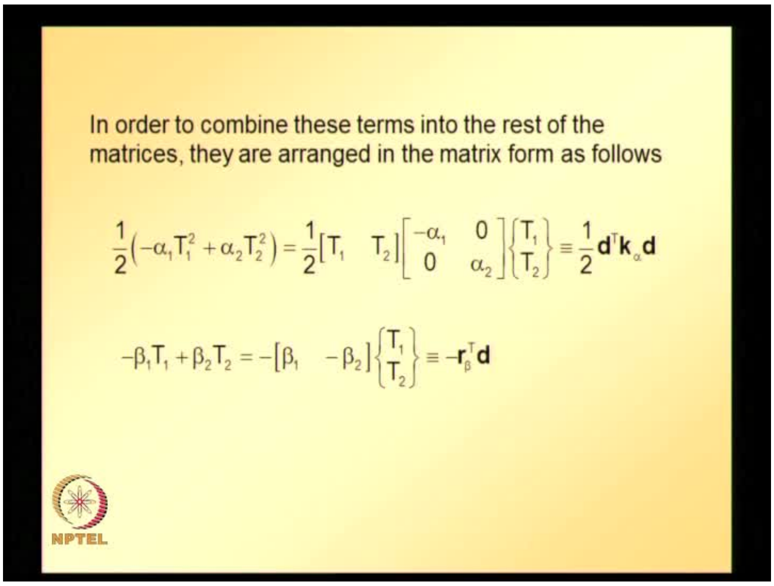

So, these boundary terms are arranged in a matrix form in this manner. And also, the other term is arranged like this, where k alpha and r beta are defined like this (Refer Slide Time: 35:10).

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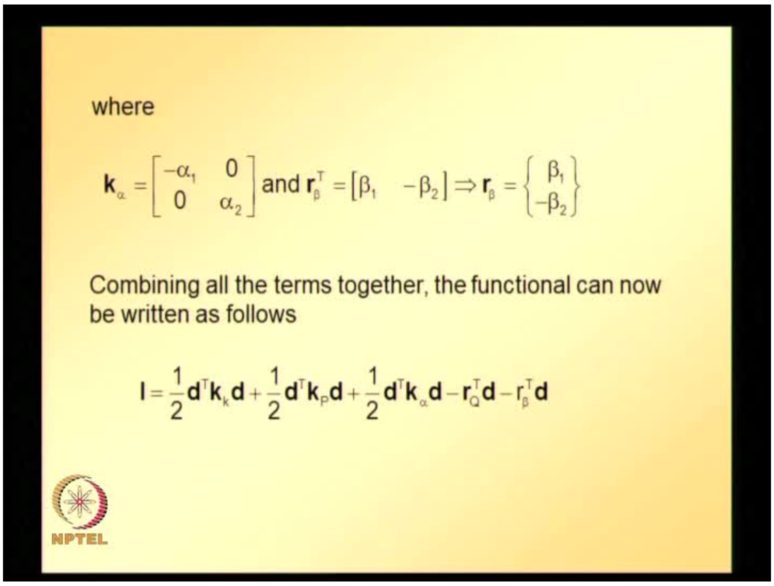

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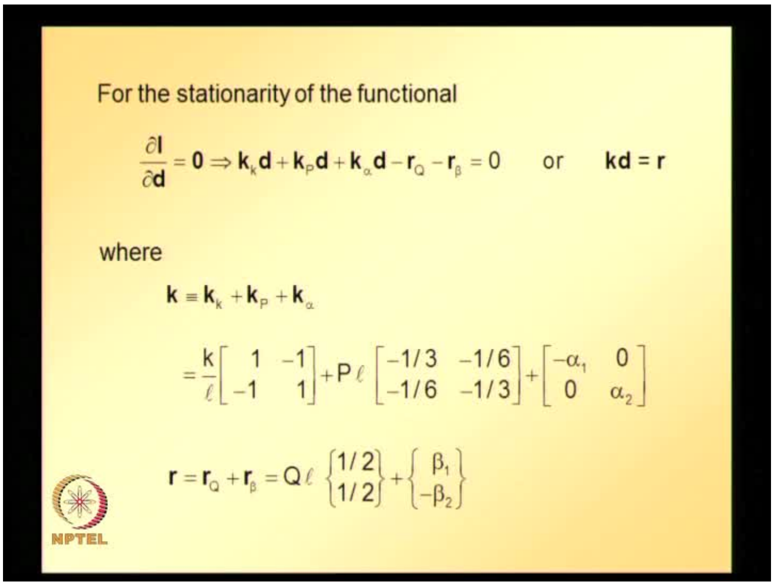

Now, all the terms in the equivalent functional, we express them in matrix form and combining all the terms together, the equivalent functional can be written in this manner (Refer Slide Time: 35:39). Now, we need to apply the stationarity condition, variation of I is equal to 0 is possible only when partial derivative of I with respect to the unknown parameters which is here nothing but d vector partial derivative of I with respect to d is a stationarity condition. Applying that condition, we get this equation which can be compactly written as, k d equal to r, where k is defined as sum of k k k P and k alpha and making substitution of k k k P k alpha, we get this and also r is defined as r Q plus r beta (Refer Slide Time: 36:30).

Please note that, this equation is obtained by assuming k is constant - k, P, Q are constants over element, and note that alpha and beta terms results from natural boundary conditions specified at the ends. For example, a natural boundary condition is specified only at node 1 of an element, then alpha 1, beta 1 terms must come from natural boundary conditions and alpha 2, beta 2 are going to be 0.

Also remember that, the explicit expressions for element equations assume that k, P, Q to be constant over an element; if these values or not constant, then one can determine new element matrices by carrying out the integration keeping k, P, Q inside the integrals and integrating and carrying out the integrations or the solution domain can be divided into large number of elements.

Over each element an average value of coefficients of these parameters k, P, Q can be used and then we can use these equations. But, this kind of approximation is going to be in addition to the already approximation that we are using for finite element discretization, which is linear element. So, there may be some error in the solution however, as the number of elements increases this error due to approximation becomes negligible.

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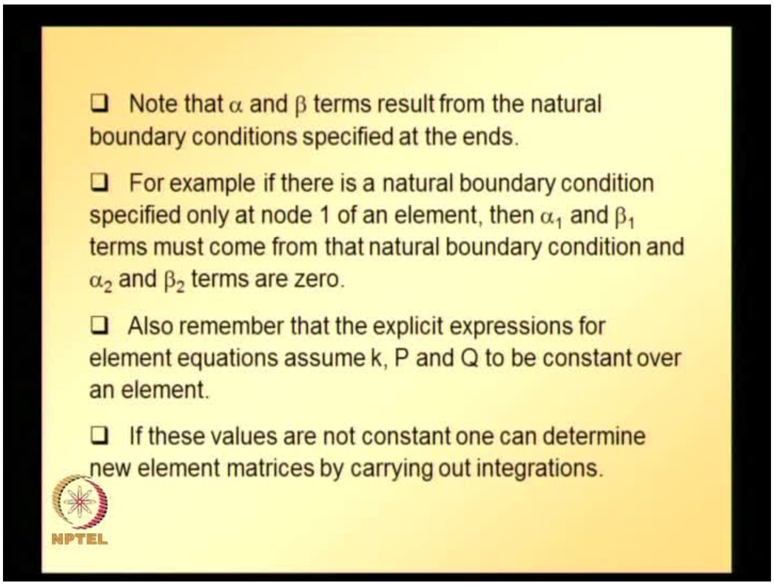

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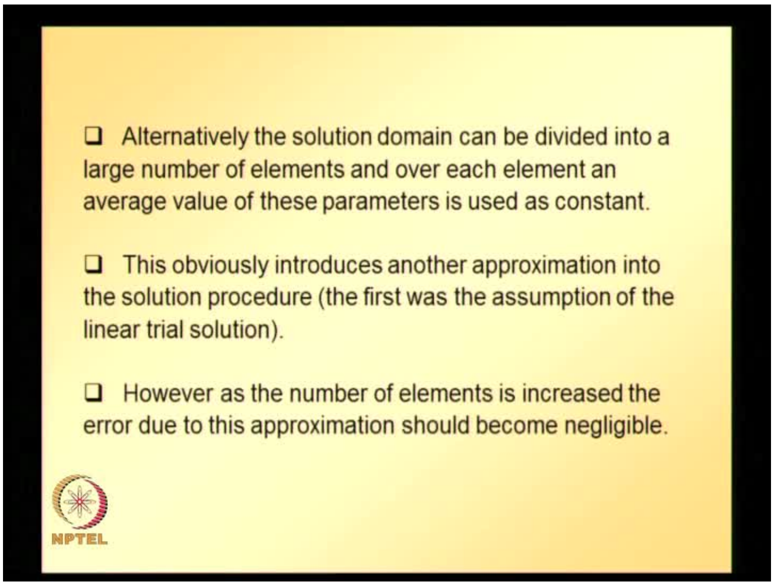

So, this is what is mentioned. Now, what we will be doing is we will be taking an application of this general one-dimensional boundary value problem which is one-dimensional heat flow.

So, basically what we did is, we derived the element equations for a linear finite element, for a general one-dimensional boundary value problems subjected to both essential boundary condition, natural boundary conditions. For a specific problem related to any application, what we need to do is we need to identify what are these coefficients k, P, Q and also the from the boundary conditions, we can identify what are alphas and betas and we can directly use the element equations that we derived based on linear finite element approximation. We can plug in into these and we can get the element equations for corresponding problem, which will be very advantageous.

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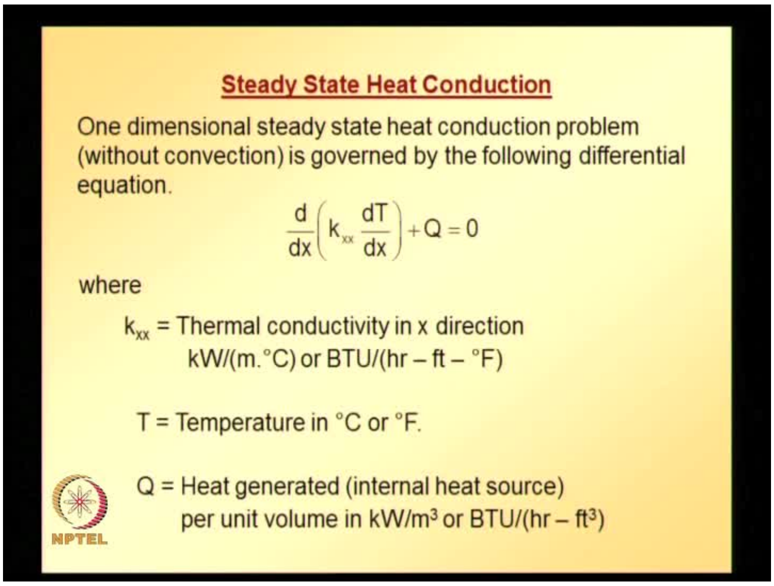

Now, let us look at application, steady state heat conduction. So, the one-dimensional steady state heat conduction problem without convection is governed by the following differential equation, where k x x is thermal conductivity in x direction. Please note that this is one-dimensional problem and the units of k x x are given and temperature T is the temperature field variable.

Now this temperature filed variable which we have in general one-dimensional boundary value problem is now temperature and Q is heat generated, internal heat source per unit volume, units are given. So, this is a linear second order differential equation and which is similar to what we have seen general one-dimensional boundary value problem, the differential equation corresponded to general one-dimensional boundary value problem.

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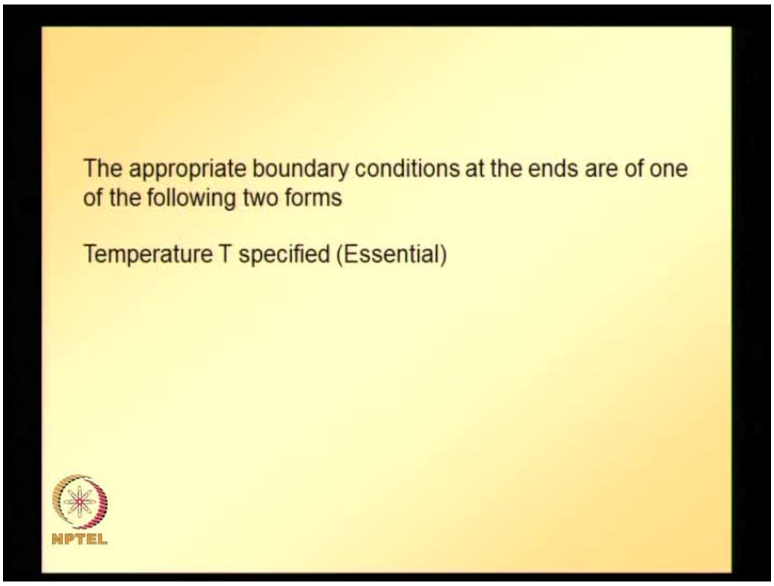

Now, let us look at what are the boundary conditions, the appropriate boundary condition at the ends are one of the following two forms: essential boundary condition or natural boundary condition.

So, what we need to do is we need to make a comparison between the this differential equation and these boundary conditions and the differential equation and boundary condition that we have taken for getting the element equations for general one-dimensional boundary value problem.

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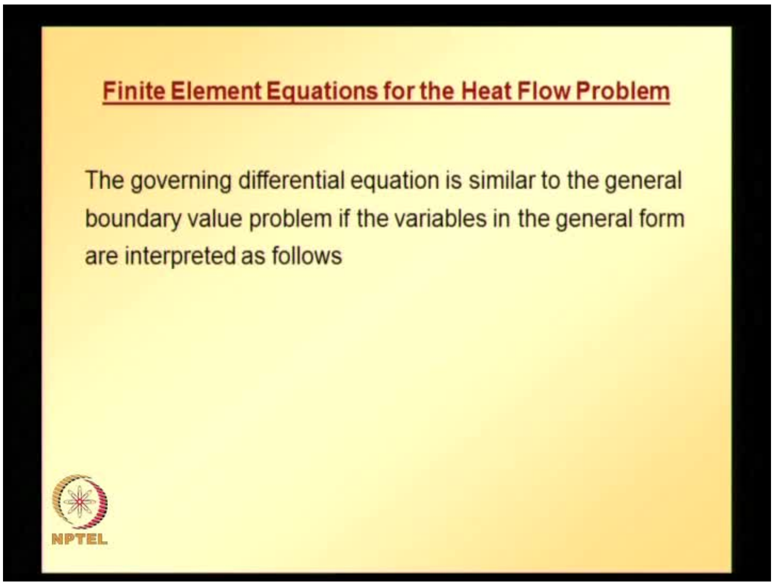

So, the finite element equation for heat flow problem are obtained by first looking at the similarities between the governing differential equation and the one-dimensional general boundary value problem that we already looked at.

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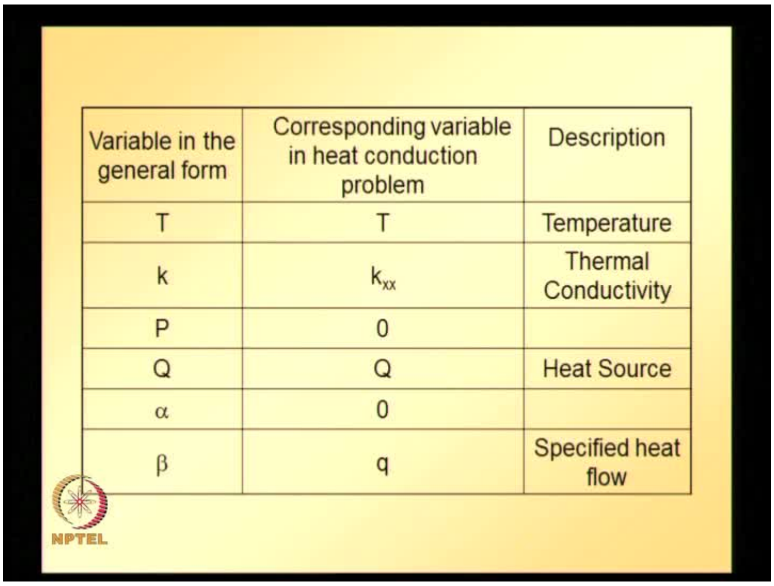

Variable in the general form	Corresponding variable in heat conduction problem	Description
T	T	Temperature
k	k_{xx}	Thermal Conductivity
P	0	
Q	Q	Heat Source
α	0	
β	q	Specified heat flow

We can make a table like this, what is the variable in general one-dimensional boundary value problem and corresponding variable in the heat conduction problem and what the corresponding variable in the heat conduction problems actually stands for we can make a table like this.

So, this table helps us, we already have the element equations, assuming linear finite element and a linear finite element, we have the element equations for general one-dimensional boundary value problem. So with this table, we can actually directly get the substitute the corresponding coefficients values and get the element equations for heat conduction problem or heat flow problem.

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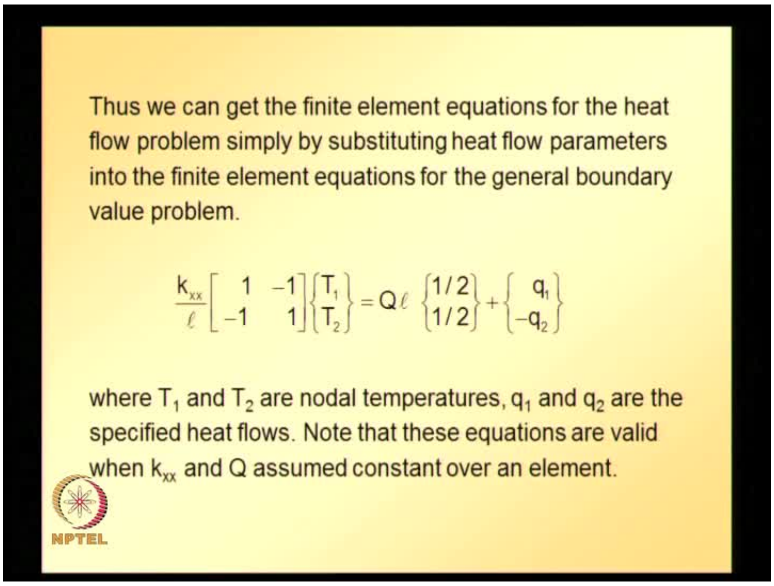

So, looking at the table, we can get finite element equation for heat flow problem simply by substituting, heat flow parameters into finite element equations for general boundary value problem.

So, doing that we get these element equations; so, for one linear finite element for this problem, these are the element equations. If you have more number of elements, for each of the element, we need to get equations based on this and then we can assemble the global equation system, apply the boundary conditions, and solve for the nodal unknowns.

So, in this equation - element equation T 1, T 2 are the nodal temperatures because now T in the general boundary value problem, which is field variable. Now is temperature so, T 1, T 2 are the nodal temperatures, Q 1, Q 2 are the specified heat flows or heat flux and note these equations are valid when k x x and Q are assumed constant over an element. Because that is what we assume, when we are actually deriving element equations for general boundary value problem.

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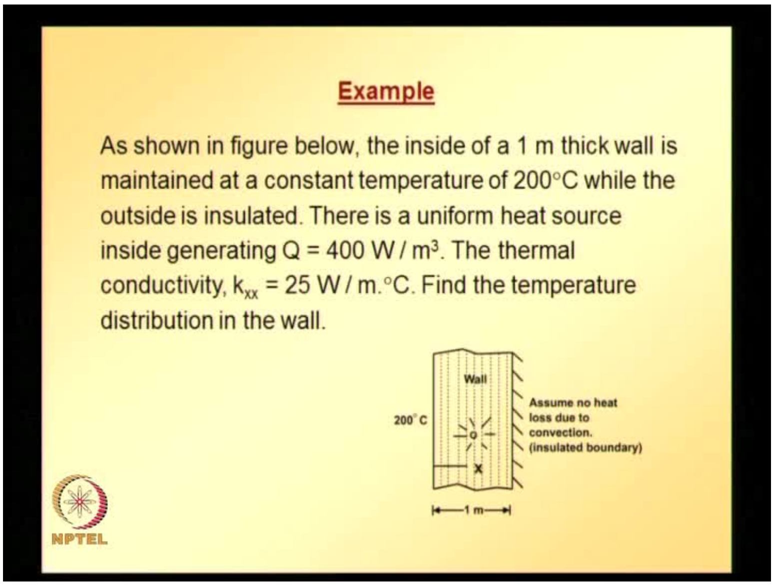

Now to get more understanding, let us take an example and solve it as shown in figure below (Refer Slide Time: 46:36). Figure is shown there, the inside of 1-meter thick wall is maintained at a constant temperature of 200 degrees centigrade, while outside is insulated. There is a uniform heat source inside generating heat which is given Q value and thermal conductivity k x x is given and we need to find temperature distribution in the wall. A schematic showing the wall and the inside temperature and outside insulation is shown, the thickness of wall is 1 meter.

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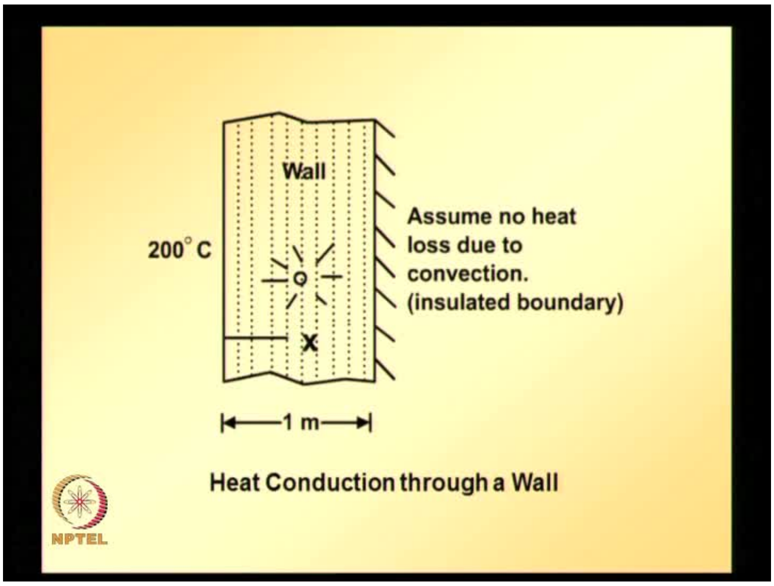

Assuming the wall is very high, it is possible to model a unit slice of wall as one-dimensional problem; so, this is the problem. So, assuming the wall is very high using a one-dimensional model, we can model unit slice of wall, no heat can flow from top and bottom surfaces, bottom phases of the model.

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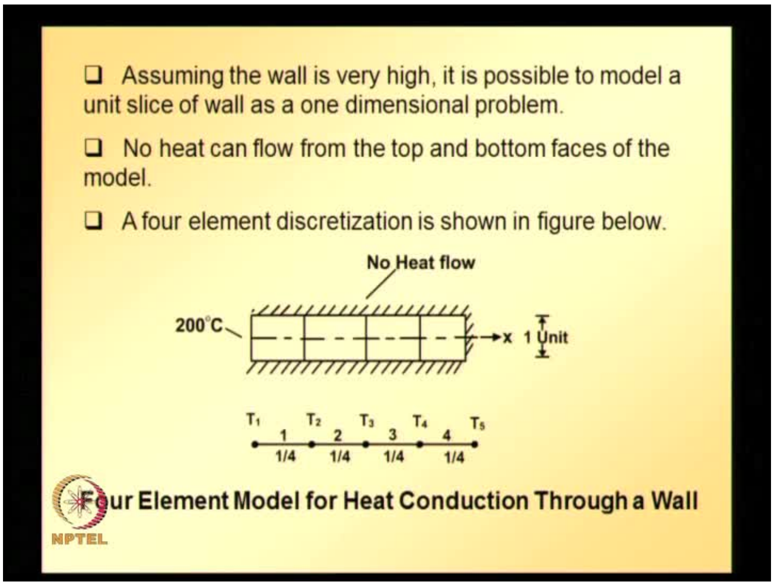

So, this is what the schematic shows the meaning of the statement, no heat can flow from the top and bottom phases of the model and the entire wall thickness is discretized using four elements.

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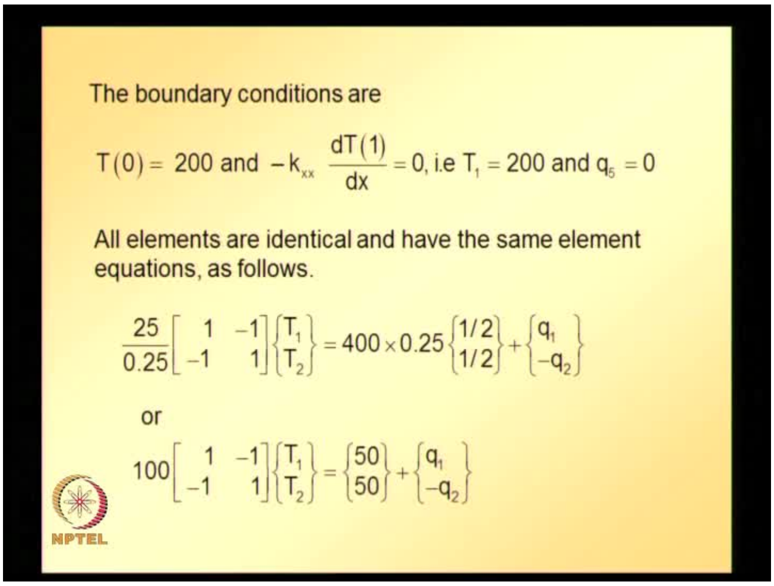

So, each element - four elements of equal length so, each element - length of each element is one fourth meter or 0.25 meters and four elements are taken, each element has two nodes. The values of temperature at each of these nodes is also indicated in the figure T 1, T 2, T 3, T 4, T 5 and now the boundary conditions are T at x is equal to 0 is 200 and because of insulation heat flow is 0 and x is equal to 1.

Here x is equal to 0 corresponds to node 1, x is equal to 1 meter corresponds to node 5. So, T 1 is equal to 200 Q 5 is equal to 0; so, substituting all these things into element equations. Since all the elements are of equal length, all elements are identical and have same element equations which are shown there, which can be simplified.

So, these are the element equations for one element and all elements will have similar kind of or same identical equations.

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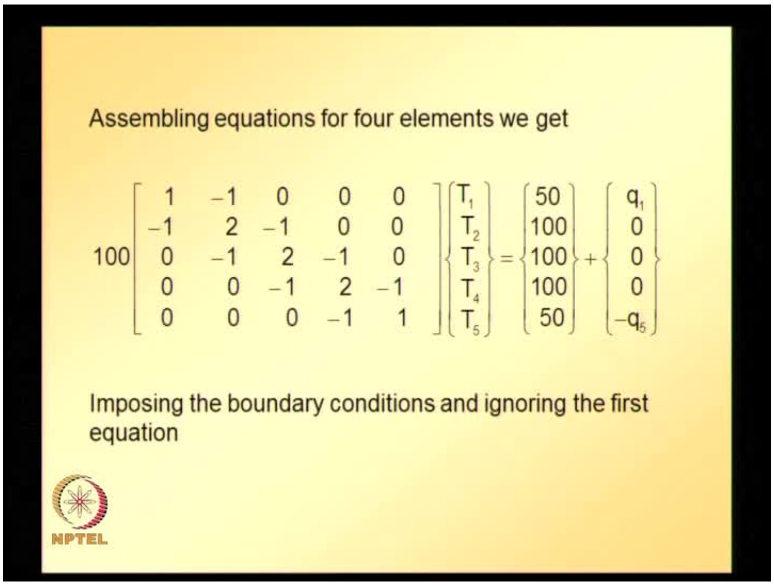

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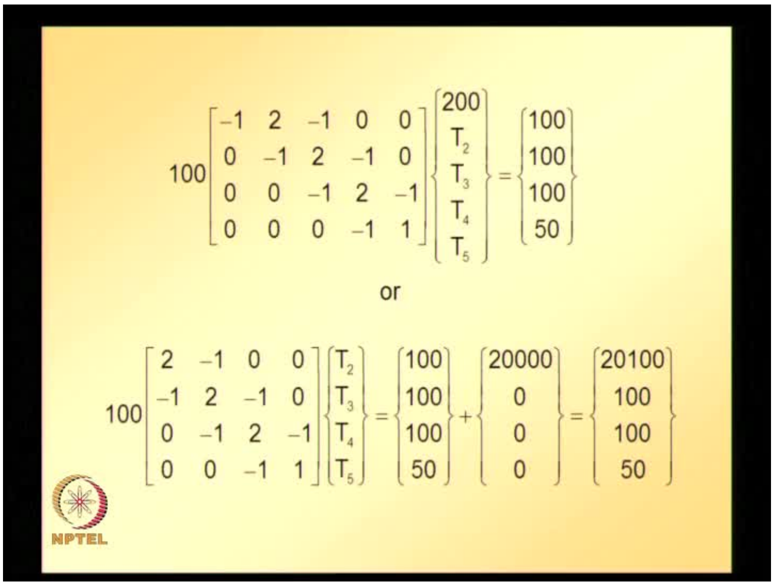

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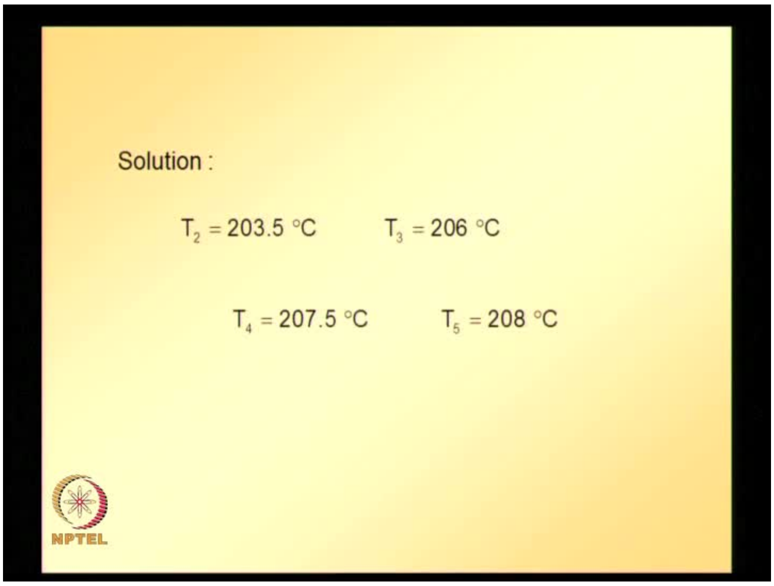

So, assembling using these one element equations, we can assemble equations for all the four elements; assembling equations for four elements, we get this one (Refer Slide Time: 50:50). Now making substitution that, T 1 is equal to 200 and Q 5 is equal to 0 and doing partitioning of matrices and rearranging, imposing the boundary conditions and ignoring the first equation, we get this and rearranging this equation, we can solve for T 2, T 3, T 4 and T 5.

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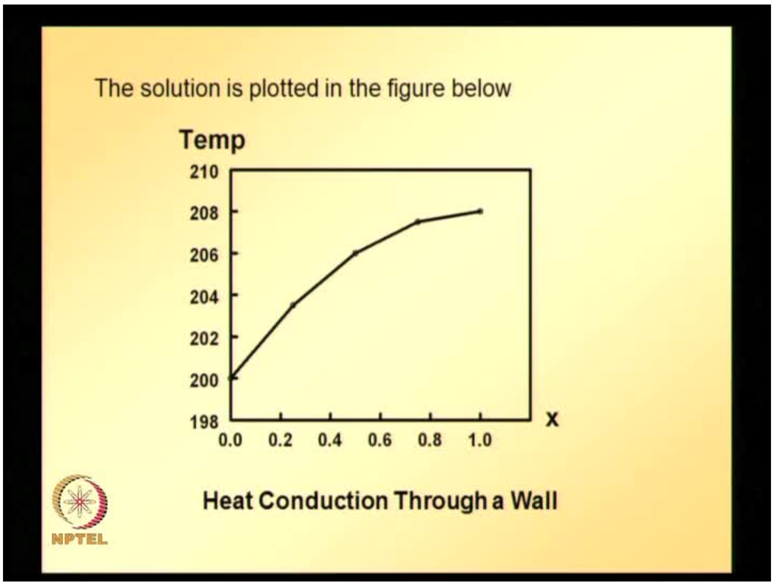

The solution of this equation system is this T 2 is equal to 203.5 degree centigrade, T 3 is equal to 206, T 4 is equal to 207.5, T 5 is equal to 208, and T 1 is already given which is the boundary condition. Essential boundary condition given that is P 1 is equal to 200, now we can plot the solution. Solution is plotted in the figure below, so this gives us heat conduction through the thickness of wall.

So, this is how the element equations that we developed for general one-dimensional boundary value problem can be used for solving some of the problems, this is one application and in the next class, we will be looking at some more applications.