Finite Element Analysis

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Module No. # 01

Lecture No. # 15

In the last class we were looking at this 3-D space frames; let me summarize what we have done: we have looked at the element equations - the derivation of element equations; also, in the last class we looked at the transformation matrix from local to global coordinate system, which is required for transforming these element equations which are in the local coordinate system to the global coordinate system.

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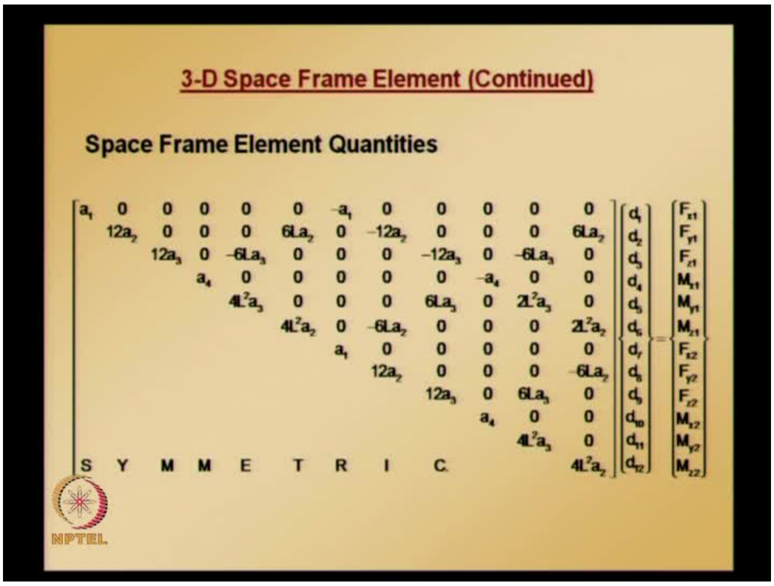

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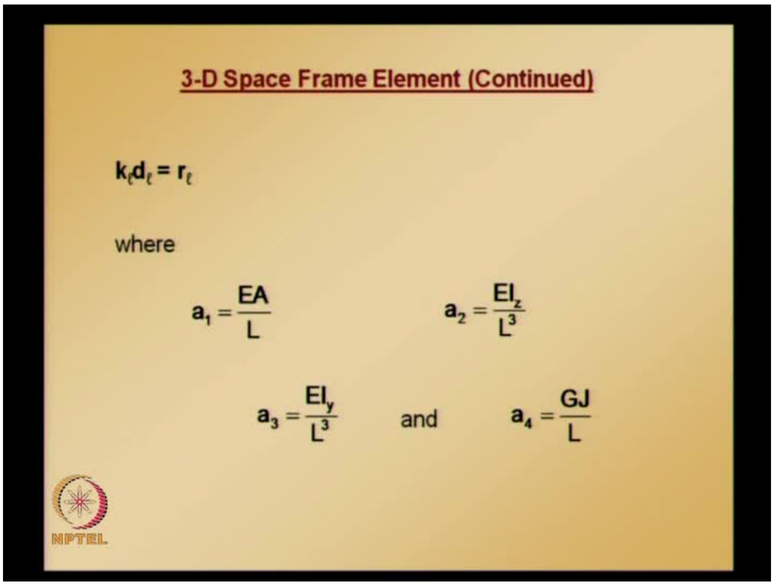

Let us briefly look at those things once again. These are the element equations which are based on axial force effects, bending effect in xy plane, bending effect in xz plane and also torsional effect; all these effects put together and assuming all these effects are uncoupled and are small deformation theory assumption, we obtained this element equations; here, these coefficients a 1 a 2 a 3 a 4 are defined like this, the previous equation element equation can be compactly written as k L d L is equal to r l.

Since this is element equations system in local coordinate for particular element, using this equation we can assemble element equations for all elements, once we know the material properties and also geometrical properties. Once we have all the element equations in the local coordinates - in the respective local coordinates - we need to convert them into the global coordinate system, before we assemble the global equation system for solving the nodal values.

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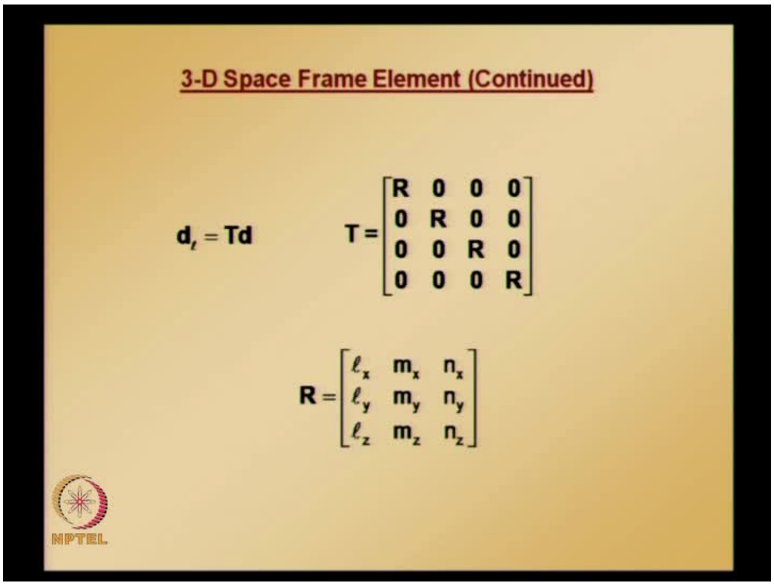

This is the transformation for the displacement vector in the local coordinate system, with the displacement vector in the global coordinate system, where R here is the rotation matrix - 3 by 3 rotation matrix - and 0 is also 3 by 3 null vector. So, transformation matrix is going to be 12 by 12 matrix and this rotation matrix comprises of all the direction cosines - L x m x n x L y m y n y L z m z n z.

In the last class, we have looked at two kinds of methods to calculate these components of direction - components of this rotation vector - which are the direction cosines. The two methods that we looked in the last class are - orientation angle method and under orientation angle method we looked at a special case in which a local y-axis lies in the same plane as global xy; also, we derived for a general case of arbitrarily oriented frame element using orientation angle method.

But, the problem with this orientation angle method is calculation of orientation angle is not easy in many practical situations; so, that is why we have the other alternate method, which is the third node method. Once again this orientation angle method - the data that is required is for the case of arbitrarily oriented frame element, the data that is required is angle alpha, which is measured in the counter clock wise direction from global y axis to global x axis and that is an angle by which the element needs to be rotated about local x axis, such that, local y axis is parallel to the global y axis.

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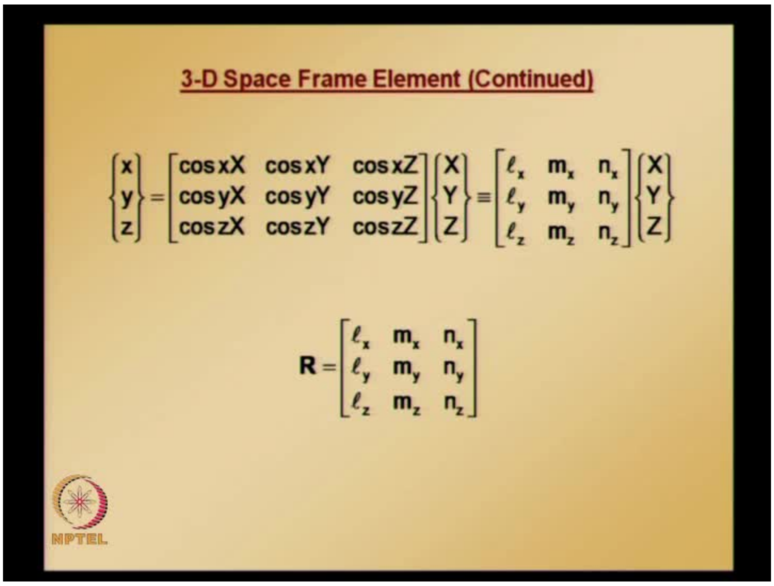

We require that in addition to the coordinates of the n points of the space frame element; so, this is the data that is required for orientation angle method. Now, let us briefly look at that method - this rotation matrix consists of direction cosines, these are coming from the coordinate transformation from the local x y z to global x y z consisting of all direction cosines.

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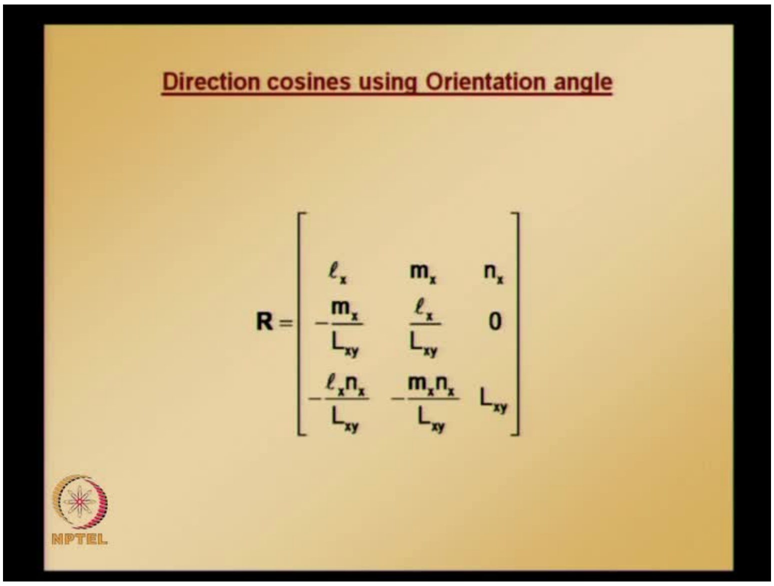

This is r matrix and this is the formula for obtaining direction cosines using orientation angle method. For the case in which a local x axis lies in the same plane as global xy, and here you can see L x y is in the denominator; by definition, L x y, if you recall, it is square root of L x square plus m x square.

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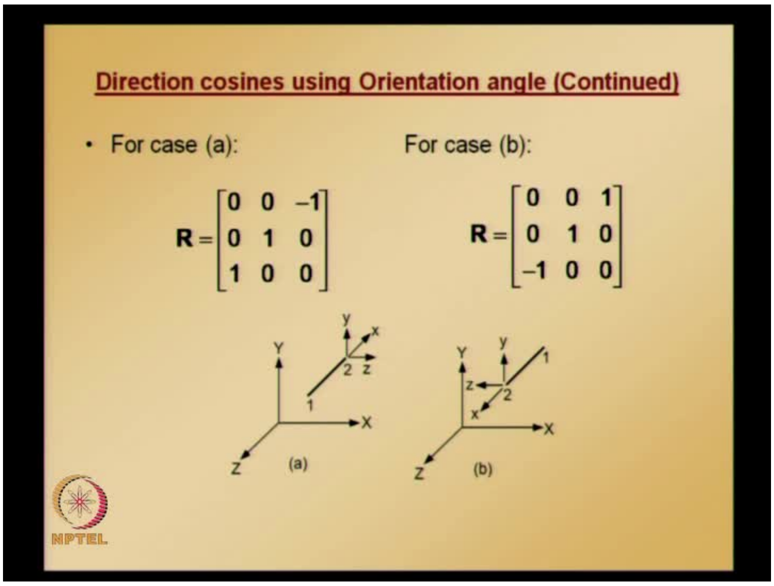

In case, L x y is 0, then will have problem in calculating; so, in that case we need to look at the element - space frame element - and by inspection we can write the rotation matrix components, in case L x y is equal to 0. L x y is going to be 0 in case the frame element - space frame element - is oriented or is parallel to the global z axis; so, for these cases - rotate the components of rotation matrix can be obtained by inspection; case (a) is shown there in the figure.

You can easily see by inspection, what is cosine of angle between small x axis capital X axis small x axis capital cosine of angle between small x axis and capital Y axis; similarly, other components - that is cosine of angle between small x axis and capital z axis; so, L x m x n x similarly you can find L y m y n y L z m z n z.

For the element - space frame element - which is shown in figure a, the element is parallel to global z axis; in that case, we can easily check the rotational matrix by inspection - is going to be whatever is given in the slide. And for figure b if the element - space frame element - is oriented in the manner which is shown in figure b; the difference between figure a and figure b is the way the nodes are numbered. In figure a whatever is numbered as node 1 is 2 in figure b and whatever is numbered as 2 in figure a is node 1 n figure b; because of that the orientation of local x axis y axis z axis are different, so even the rotation matrix is going to be different.

For case b, rotational matrix again by inspection and just observing what is going to be the cosine of angle between the local axis and global axis, we obtained this one; so, using these we can calculate the direction cosines using orientation angle.

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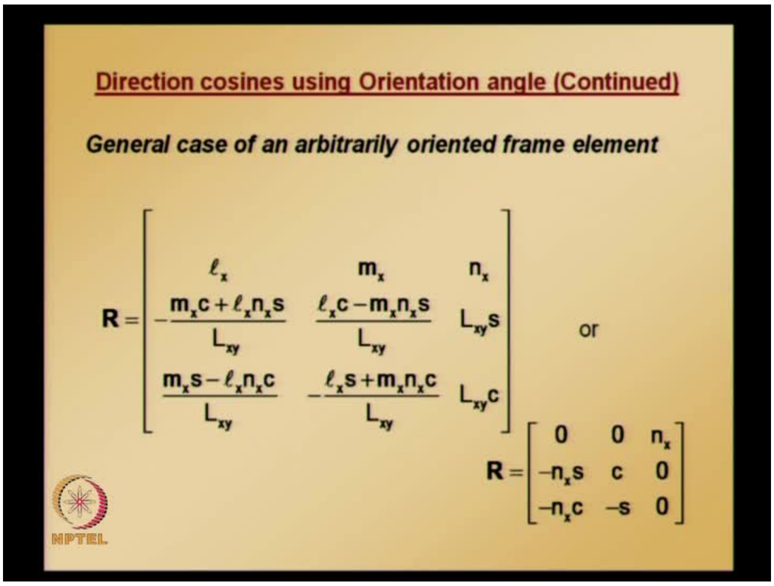

This what we saw in the last class; the last case is general case of arbitrarily oriented frame element; in this case, first thing what we need to do is we need to rotate the given frame element about local x axis such that local y axis is parallel to the global y axis.

In this case, of an arbitrarily oriented frame element, rotation matrix is given by this one. Again, in the special case where L x y is 0 we get rotation matrix by this one. It can be easily a checked that in the limit L x y tends to 0 the components which are there in the first rotation matrix reduces to what is given in the second rotation matrix. In these matrices you have c and s c is cosine of alpha s is sine of alpha, alpha is nothing but angle between global y axis and local y axis measured in the counter clock wise direction or the rotation by which we need to rotate about x axis to make the global or local y axis parallel to the global y axis; cosine of that and sine of that is what is c and s respectively.

So, using this rotation matrix formula we can obtain, for any arbitrarily oriented frame element, the rotation matrix.

This is by rotation angle method, and as I mentioned calculation of rotation angle is not easy in many practical situations; so, the other alternate method is third node method in which actually - third node is - we need to define a third node; because, this third node is going to be used to define either local x y or x z plane, and whatever formula we have seen in the last class it is based on third node being used to define local x z plane.

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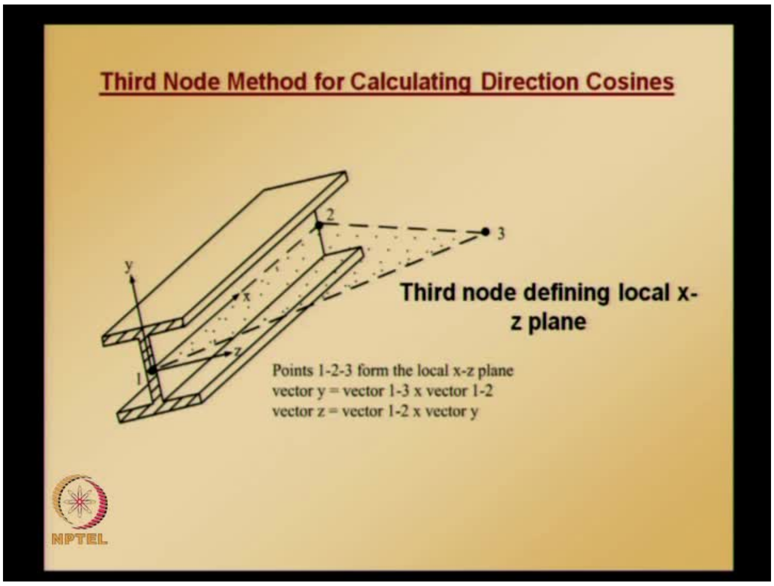

Schematically, it can be explained using this figure; here, an arbitrarily oriented space frame element is shown in which local x axis is in such a way that it is a vector going from node 1 to node 2 and local z axis and local y axis are also shown; in addition, a third node is also shown in the figure to define x z plane.

Once we know the vector 1 2 and 1 3, cross product of vector 1 3 and 1 2 gives us local y axis or vector along local y axis; when we normalize it we get unit vector along local y axis. Once we know the vector along y axis and using the nodal coordinates of one and two nodes we get local x vector along local x axis, and cross product of vector along local x axis times local y axis gives us a vector along z axis, which we can normalize to get unit vector along z axis; once we have the vectors - unit vectors - along local x axis, local y axis and local z axis we can easily write what this rotation matrix is, because the components of this vector are going to be the components of the rotation matrix.

This is about the third node method; what additional information we require? We require a third node to define - either plane xy or xz, but we usually take one of the existing nodes in the model as third node, which will be clearer to you when we solve a problem.

So, using this or using any of these two techniques - that is, either rotation angle method or third node method we can calculate the rotation matrix; and, once we get the rotation matrix the element equations in the local coordinate system can be transformed into the element equations in the global coordinate system.

So, the assembly and solution procedure for nodal degrees of freedom - the procedure is standard procedure, which we are following till now; after nodal solution is obtained the displacements, axial forces, moments and shear at any point along the 3-D frame element can be computed using shape functions and its derivatives as we are doing for the plane frame or trusses or beams that we have seen till now.

For each element, global displacements can be transformed into the local displacements; because, when we want to calculate these axial forces, moments and shear whatever when we solve the global equation system we get global displacements in the global coordinate system we need to convert back into the displacements in the local coordinate system; because, using this we will be calculating axial forces, moments and shears or even displacements at any point along the frame element.

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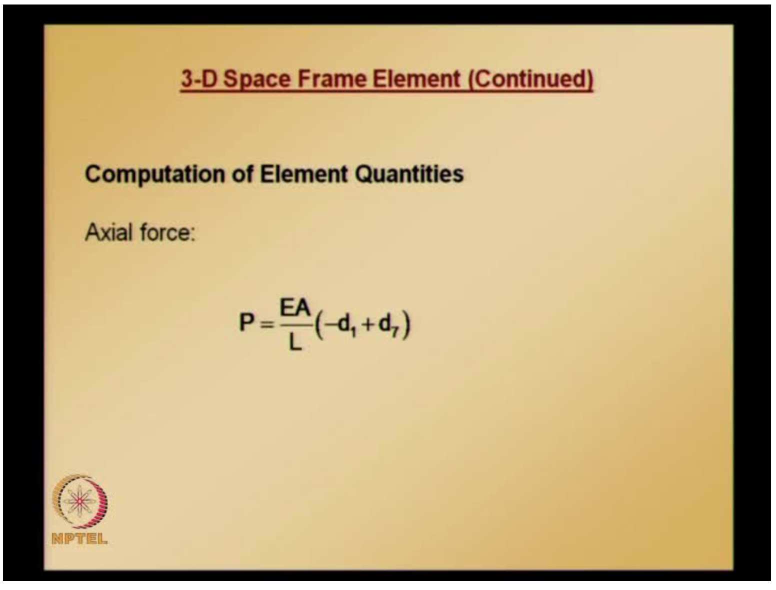

For an element with a uniform load - q y q z, q y in the xy plane q z in the xz plane; the equations in terms of local coordinates, that is, s is equal to 0 corresponds to node 1 s is equal to 1 corresponds to node 2, in that case, we can calculate; once we get the local displacements or displacements in the local coordinates system for that particular element we can calculate axial force using this formula.

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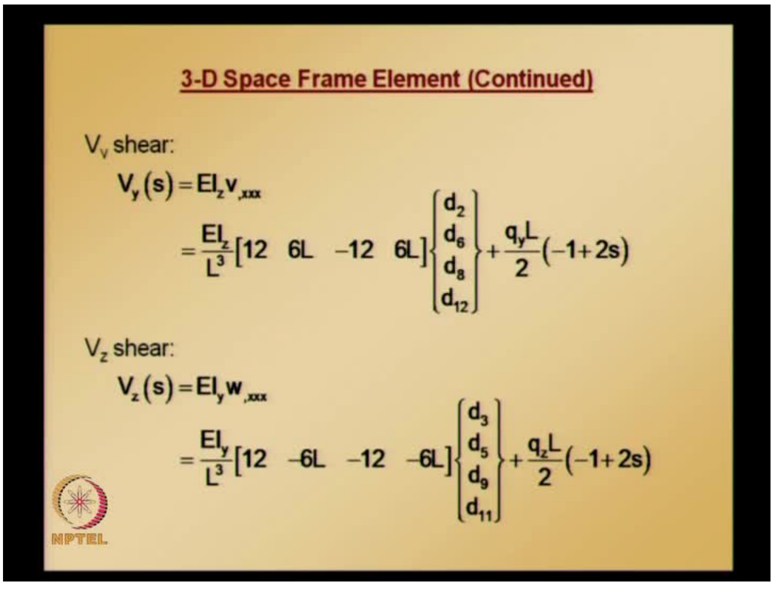

Shear - v y capital V is used to denote shear as a function of s again s is equal to 0 corresponds to node 1 s is equal to 1 corresponds to node 2 is given by this; and, you can see the second part in in this equation it is correction term for fixed end solution; here, the correction term that is I have shown in the slide is corresponding to uniformly distributed load case.

If it is some other load case we need to use the corresponding fixed end solution for shear, and shear v z is given by this one; again, here we have fixed end solution correction for uniformly distributed lode case.

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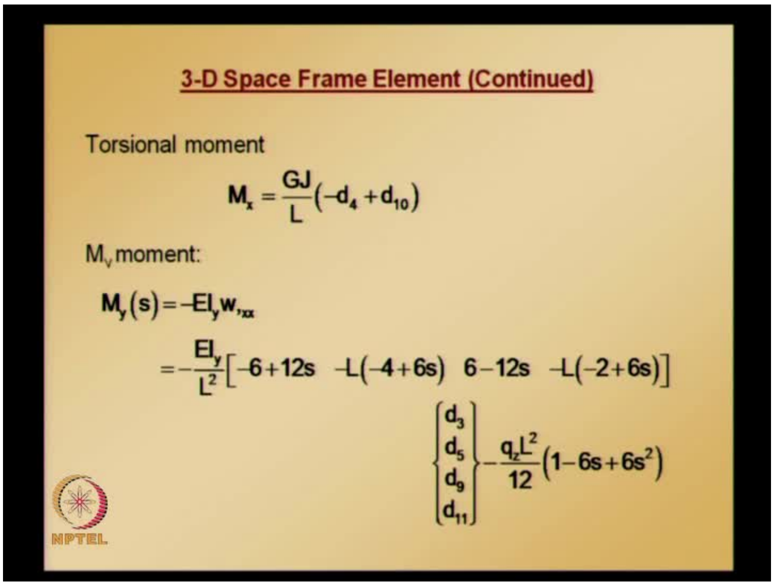

Using these formulas we can calculate axial force components of shear and torsional moment; here, in the formulas we have the nodal values d 1 to d 12 and these have the same meaning as what we have seen in the last class - they comprise of all the 3 translation degrees of freedom at node 1; 3 rotational degrees of freedom at node 1; again 3 translational degrees of freedom at node 2; 3 rotational degrees of freedom at node 2.

And moments - m y is obtained using this formula; please note, that for axial effects and torsional effects no correction is required - fixed end solution; because, we are actually writing formulas for the case of uniformly distributed load.

If it is a problem in which temperature change is involved, in that case, correction is required for axial force effects; but, here we are writing for uniformly distributed place, so you do not have correction for axial force effect and torsional axial force and torsional moment, but we need to apply correction fixed end solution correction for shears and moments.

So, moment m y is given by this - the second part of this equation is fixed end solution correction coming from the case for uniformly distributed load; moment m z is given by this (Refer Slide Time: 20:20).

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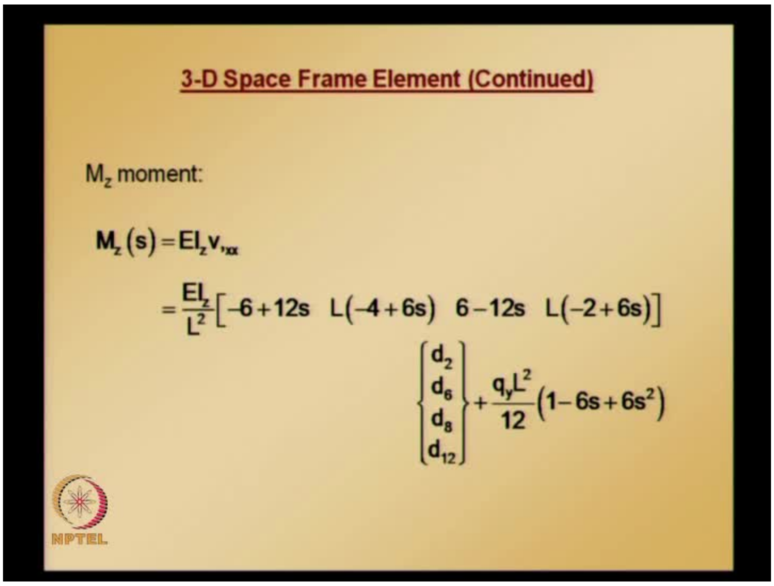

So, what we need to do is - given any problem we need to assemble element equations in the local coordinate system and use transformation method and get the element equations in the global coordinate system based on the nodal connectivity; assemble the global equation system, apply the displacement boundary conditions and solve for the nodal values; once we get the nodal values we need to again find the corresponding displacements and rotations in the local coordinate system before we calculate axial force or shear moment and torsional moment.

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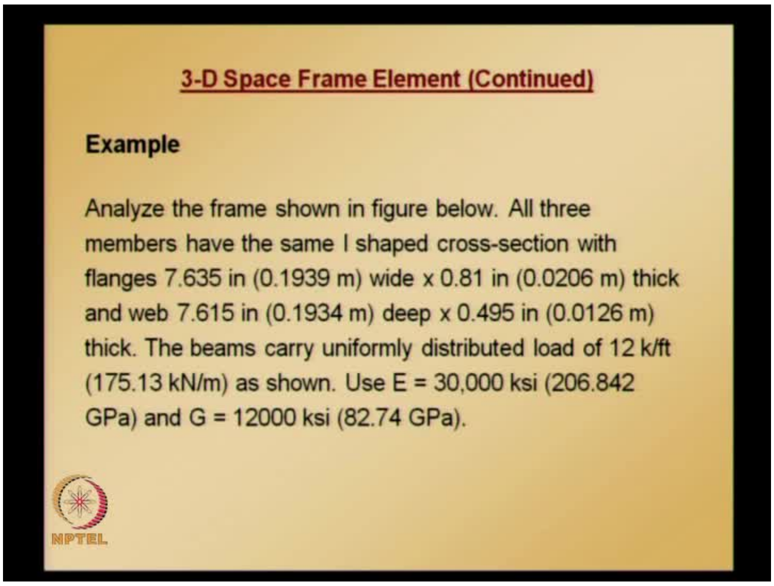

This is the procedure; now, let us take a numerical example: analyze the frame shown in figure - figure will be shown in the next slide - all members have same I shaped cross section and the flange dimensions web dimensions and depth are given both in f p s units and s i units and beam carries uniformly distributed load which is shown in the figure and material properties are also given.

Here, since the problem is given both in f p s units and s i units we can work in any of the units, but here the workout is shown for f p s units case because f p s units are round numbers shear.

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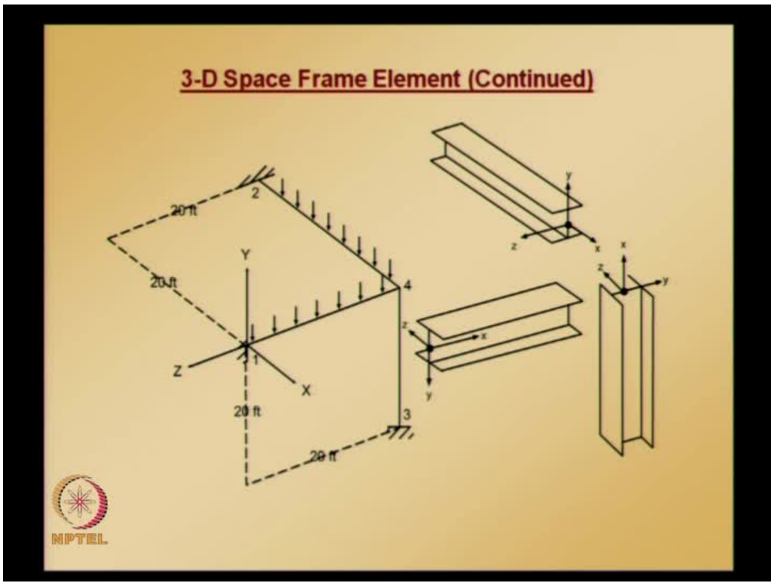

This is the frame we need to analyze - three-dimensional space frame. If you see here, all the dimensions are given, length of these members are given and also global coordinate system is indicated in the figure; also, for each of the members local coordinate system is indicated; also, nodes are numbered. Please note, that in this 3-D space frame problem for each element the equation size is going to be 12 by 12 and here we have 3 members, so at each node you have total 6 degrees of freedom and totally 4 nodes are there; so 4 times 6, 24 by 24 global equation system we are going to get. In case, we are going to assemble entire global equation system, but instead of assembling the entire global equation system since here nodes 1 2 3 are restrained all the degrees of freedom are 0.

Anyway, in the global equation system which is going to be 24 by 24 we are going to eliminate rows and columns corresponding to the degrees of freedom of these nodes 1 2 3; so, we are going to eliminate in total eighteen rows and columns finally we are going to get 6 by 6 reduced equation system.

Instead of assembling the entire 24 by 24 global equation system and reducing we can smartly assemble the reduced equation system directly by noting down the contribution from each of the elements 1 2 3 to the locations corresponding to the degrees of freedom of node 4; because, node 4 is the one for which we need to calculate all the degrees of freedom - 3 translational degrees of freedom and 3 rotational degrees of freedom at node 4.

We will try to just assemble the matrix, even the element matrices in such a way that we carefully make sure that we have only the contributions which go into the global equation system to calculate the corresponding degree of freedom of node 4.

If you see this figure - in the figure for each of the elements - even the local coordinate system is given and since global coordinate system is defined we can easily find what are the coordinates of nodes 1 2 3 4, and since the local coordinates system is also shown for each of the members we can easily find what is the angle between local y axis and global axis - global y axis - what is the angle between local y axis and global y axis for each of the members.

Because, if you want to use orientation angle method to calculate the rotation matrix you require this angle, which is angle between local y axis and global y axis measured in the counter clockwise direction.

If you see for - let us say, element 1 is the one which is connecting nodes 1 and 4 and if you see for that element the angle between local y axis and global y axis is - you can check - it is 180 degrees or pi radiance or 180 degrees.

Using orientation angle method we can easily find what the rotation matrix is; once we know this angle between local y axis and global y axis for this particular element and also once we know the coordinates of nodes 1 and 4 - similarly orientation angle alpha for element 2 which is connecting nodes 2 and 4 it is, you can easily check, the angle between global y axis and local y axis it is equal to 0.

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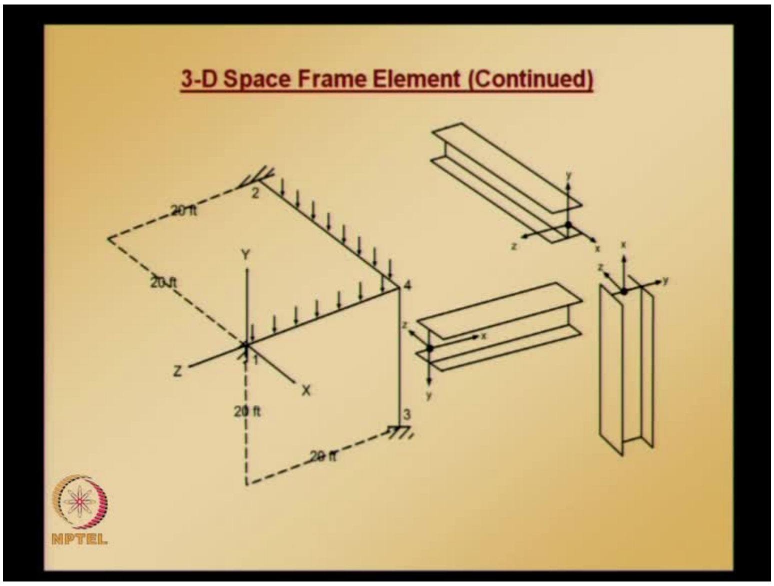

Using this information we can find - and once we know the coordinates of nodes 2 and 4 you can easily find what is this rotation matrix; similarly, for node 3 - so for element 3 which is connecting nodes 3 and 4, the orientation angle is 1.5 times pi, that is, 3 pi by 2 radiance or 270 degrees.

Using this information we can easily write what is the rotation matrix, and from there we can get the transformation matrix for the local to global coordinate system; now, that is how you can use rotation angle method or we can use third node method. Here, for illustration purposes, for this particular problem, we will use third node method to calculate the rotation matrix.

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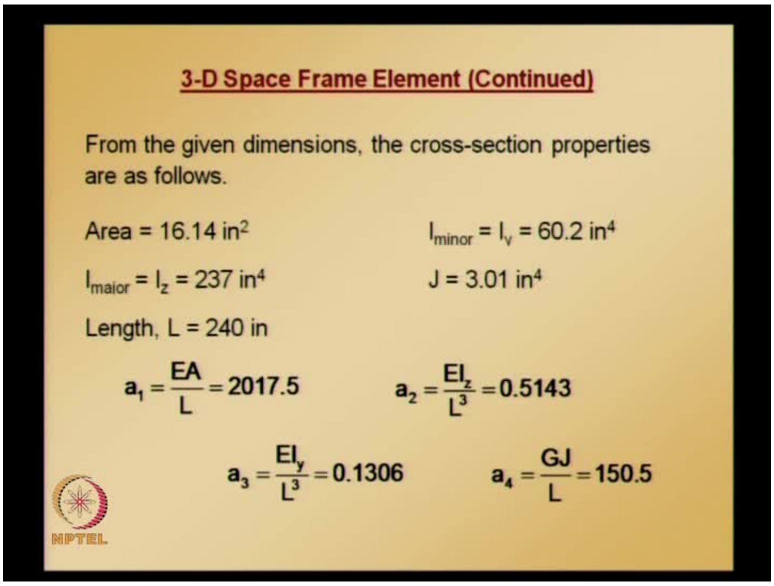

Now, let us proceed with this information and from the given dimensions and cross sectional properties - and also if you see each of these members y axis is minor axis and z axis is major axis.

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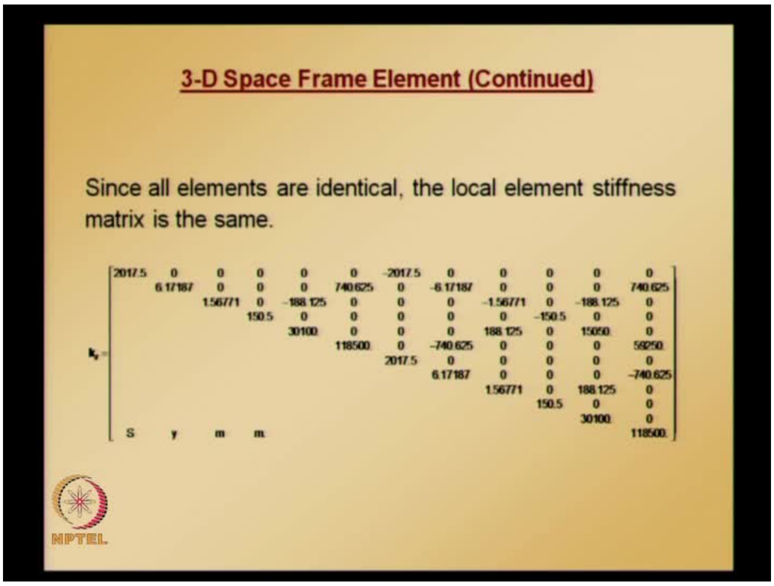

From the dimensions and cross section properties we can get these values that are given, and from this we can calculate what are the coefficients - a 1 a 2 a 3 a 4, for each of the members; but, in this particular problem all the members are identical and since all elements are identical local elements stiffness matrix for all elements is same, which is going to be 12 by 12; so, this matrix is obtained based on the orientation of the members in such a way that y axis is minor axis and z axis is major axis, and for all elements this is going to be the local stiffness matrix; because, all the elements are identical.

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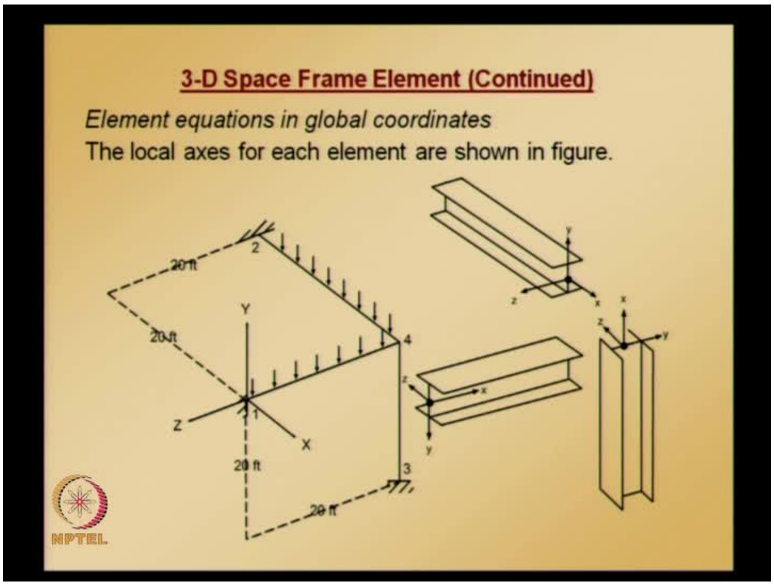

Now, element equations in global coordinates - we obtained element equation in local coordinate system; to get element equations in the global coordinate system we need to apply transformation method.

For that, as I mentioned, will be using third node method to get the rotation matrix. The local axis for each element is shown in the figure. You can see here that we need to identify third node - element 1, element 2 and element 3; third node is not something - is not going to be a point something which is outside the structure; it can be, but in this particular place let us make sure that the third node is defined based on a point which is there already existing in the model.

So, for element 1 which is connecting nodes 1 to 4 third node is going to be node 2; for element 2 which is connecting nodes 2 to 4 third node is going to be 1; for element 3 which is connecting nodes 3 to 4 third node is going to be 2; please note, that element 1 is oriented in such a way that its going from - its along a vector going from nodes 1 to 4 and element 2 is oriented such a way that its along vector which is going from node 2 to node 4 and element 3 is oriented in such a way that it is a long vector going from node 3 to node 4.

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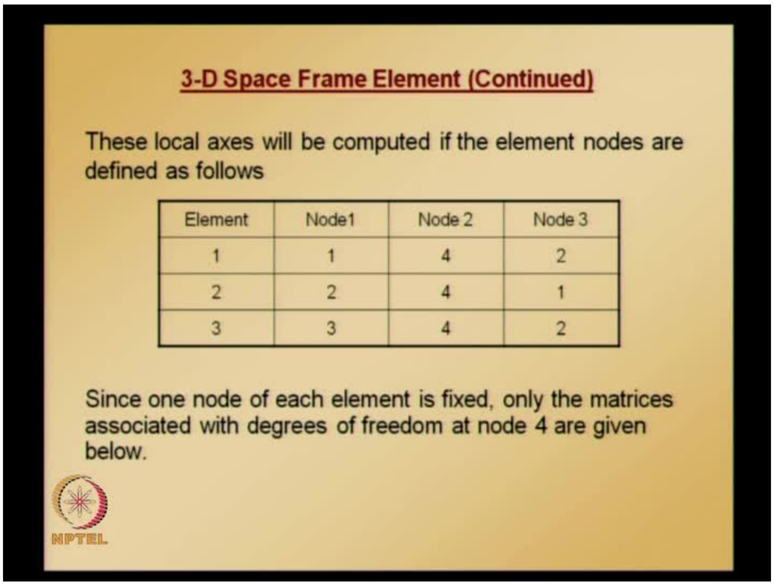

We just identified what are the points which are going to act as third node for each of these members; we can put this in a table - these local axes using third node method will be computed if element nodes are defined as follows: since one node of each elemen is fixed only matrices associate with degrees of freedom at 4 are given below; this is what I mentioned - since nodes 1 2 3 are fixed or restrained for all degrees of freedom we will be assembling the contribution which goes into the global equation system which corresponds to the degrees of freedom at node 4.

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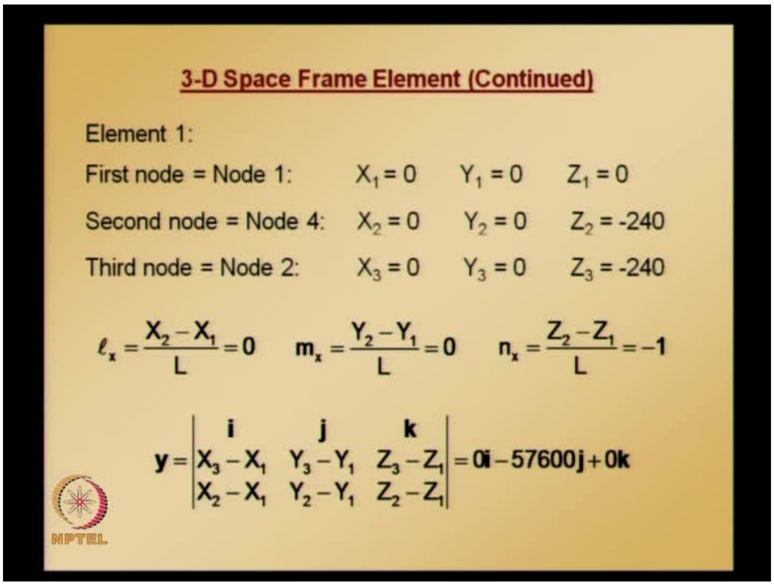

Now, we need to assemble the element equations for each of these elements, for that we need to get the rotation matrix; we will start element 1 node 1 the coordinate values; node 2 coordinate values; and, we identify what is node 3 for third node for element 1 that is node 2; the values of coordinates are given, so using this we can calculate L x m x n x.

Cross product of vector, which is joining nodes 1 and 2 - first node and third node, and cross product of that a vector which is joining first node and third node with a vector which is joining first node to second node gives us local y axis; and, the cross product is shown there in the slide.

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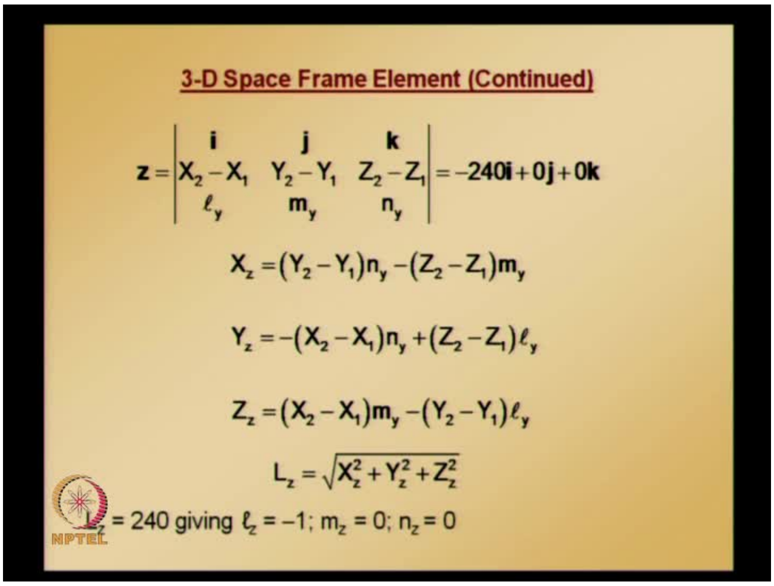

L y is equal to 0, m y is equal to minus 1 - this is the vector in the local y axis, and when you normalize it you are going to get 0 i minus 1 j plus 0 k so L y is going to be 0, m y is going to be 1, n y is going to be 0; and, we require another quantity L y, which is going to be square root of X Y square Y Y square Z Y square - where X Y, Y Y and Z Y are defined based on the coordinates of first node, second node and third node so L y if we calculate it turns out to be this and L y m y n y values are given there.

Now, we have got the local y axis - cross product of local x axis and local y axis gives us local z axis; so, that is what is cross product is shown here. Also, we require a quantity called L z, which is X z square plus Y z square plus Z z square that is also given; L z m z n z the vector z which is minus 240 i plus 0 j plus 0 k, if you normalize with the respect to the magnitude of this vector we get unit vector along local z axis, and the components of that, which is going to be l z which is going to be minus 1 m z is going to be 0 n z is going to be 0.

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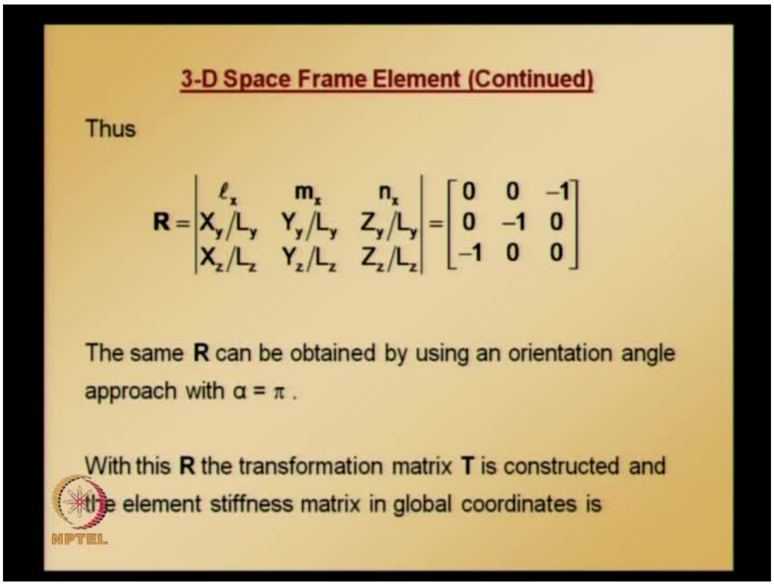

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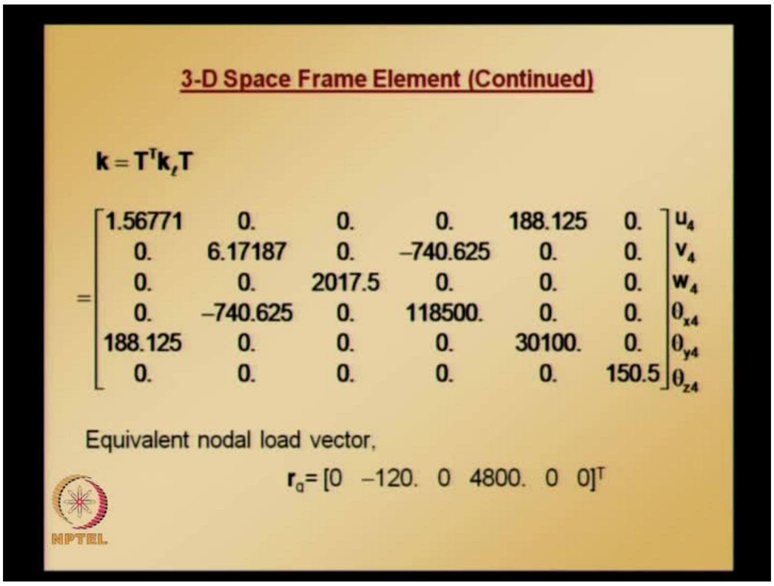

So, we got L x m x n x l y m y n y L z m z n z, so we are ready to write the rotation matrix corresponding to element 1. This rotation matrix - the same rotation matrix can also be obtained using orientation angle method noting that the angle between local Y axis and global Y axis for this particular element is pi degrees; the same R can be obtained using orientation angle approach with alpha is equal to pi radiance; with this R transformation matrix T is constructed and element stiffness matrix in global coordinates can be obtained by this formula: k is equal to T transpose k l T.

Where, the contribution to the global equations global element equations corresponding to the degrees of freedom at 4 - node 4 - are shown a beside this matrix u 4 v 4 w 4 theta x 4 theta y 4 theta z 4; please note, that here in this particular problem for element 1 which is along a vector going from nodes 1 to 4 the load, which is uniformly distributed load is applied in xy plane.

So, using the formula corresponding to the equivalent load vector for distributed load - applied in xy plane - which is q i q i L over 2 q i L square over 12 q i L over 2 minus q i L square over 12, transpose of that - that is going to be local load vector this load vector in the local coordinate system can be converted into the load vector in the global coordinate system using transformation method to obtain equivalent load vector in the global coordinate system.

r q in the global coordinate system is given by this one; this is equivalent load vector for element 1 in the global coordinate system; similarly, we need to assemble stiffness contribution to the global equation system from element 2 and element 3 and also for element 2 we have to assemble this equivalent load vector, because distributed load is applied on element 2.

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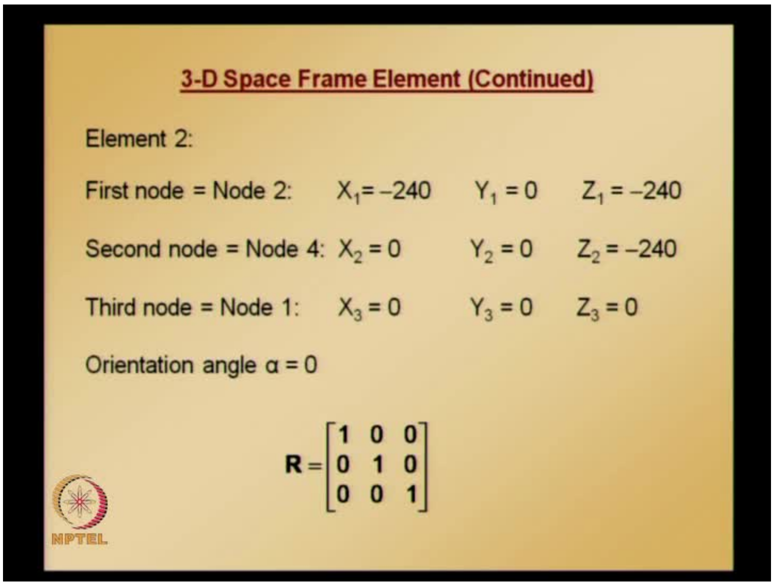

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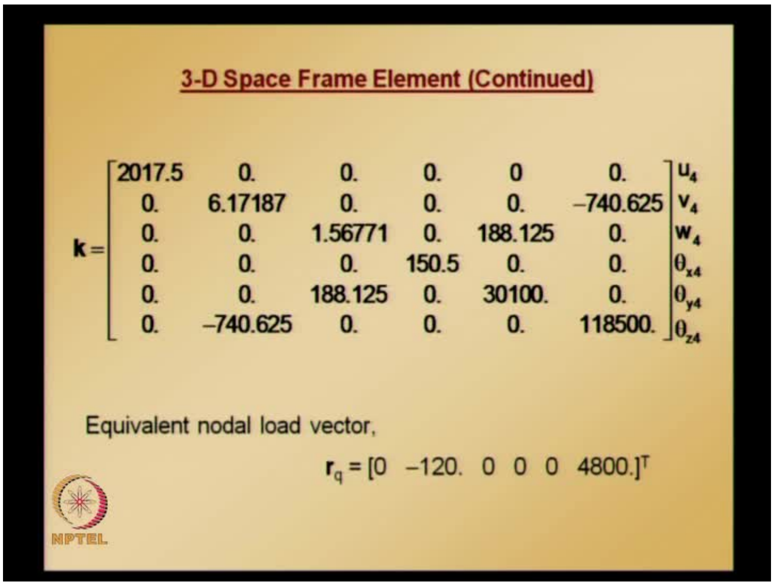

We need to repeat calculations for element 2 noting what is the first node, what is second node and what is third node for calculating the direction cosines using third node method; if we wish, we can obtain this rotation matrix using orientation angle method by noting that orientation angle between local Y axis and global Y axis is 0 for this particular element.

This is going to be the rotation matrix and contribution to the global stiffness matrix is - from this element - is this, and this is only a partial matrix because here we are noting only the contribution that goes in to the global equation system corresponding to the degrees of freedom for node 4 and equivalent nodal load vector.

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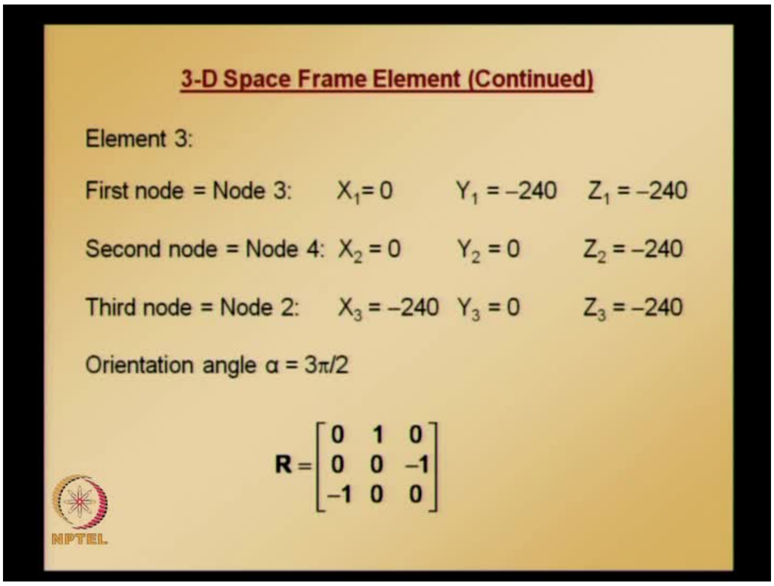

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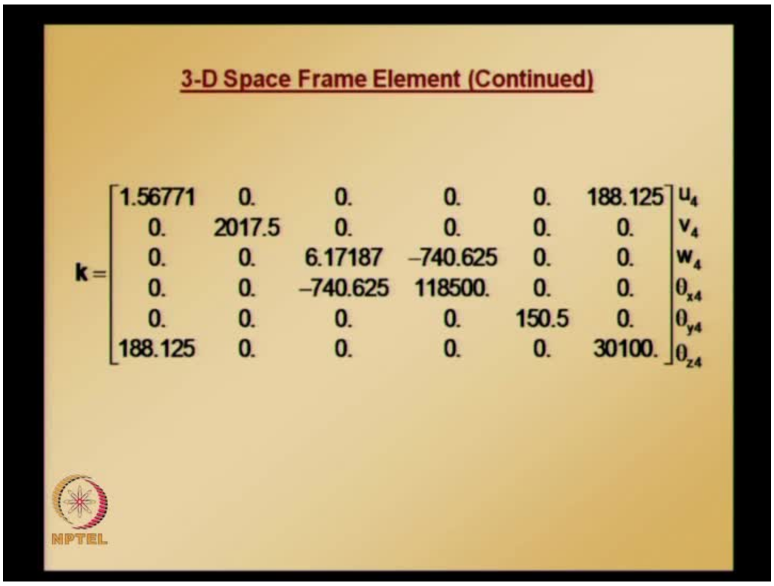

Similarly, we need to get for element 3 - first node, second node and third node; using this we can easily calculate what is rotation matrix, same thing we can also calculate using orientation angle method and stiffness contribution to the global equation system is this one.

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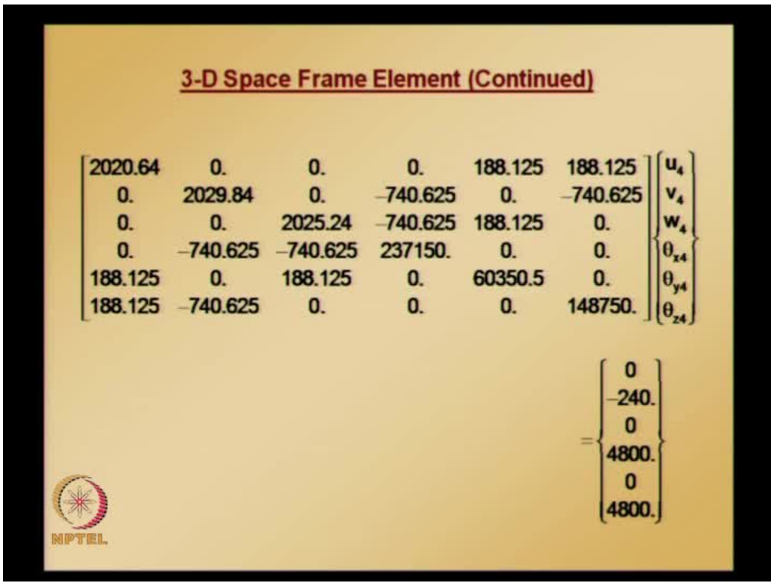

Summing up all the contributions - a global system of equations, after imposing boundary conditions, is a 6 by 6 matrix involving degrees of freedom corresponding to node 4; these equations can be obtained by simply adding the 6 by 6 element matrix associated with node 4 or local node 2 of each of these elements, thus global equations are obtained by this.

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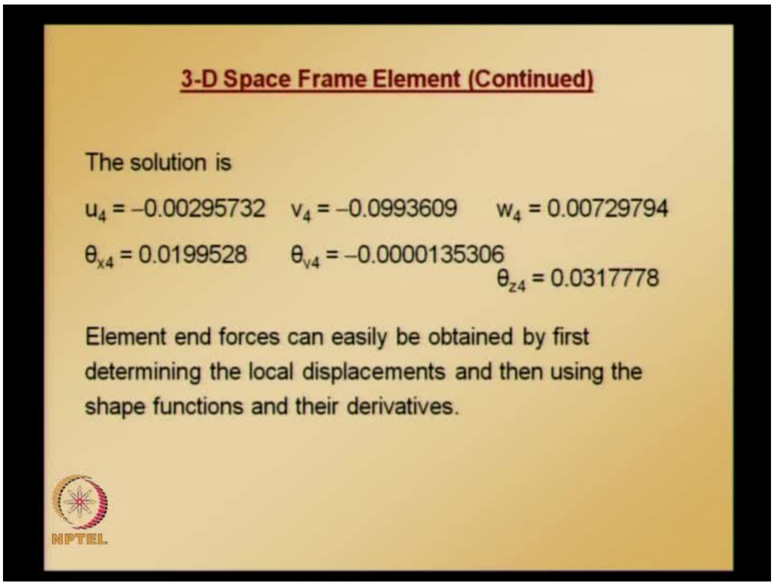

That is, whatever stiffness contribution we got from element 1 element 2 element 3, which is sum up we get this stiffness matrix which is shown in the slide. Force vector is also obtained by summing up equivalent load vector from element 1 and 2 and solving the 6 by 6 equation system we get the nodal solutions corresponding to node 4; the solution corresponding to node 4 u 4 v 4 w 4 theta x 4 theta y 4 theta z 4 are given there.

These are the values in the global coordinate system; element end forces can easily be obtained by - first determining local displacements, and then using shape functions and their derivatives.

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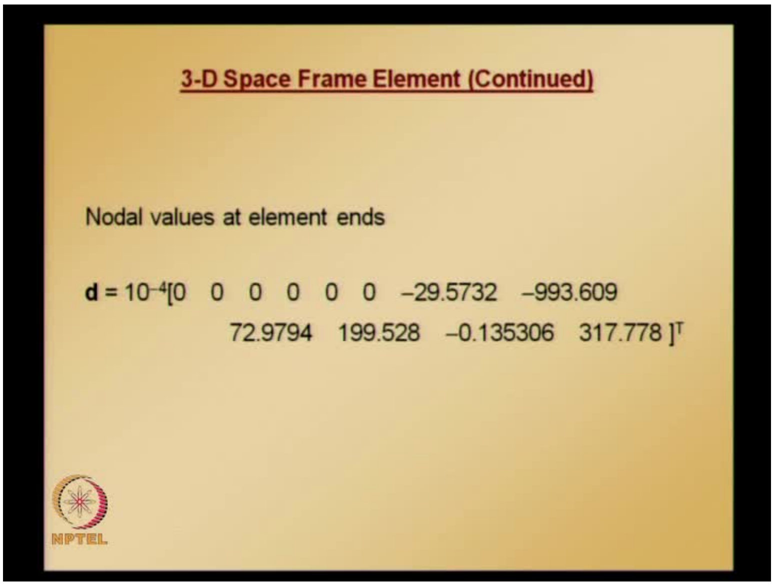

Whatever values given here are in the global coordinate system - so we need to calculate what the corresponding values are in the local coordinate system. Please note, that we numbered elements such or we numbered node in such a way that element 1 goes from node 1 to node 4, element 2 goes from node 2 to node 4, element 3 goes from node 3 to node 4; so, the nodal values for each of these elements in the local coordinate system turns out to be same nodal values at element ends using transformation matrix and the global displacements, we get nodal values at element ends.

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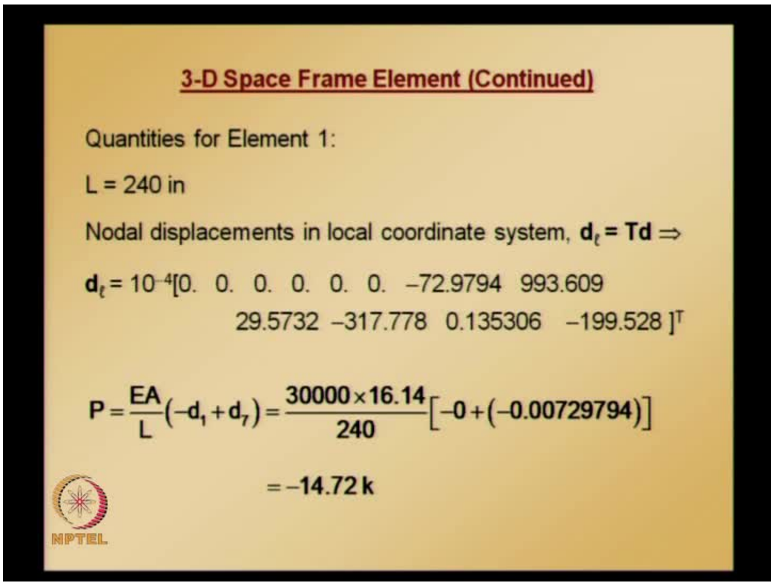

These are the nodal values in the global coordinate system for each of the elements 1 2 3; once we get this global displacement vector we can use transformation matrix to get the local displacement vector. The quantities for element 1 - length, using transformation matrix we can get nodal displacements in the local coordinate system and the displacements in the local coordinate system turns out to be this, and once we obtain the displacements in the local coordinate system we can calculate axial forces, shear, moment and torsional moment at both ends.

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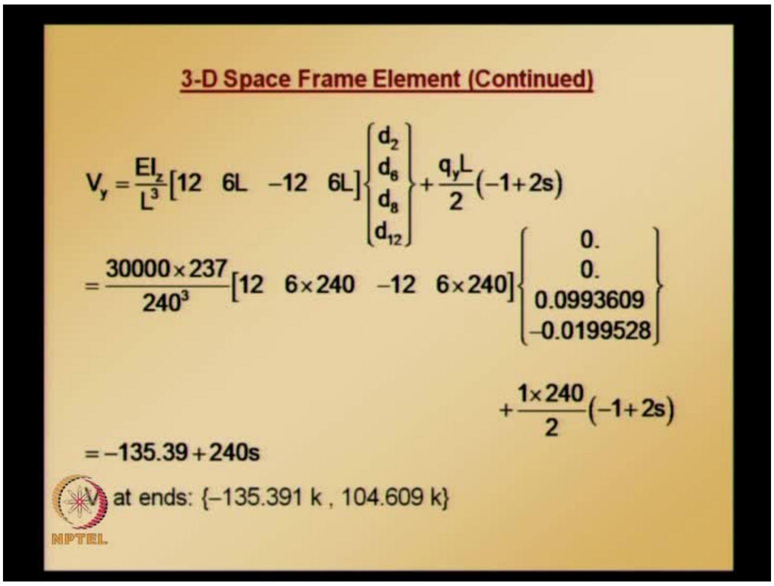

So, for element 1 axial force is obtained using this; all units are in f p s units and shear - we need to apply the fixed end correction corresponding to distributed load and this is going to be function of s.

Substituting s is equal to 0 we get shear at node 1 and substituting s is equal to 1 we get shear at node - local node 2 or node 4 in the global node numbering.

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3-D Space Frame Element (Continued)

$$V_z = \frac{EI_y}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\end{bmatrix}\begin{Bmatrix}d_3\\ d_5\\ d_9\\ d_{11}\end{Bmatrix} + \frac{q_z L}{2}(-1+2s)$$

$$= \frac{30000\times 60.2}{240^3}\begin{bmatrix}12 & 6\times 240 & -12 & 6\times 240\end{bmatrix}\begin{Bmatrix}0.\\ 0.\\ 0.00295732\\ 0.0000135306\end{Bmatrix}$$

$$= -0.0072$$

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3-D Space Frame Element (Continued)

$$= -\frac{30000\times 60.2}{240^2}\begin{bmatrix}-6+12s & -240(-4+6s) \\ 6-12s & -240(-2+6s)\end{bmatrix}\begin{Bmatrix}0\\ 0\\ 0.00295732\\ 0.0000135306\end{Bmatrix}$$

$$= -0.759981 + 1.7236s$$

M_y at ends: {−0.76 k-in, 0.96 k-in}

Similarly, the other shear component V z and its constan,t and moment m y with fixed end solution correction; simplification of that gives in terms of s, s is equal to 0 corresponds to node 1 s is equal to one corresponds to global node 4 m y value at these nodes 1 and 4.

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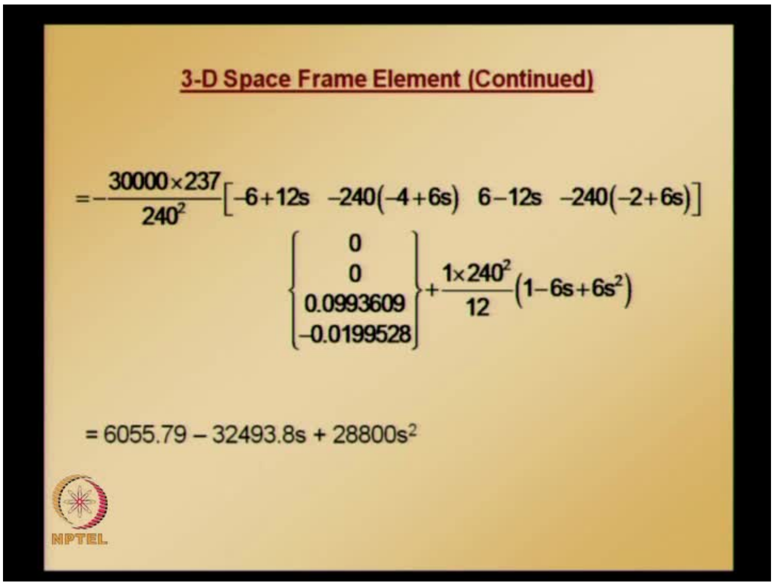

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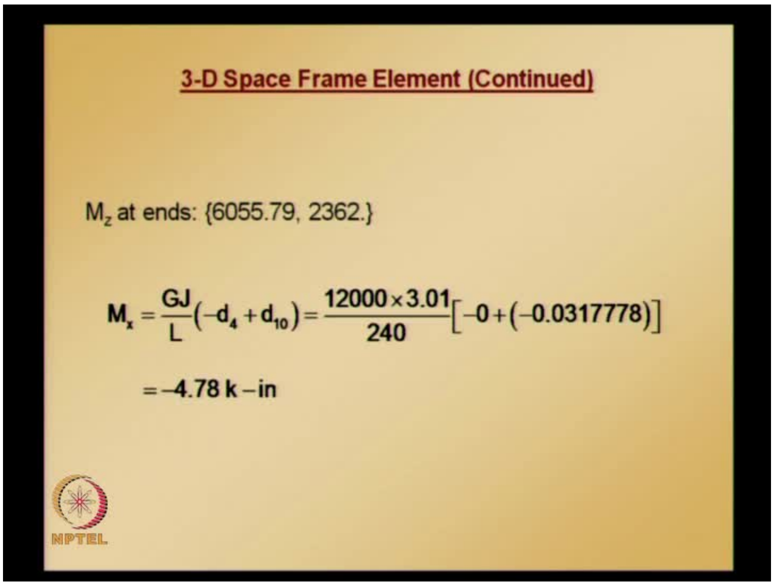

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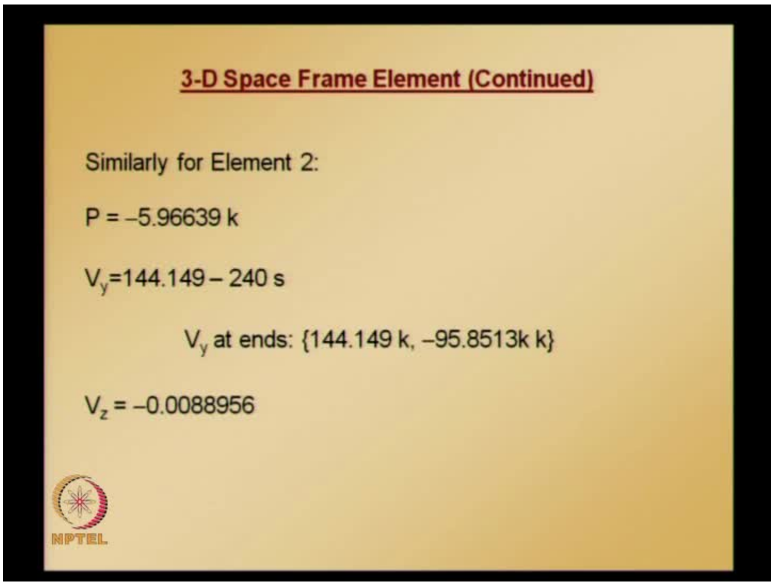

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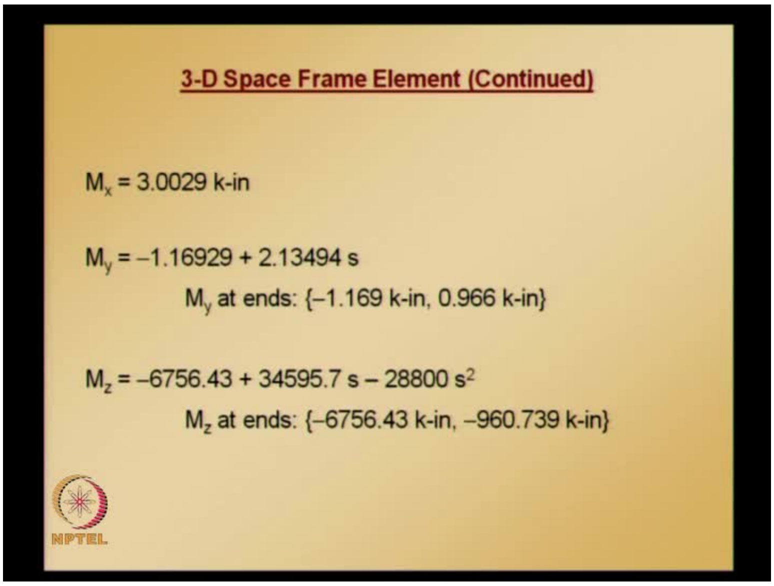

Similarly, m z component - simplification of that gives us m z component in terms of s so by substituting s is equal to 0 and s is equal to 1 we get m z values at nodes 1 and 4 and torsional moment m x, and these are the axial forces, shear moment and torsional moment for element 1; similar calculations can be repeated for element 2. Details are not given here - final values are given; this is a torsional moment M x M y M z.

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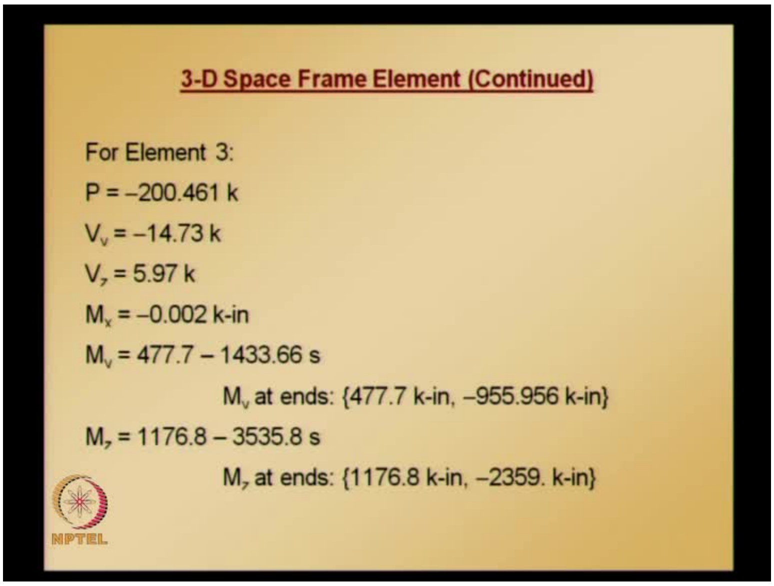

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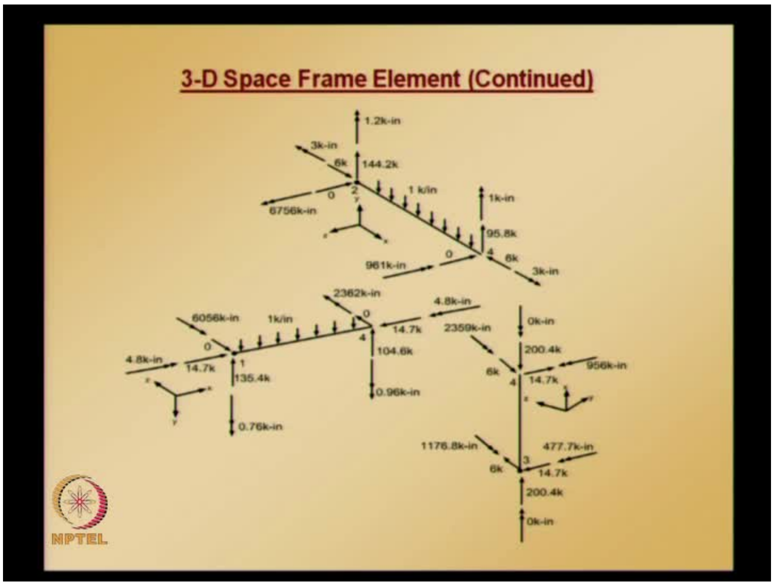

Similarly, for element 3 axial force, shear components, moments; once we obtained axial forces, shears, moments we can draw free body diagram - the forces can be shown on element free body diagram like this and we can check the equilibrium of each of these elements for numerical accuracy, and also we need to make sure that all the elements put together are in equilibrium.

So, that helps us to check the numerical accuracy and it can be easily verified; indeed, whatever values are obtained here satisfy all the equilibrium conditions.