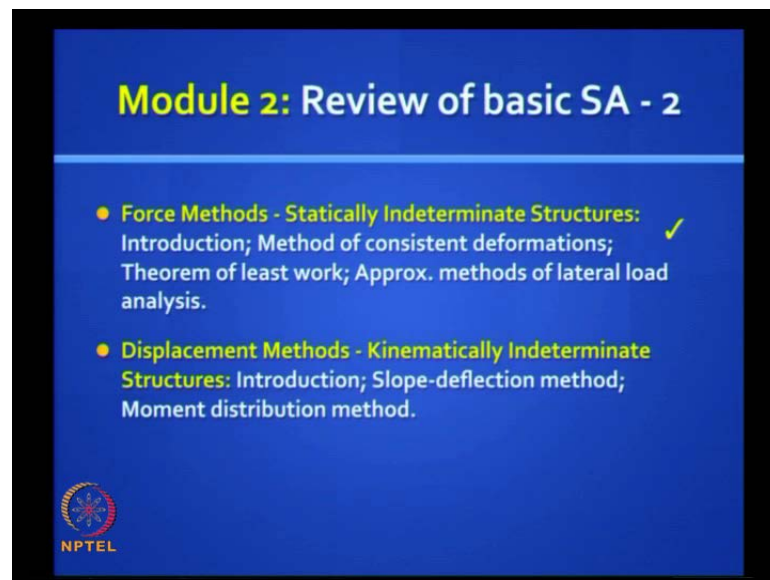


Advanced Structural Analysis
Prof. Devdas Menon
Department of Civil Engineering
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Module No. # 2.2
Lecture No. # 08

Review of Basic Structural Analysis-2

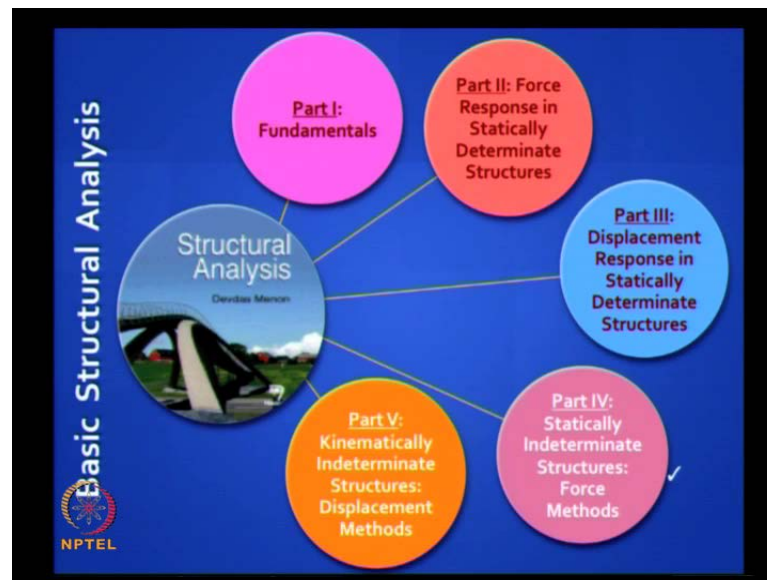
Good morning to you. We are now on to lecture 8 in the second module on Review of Basic Structural Analysis.

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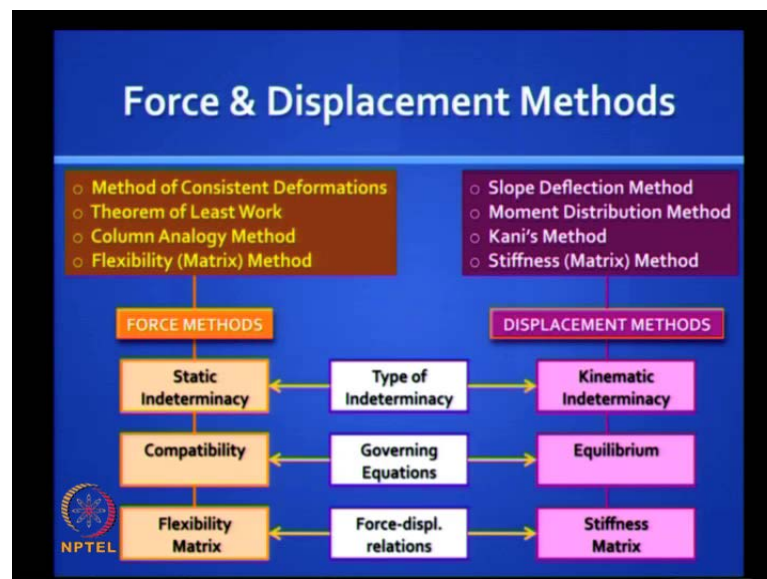
We are still on to force methods; this is the second lecture on solving force methods.

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This is covered in part IV of this book on structural analysis.

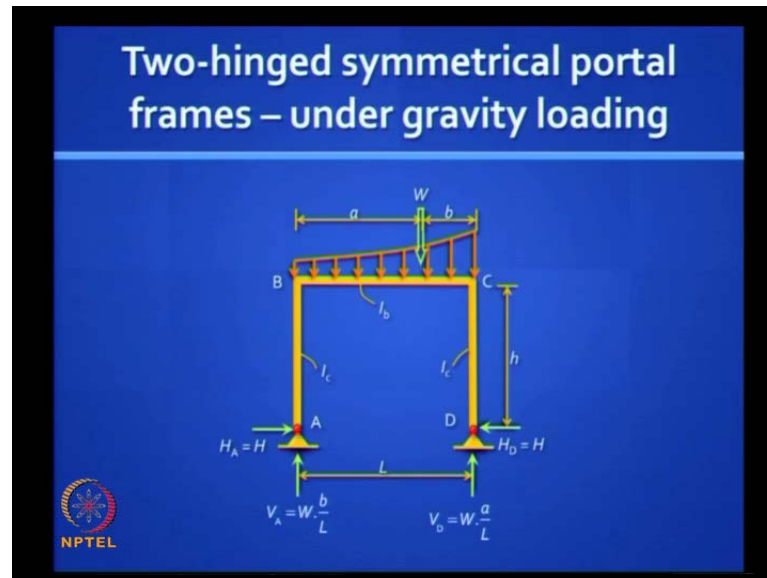
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In the last class, I broadly discussed the basic conceptual difference between force and displacement methods. In this module, we will be focusing on two force methods, namely, the method of consistent deformations and the theorem of least work. If you recall, we had already demonstrated the application of these two methods to solving statically indeterminate beams, continuous beams and fixed beams to frames; we also applied it to trusses. In this session, we will extend that application to portal frames,

specifically two-hinged portal frames, two-hinged arches, fixed arches and also frames with elastic supports. In the next section, tomorrow, we will cover approximate methods of analysis under lateral loads.

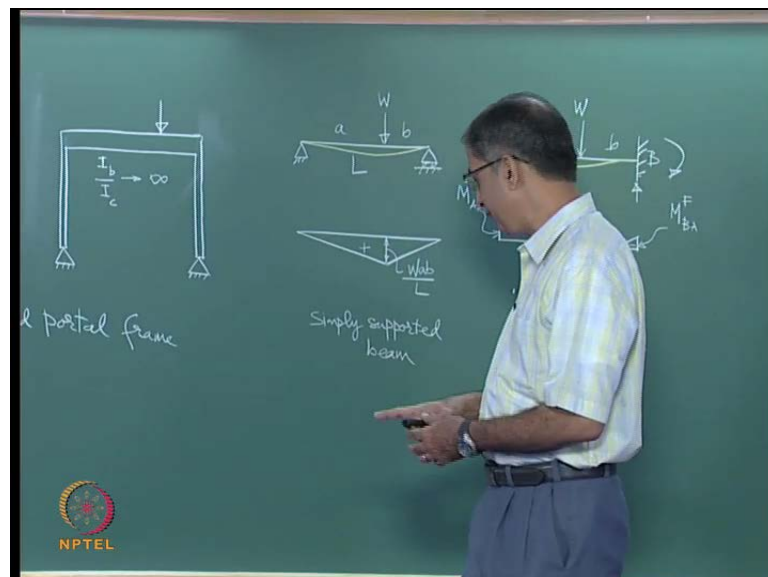
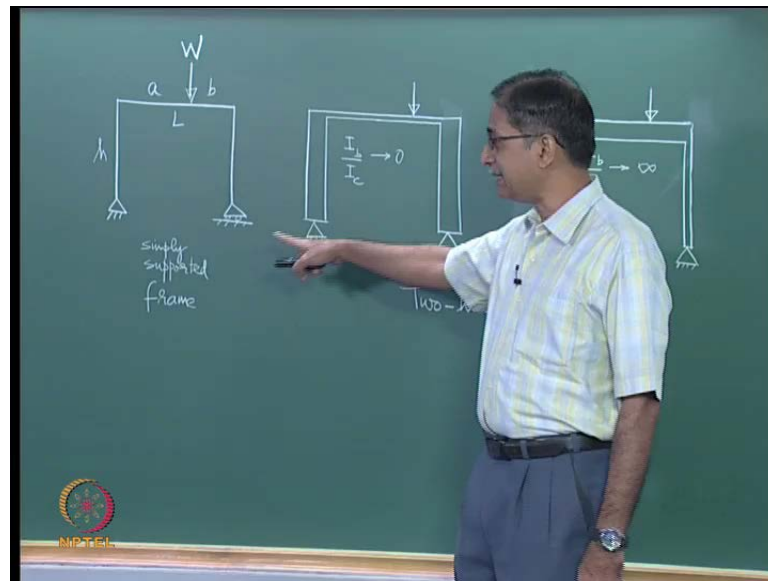
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I want you to look carefully at this simple problem. What you see in the picture is a two-hinged portal frame, two-hinged because you have two hinge supports at the bottom, and subjected to some arbitrary vertical gravity loading. The frame is symmetric; that brings in some special features, which I want you to understand. Let us be as generic as possible; let us say that the beam has a second moment of area I_b , which means it has a flexural rigidity $E I_b$ and the two columns have the same flexural rigidity $E I_c$. Is it clear?

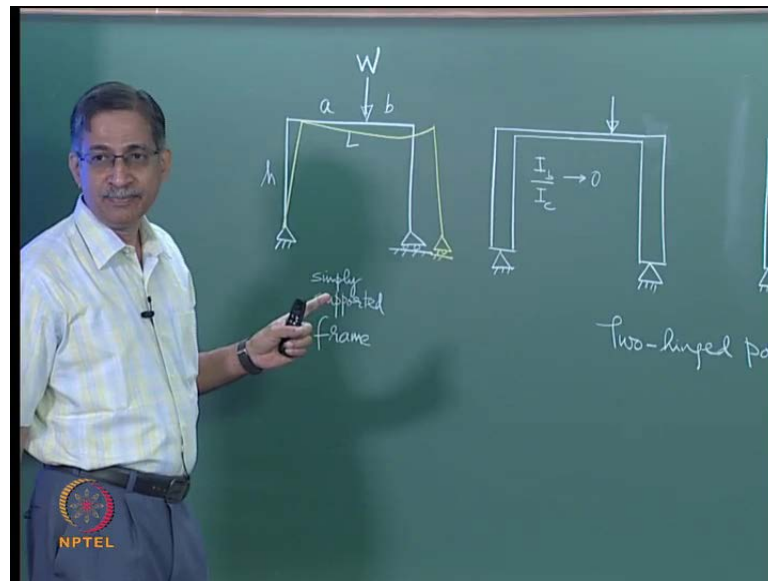
How will this frame behave? First of all, can we calculate the reaction? You will notice that yes, if the resultant vertical load is W , we can get the vertical reactions as you would get in a simply supported beam. What about the horizontal reactions? What about the bending moment diagram? What is the degree of static indeterminacy in this beam? 1. Is it internally or externally **determinate**? Either way; you can treat it as externally indeterminate by taking the horizontal reaction to be redundant or you could take maybe the bending moment at the junction of the beam and column as a redundant.

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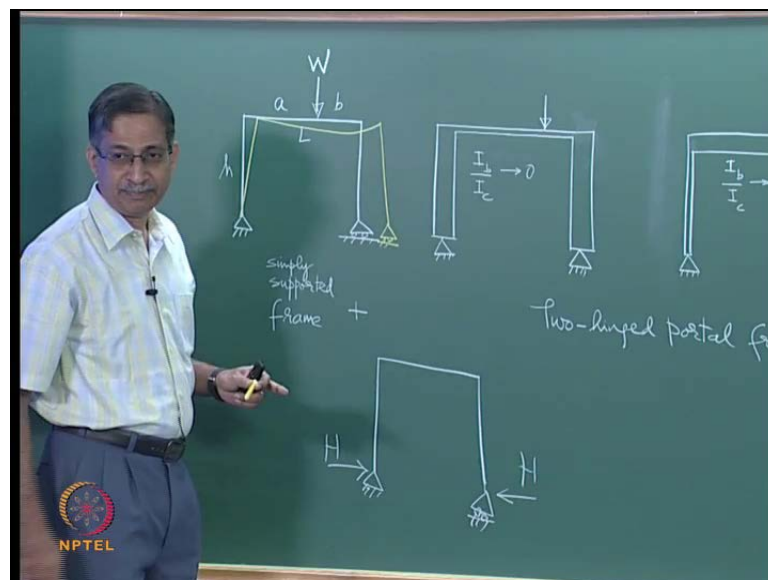
Now, I want you to look at these pictures which I have drawn. Take look at this. This is that same frame but it is simply supported; that means one of the supports is a roller support. In what way is the behavior of this frame different from the original frame? There will be no bending moments in the column and, in fact, this will behave exactly like a simply supported beam. If you look at the bending moment diagram, it will be as shown here (Refer Slide Time: 03:50) and these legs will remain straight.

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You have to basically copy that diagram but it will sway; so, it is going to probably look like this. If you were to draw the bending moment diagram of this frame, the vertical elements will have no bending moment and the beam will have the same bending moment as shown here (Refer Slide Time: 04:23). This is very straightforward; it is a sagging moment diagram. If you arrest this movement, you are going to get a horizontal reaction which means shear in these vertical columns. You will get this (Refer Slide Time: 04:37) plus an additional effect. What is that additional effect?

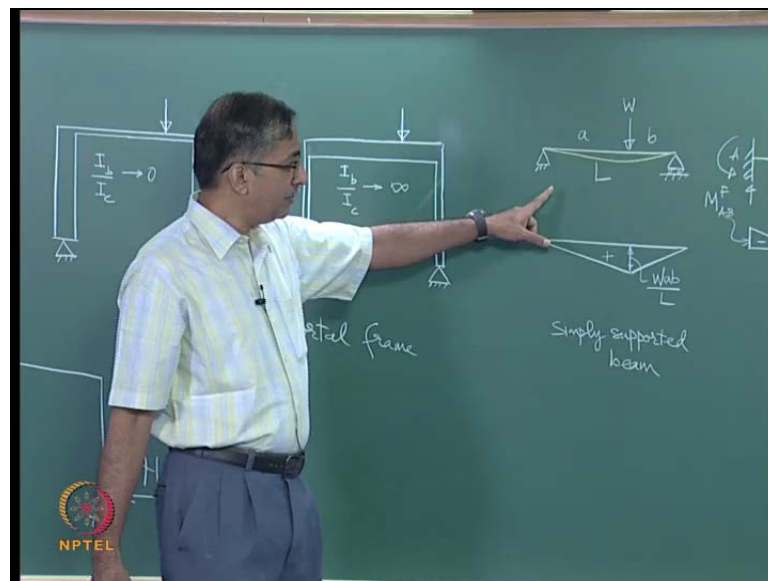
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It is as though you had the same frame but you also had this horizontal force H . This plus this is that frame. Now, how do you analyze this frame? How do you understand its behavior? As far as the bending moment diagram of this frame was concerned – bending moment or shear force or axial force (Refer Slide Time: 05:23), you will find that the force response does not depend on I_b and I_c ; it does not depend on the relative stiffness of the members. It is statically determinate; the statics do not depend on the member sizes, the material property and all that. It is a straightforward equilibrium calculation, but the response of the portal frame, which is two-hinged, does depend on I_b and I_c .

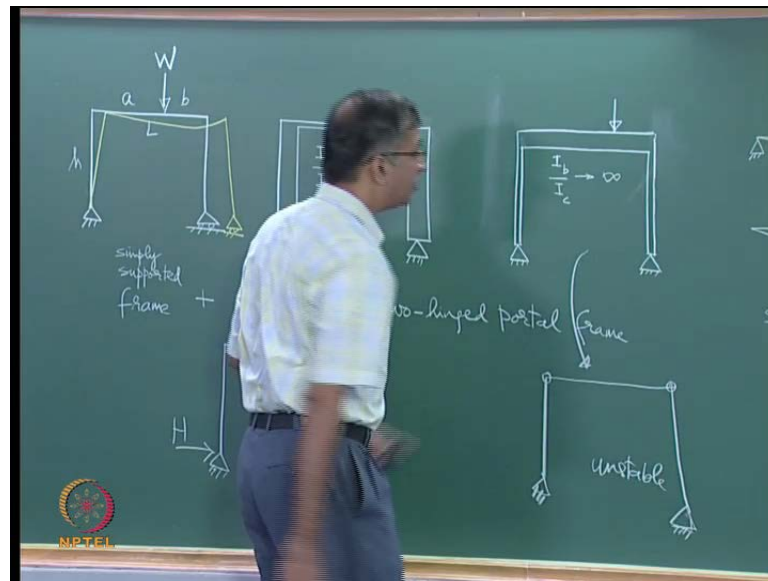
Now, the ratio I_b by I_c has a big role to play and I want you to see that. Let us say the beam is very slender compared to the column; that means the column is very wide – maybe it is a wall. Then, you could say I_c tends to infinity or I_b by I_c tends to 0 (Refer Slide Time: 06:15). So, the picture would look like this. Of course, you treat them still as line elements. On the other hand, you can have a rigid beam and the column relatively slender. So, I_b by I_c here tends to infinity (Refer Slide Time: 06:31).

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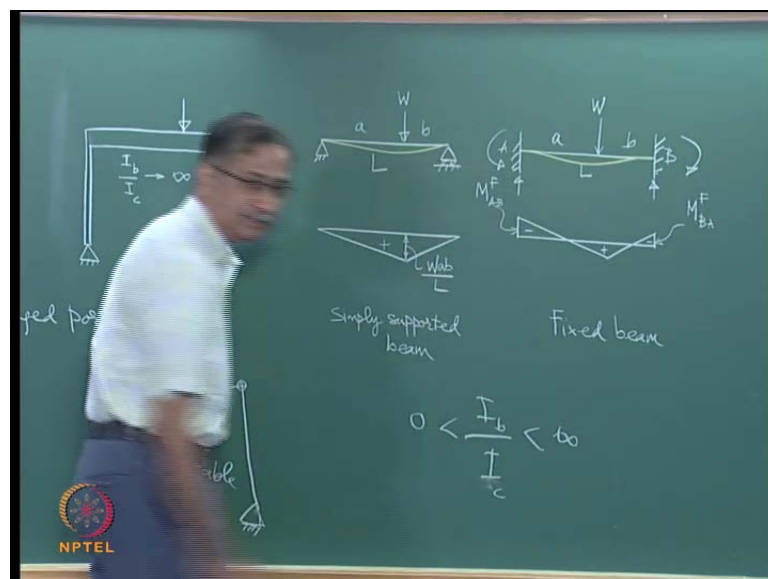
Similarly, if you were to just isolate the beam part, one extreme is a simply supported beam. When would you have this condition? When you have this condition, with an important difference.

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If this is really true, it would degenerate to a beam with a hinge there. What is the problem with this? If it is really infinity, if I_c is really 0, then you get a hinge, an internal hinge, here. This is not stable; this is unstable; this is not a good condition to have. On the other hand, this is an extremely good condition to have; this is very stable (Refer Slide Time: 07:43).

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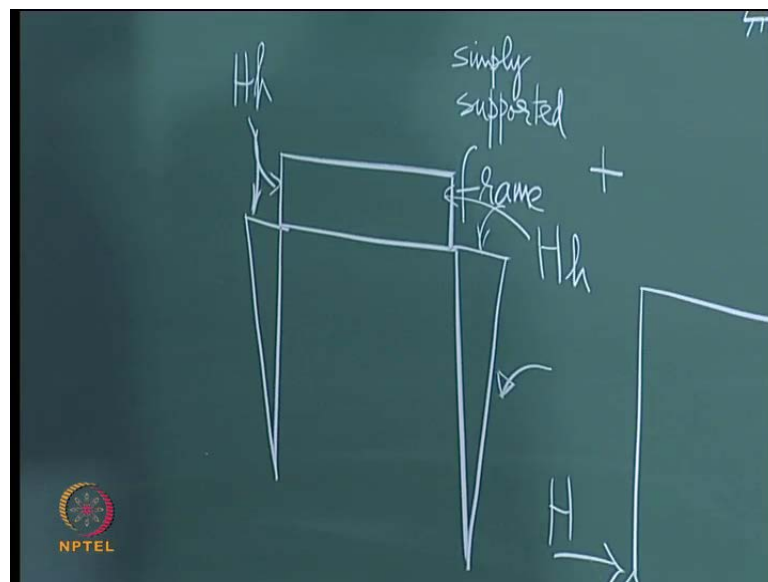


The reality is that I_b by I_c will lie between two extremes. They are limits; they are not physically realizable; they are limits. If you really want to model it as a beam, how

would you model it? Tell me. This model is one limit (Refer Slide Time: 08:14). This model is perhaps another limit where there is no rotation possible. There is no rotation possible at the beam-column joint because the columns are very stiff. So, it is approaching fixity conditions. If it is a fixed-fixed beam, you will find the fixed end moment here and the fixed end moment here are going to be different.

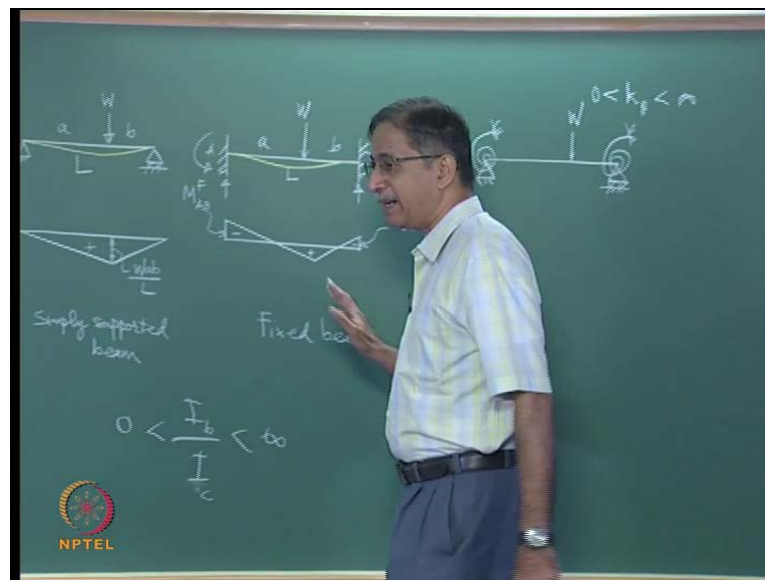
Actually, this fixed end moment (Refer Slide Time: 08:44) is going to be more if b is $((.))$ less than a ; they are not going to be equal. Would this also simulate that behavior? Or will there be a small difference? What is the difference? That is the crucial point to know. What is the difference? $((.))$ These two moments have to be equal in this situation because that is nothing but H into h .

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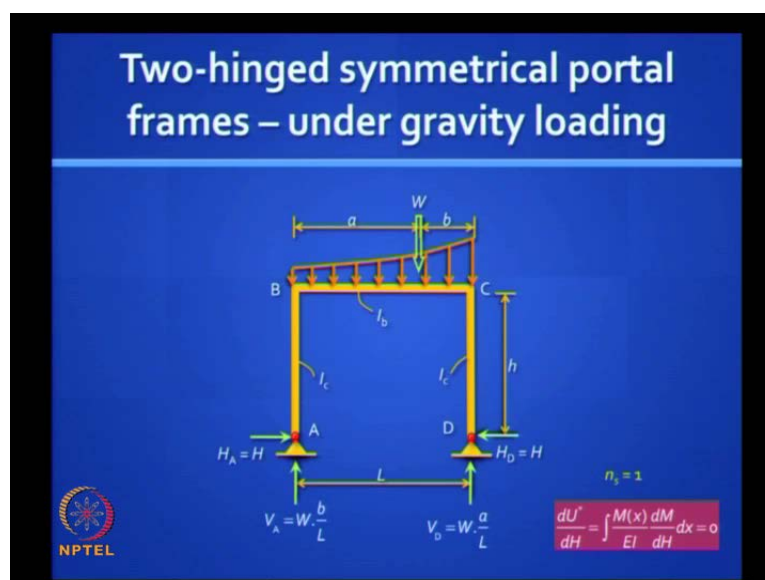
The bending moment you get from this diagram will look like this. Do you agree? This is going to hog and the moment on this side and this side will be equal to H into h . We will see later that if it is symmetric, this H into h turns out to be an average of these two moments; very interesting; it averages out, but that is the other extreme. If you had to **model isolate the beam**, what is the boundary condition you will apply?

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What kind of hinges? This cannot move. So, it is a simply supported beam with rotational springs. Very good. With rotational springs. The rotational spring has a stiffness k_{theta} . If it is simply supported, then one limit is 0; if the columns are infinitely rigid, then the other limit is infinity; broadly, you must have these feelings. Now, let us look at the solution to this problem.

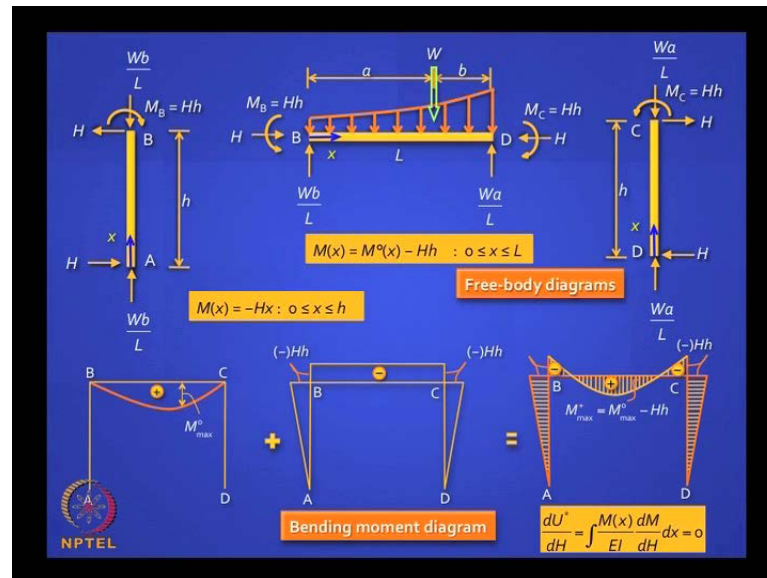
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The energy method is particularly useful in solving such problems. You know that the energy method says, the theorem of least work says that if you treat H as a redundant dou

U star or d U star by dH which is integral M into dM into dx by EI , which you see here, is equal to 0. Now, what is n ? You have to integrate over the full length of the frame; you have to integrate separately for the two columns and for the beam. Will you try that and give me an expression for capital H? A generalized expression which accounts for I_b and I_c variations.

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In this picture, I am showing you the free body diagrams where M_B and M_C are capital H into small h. Does the diagram make sense? Your redundant is capital H; if you know capital H, you know everything. Can you draw the bending moment diagram? What will it look like? It will be nothing but this plus this (Refer Slide Time: 12:22). That is what the diagram will look like because it is a superposition of two simply supported frames; so, it is going to look like that. Of course, the picture that I have shown you there is a distributed load and so you will have a curved bending moment diagram. The picture I have shown you on this blackboard has a concentrated load; so, it has a linearly varying free moment diagram (Refer Slide Time: 12:53). That is the only difference. Clear?

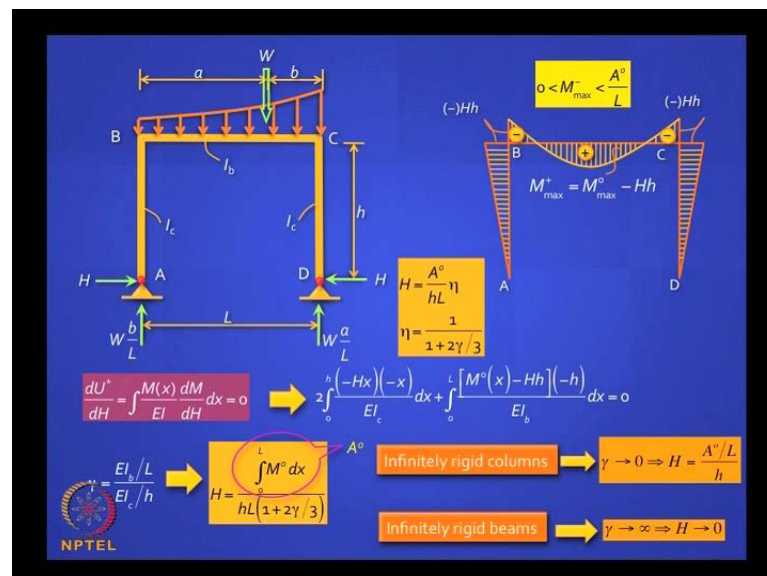
Now, you have to write an expression for M in the beams and in the columns. What will be the expression in the column? It is a linearly varying bending moment. So, this is your resultant bending moment diagram (Refer Slide Time: 13:13). Can we write this expression for the column? That is, if you take the column and you say starting from the base x pointing upward, is this expression correct? Now, whether you write plus or

minus depends on your convention. But, let us say, we are sitting inside that frame; so, it is outside; we can treat it as hogging. For vertical elements, it is not right to use words like hogging and sagging; for horizontal and inclined elements, it makes sense. Clear? Do you agree that this is the expression and the limits are from 0 to small h ? This is for both column AB and column DC.

Now, can you write an expression for the beam? Can you write it in terms of M naught? M naught is the free bending moment at any location x in a simply supported beam. What will it be? $((.))$ The sagging moment is M naught of x and minus capital H into h because that is a constant moment throughout the length of the beam. Can we write that? Got it? Here, the limits are from 0 to L .

Now, you invoke the theorem of least work. Can you substitute these values and derive a formula for capital H ? You have to integrate for two columns and one beam. If you differentiate this, what do you get with respect to $((.))$? You get minus x . If you differentiate this, what will you get? What do you get? You are differentiating with respect to capital H . Which is the function of capital H here? This is not a function; this is the function, is it not? It is pretty easy to do.

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This is what you get. Agreed? This is the first expression for the two columns; so, 0 to h . This is the bending moment; this is the derivative of that bending moment; this is for the beam – 0 to L ; this is the moment which you can calculate minus h and the derivative of

this $((.))$. Do you agree to this? It is a straightforward calculation. Will you expand this and simplify?

Now, let us introduce a parameter which you can call as stiffness parameter where I take the relative stiffness of the beam to the column; so, EI_b by L to EI_c by h . That gets rid of all those variables and makes it generalized. What are the limits of gamma? gamma will tend to 0 when the column is infinitely rigid; gamma will tend to infinity when the beam is infinitely rigid. If you solve this equation in terms of gamma, do you get this expression? It is a very straightforward thing to do. What does this remind you of? What is this – integral M naught into dx ? It is an area. Area of what? Area of the free moment diagram, right? Area of the free moment diagram. It is a useful relationship to remember and we call that A naught.

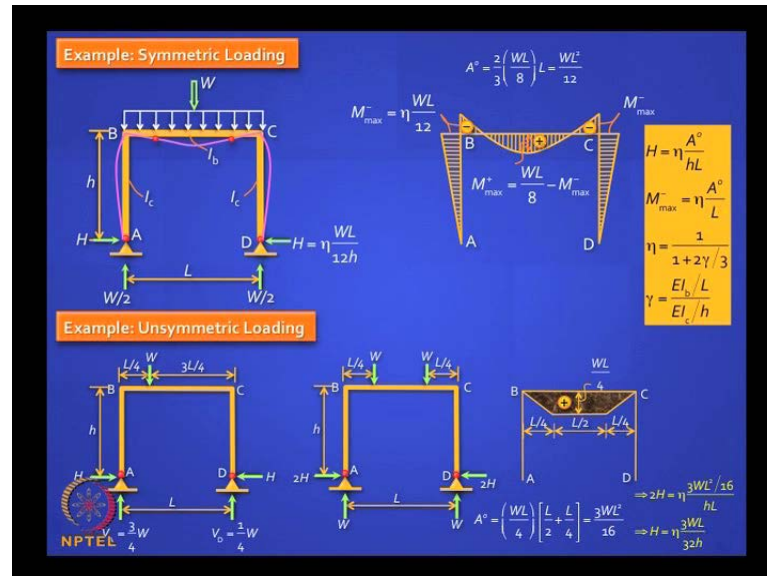
Let us look at two extremes. If you have infinitely rigid columns, which is this case (Refer Slide Time: 17:19), if you have infinitely rigid columns, gamma will tend to 0 and that equation reduces to H equal to A naught by L . On the other hand, if you have infinitely rigid beams, gamma tends to infinity and H tends to 0. That is obvious because if you are approaching this condition, you do not get any horizontal force because there is no moment transferred from here to here (Refer Slide Time: 17:49). There is no moment in the column; the horizontal force tends to 0.

Now, let us use a parameter η to take care of the gamma. If we define η as 1 divided by 1 plus 2 into gamma by 3, which is essentially the denominator here, then capital H is A naught divided by hL into η . Let us take simple problems; let us take symmetric loading problems. Is this clear? It is quite simple. If you take a symmetric. You will find that this moment here, capital H into h , when you substitute in this equation will be A naught divided by L . What does A naught divided by L remind you of? It is the average value of the bending moment in the free moment diagram; so, it is interesting.

The value of the moments you get here, the maximum value of that moment is A naught by L because the highest value of η is 1. So, that bending moment will vary from 0 to A naught by L . As a structural engineer, you must be sensitive to the bounds of your results; you know that the answer is going to lie between this extreme and this extreme; the deflected shape will be from one extreme to the other extreme. You get a clear

picture of the overall behavior in this procedure. Let us take a simple example. Will you solve this problem?

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Let us take a simple example. Will you solve this problem? Here, I have a symmetric loading, a uniformly distributed load. Can you draw the bending moment diagram in terms of eta? If you recall, this is the general bending moment diagram (Refer Slide Time: 19:52). **This is the general bending moment diagram.** Here, you have a parabola due to a UDL. What will it look like? These are your general expressions for any symmetric loading (Refer Slide Time: 20:06).

Do you agree that this is the bending moment diagram? The bending moment diagram in the beam will be a parabola which gets lifted up by capital H into small h; the area of that free moment diagram is two-thirds into WL by 8; so, it is a parabola. Do you agree to this expression? Two-thirds into WL by 8 into L will be the area of the free moment diagram. That is A naught; that is all you have to calculate; that is WL squared by 12 and that divided by L is WL by 12 into eta.

Now, WL by 12 reminds you of what? The fixed end moment; so, it is the other **((.)).** So, it is a fixed end moment into a parameter eta which can vary from 0 to 1. The highest moment you can get is that of a fixed beam and the lowest is a simply supported beam. It is a very simple calculation and your bending moment diagram at the mid span will be WL by 8 minus whatever value you get here. Clear? It just gets lifted up.

Let us take a case of unsymmetric loading where W is at quarter span loading. You can do any problem; we will just take these two and see the kind of results you get. It is smart to convert the unsymmetric loading to a symmetric loading because there is a relationship between these two. What is the relationship? In a horizontal reaction when you have a single load, this h will get doubled. It will get doubled when you get this because you invoke the principles of parity and superposition, is it not?

Solve for this first, which is very easy; we are just using these equations. Find $2h$, divide by 2 and you get H ; it is a clever way to solve this problem. Will you do that? Do you agree the M naught diagram will look like that (Refer Slide Time: 22:41)? That is an easy area to calculate. If you work it out, that is the area; these are very simple and so I am going fast now. You can solve for $2H$, divide by 2 and you get H . These are very quick ways of solving such problems but the real application of this procedure is for arches which are very common in structural engineering.

Where do you come across two-hinged arches? In bridges, yes. You know the bow string girder bridge; the deck is suspended through vertical elements into an arch; you get roughly a uniformly distributed load on the arch but not truly uniformly distributed. The arch is usually parabolic for that reason but because a loading is never always funicular, you will always get some bending moments in the arch, which you want to minimize. Do you do it for buildings as well? Yes, big gateways and so on but I will show you pictures of a project which we are doing now where we have a series of arches.

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Two-hinged symmetrical arches

$M(x) = M^0(x) - Hy(x)$
 $\frac{dU^*}{dH} = \int_0^s \frac{M(s)}{EI(s)} ds = \int_0^s \frac{M^0(s) - Hy(s)}{EI(s)} \{-y(s)\} ds = 0$

$H = \frac{\int_0^s \frac{M^0(s)y(s)}{EI(s)} ds}{\int_0^s \frac{y^2(s)}{EI(s)} ds}$

$H = \frac{\int_0^{L/2} \frac{M^0(x)y(x)\sqrt{1+y'^2}}{EI(x)} dx}{\int_0^{L/2} \frac{y^2(x)\sqrt{1+y'^2}}{EI(x)} dx}$

Parabolic Arch:
 $y(x) = \frac{4h}{L^2}x(L-x)$ $I(x) = I_0\sqrt{1+y'^2}$

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Let us take a generic case of a two-hinged symmetrical arch. You know that if you are dealing with an unsymmetrical loading, it is easy to deal with it because you convert it to symmetric, find the horizontal thrust, divide it by 2 and then you take a free body ((.)). Can we use the same theory and do it here? Can you write an expression for capital H? Do you agree that capital H...?

Now, we have to integrate. There is no beam and column here; there is just one curved length. Let us say that parameter is s , small s , and the full length is capital S . Do you agree this is the expression for dU^*/dH ? There is only one redundant H and since $Mx...$ at any section, the bending moment is $M^0(x)$ minus Hy (y is the rise in the arch at any location); if you take the derivative of this, you get y . Do you agree that this reduces to this (Refer Slide Time: 25:10)?

Now, there is a problem with that integration. It is not easy to integrate because it will look like this (Refer Slide Time: 25:19). This problem was tackled historically by making some assumption because you do not get closed form integrals especially if EI is constant along s . Is there a clever way of simplifying this equation, not making much error? Any suggestions? I will give you a clue. Can we do something to integrate along the horizontal length – along x rather than s ?

What is the relationship between s and x ? ds is dx into square root of $1 + y'$ squared; I hope you know that; that is a simple expression. If you substitute that, you will

get this expression; direct substitution. Now, you tell me what is the assumption that is worth making. It has something to do with getting rid of EI from that entire equation. Now, if you do that, you still have a square root. It is not easy to integrate this expression. You do not want to see these square roots; you do not have a closed form integral for such expressions.

What is a clever thing to do without making too much of a mistake? M naught can take any shape depending on the loading; so, you cannot make an assumption on M naught. You have to use your intelligence and do it. People have done this; today, that is the way you analyze arches. It is simple mathematics – a clever move. How do I get rid of square root of 1 plus y dash square and just knock that off from that equation?

Polar coordinates is not suitable; that is suitable for segmental arches. Can we assume EI to take some variation along the span? Where do you think the arch should be thicker more – at the crown or at the supports? Crown. Why? Where is the axial compression more – at the crown or at the supports? At the supports. It makes sense to thicken the arch at the support and make it thinnest at the crown.

Let us say the EI at the crown is I_0 . Can I assume a variation of I of x in terms of I_0 which will help me crack this problem? Yes? I x is equal to I_0 into $((.))$. That is what they did. The formula for a parabolic arch is this; you can easily derive this to satisfy the boundary conditions (Refer Slide Time: 28:32). This was the assumption made. Do not worry; people do not really make arches to have exactly I equal to this. But they often thicken it at the supports; it does not matter how you do it; you do it as per your convenience. You can even make it uniform and still assume this and go ahead; you will make a small error. We will see that the error is not significant but it is a tremendous reduction in your work. If you make this assumption for analysis, how does that equation reduce?

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Two-hinged symmetrical arches

$ds = dx\sqrt{1+y'^2}$

$$M(x) = M^*(x) - Hy(x)$$

$$\frac{dU}{dH} = \int_0^s \frac{M(s)}{EI(s)} ds = \int_0^s \frac{M^*(s) - Hy(s)}{EI(s)} \{-y(s)\} ds = 0$$

$$H = \frac{\int_0^{L/2} \frac{M^*(s)y(s)}{EI(s)} ds}{\int_0^{L/2} \frac{y^2(s)}{EI(s)} ds} = \frac{\int_0^{L/2} \frac{M^*(x)y(x)\sqrt{1+y'^2}}{EI(x)} dx}{\int_0^{L/2} \frac{y^2(x)\sqrt{1+y'^2}}{EI(x)} dx}$$

Parabolic Arch:

$$y(x) = \frac{4h}{L^2}x(L-x) \quad I(x) = I_0\sqrt{1+y'^2}$$

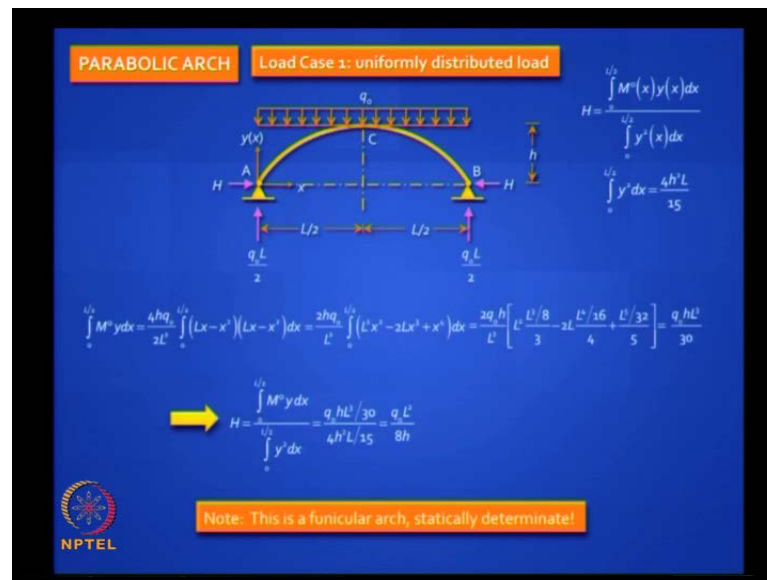
$$H = \frac{\int_0^{L/2} \frac{M^*(x)y(x)}{EI(x)} dx}{\int_0^{L/2} \frac{y^2(x)}{EI(x)} dx} = \frac{4h^2L}{15}$$

That equation now enormously simplifies; you are now integrating along the horizontal length. You have the equation for the arch and it is symmetric; so, you need to integrate only to L by 2. There is another beautiful thing that you can see which is independent of the loading. What is that? One: the denominator is a constant. Can you integrate that and tell me the value? It is very easy to do that – integral **y squared** into dx from 0 to L by 2.

This denominator is a constant which you can remember because y is this. There you are; 4 into h squared into L by 15. That is an easy integration to do. Remember that and you are ready to solve arches. The first question is let us say you have got an unsymmetric loading. Do not worry; make it symmetric because then you need to integrate only your half, find out the horizontal force and then divide that by 2 to get it for the original problem. Secondly, in this integration, the denominator simplifies to 4 into h squared into L by 15.

Let us say you have a case where you have a parabolic arch subjected to a uniformly distributed load along the horizontal span. Do you know the answer for that? Do you know the answer for this case? How can you forget? Yes? Do you know the answer for capital H? What is it? **((.))** There is something nice about this arch. This arch is funicular for this loading. That means there are no bending moments anywhere in the arch.

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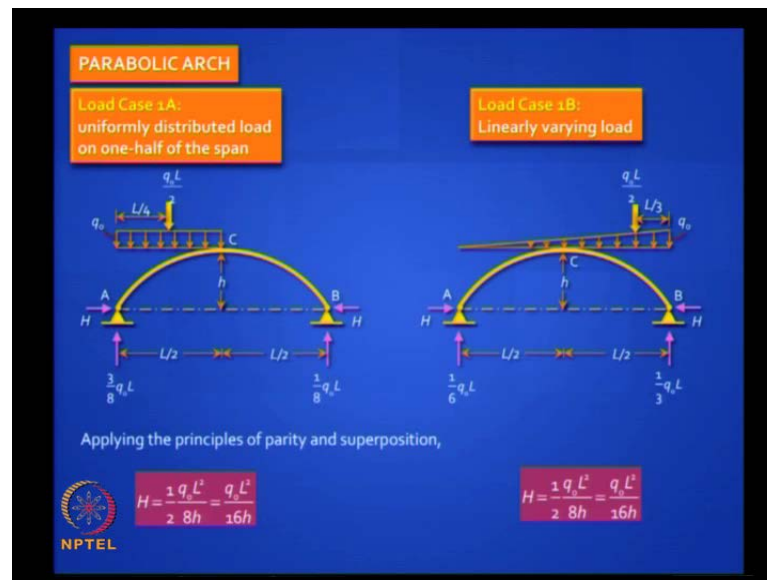


What is the bending moment at mid span in a simply supported frame? q_0 into L squared by 8. That will get eliminated at the crown. You have to divide it by h ; so, it is q_0 into L squared by $8h$; you know the answer. But let us say we pretend we do not know the answer and we do this integration making that assumption of I_x equal to $((.))$. Do you think we will get exactly that answer or will there be a small error? You would expect a small error; let us check it out.

This is an easy integration to do. **M naught.** You can see that the support reaction is q_0 into L by 2. So, if you cut a section anywhere in a simply supported arch, it will be q_0 into L by 2 into x minus q_0 into x squared by 2. You can really work this out; it will reduce to this form; please check it out at your convenience. When you complete this integration, you get in the numerator q_0 into hL cubed by 30; you can check it out.

Now, you substitute here and you get exactly, without any error, the expression q_0 into L squared by $8h$. When people saw this result, they said the error, if at all, in a generic case will be negligible. This gave confidence in using this technique; whoever suggested this was a genius because it simplified the work so much. Otherwise, you will have to resort to numerical integration, which is not worth $((.))$. This is a very simple way to solve any arch. Is it clear? This is a funicular arch, statically determinate.

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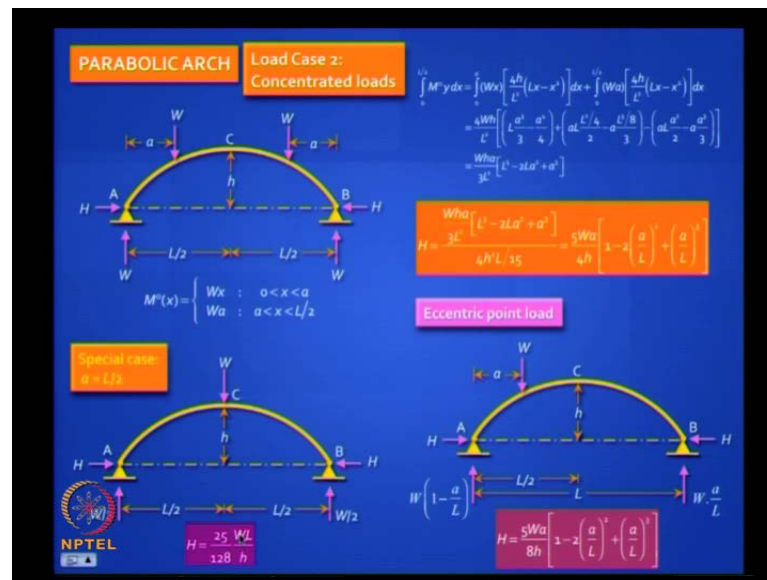


Now, let us do something more interesting. Let us say one half of the arch is loaded. Without any calculation, can you get capital H? It will be q_0 into L squared by? There you are. You should write applying the principles of parity and superposition. Let us say you have got it. Can you now draw the bending moment diagram, axial force diagram and shear force diagram? Only then is the analysis complete. Getting H makes it statically determinate but then you have to finish it.

You have done this exercise for three-hinged arches, remember? We have done this before. This is now like a three-hinged arch. In fact, in this particular case, it is a three-hinged arch because the bending moment at C will be 0. You can do this; I am not going to do this all over again; I am giving you an assignment where you have to do it; we have done this earlier.

What about this? Let us say you had a triangular loading like this (Refer Slide Time: 34:35). What do you think capital H is for this second case, load case 1B, when you have a triangular load? It is the same. It looks tricky but then if you put a triangle the other way, it will add up to a UDL. Here also, you get the same answer (Refer Slide Time: 34:52).

(Refer Slide Time: 34:55)



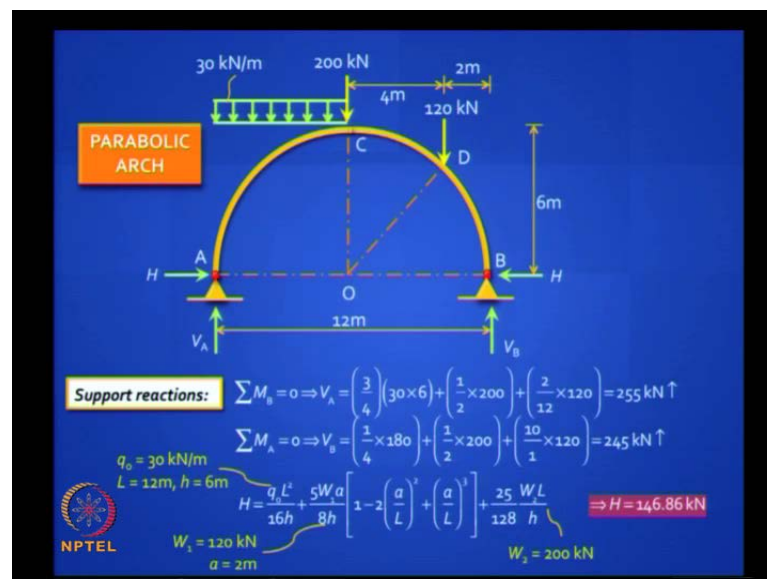
Let us take the case of concentrated loads. Can you derive an expression? In fact, these are the two standard loading cases. You will have a uniformly distributed load and you have a concentrated load. You could have the concentrated load at the crown but that is a special case of this. How will you solve this problem? Let us do this together. What is the expression for M naught? Very easy; you get pure bending in the simply supported arch between the two concentrated loads and that is W into a ; in the region up to the concentrated load, it is W into x . It is enough to integrate to one half; do not bother about the other.

Do you agree to this expression? You have got an expression for M naught. Please check this out. You just write the equation in the final form. Do you not think it is worth doing it on your own later? Check it out; it is a standard integration. You are good at integration; it is a simple integration. You know the equation for y ; it is a parabola. We wrote the equation here for M naught; you have to integrate it over two parts: 0 to a and a to L by 2 . Is it clear?

What is the denominator? It is a fixed quantity, is it not? $4h$ squared by L ; so, $4h$ squared into L by 15 . That is an equation which practicing engineers like to see because all you need to do is to substitute the value of a by L , a by L is a ratio, and you have got the answer. This is a standard solution. You can use this; you do not need to reinvent the wheel but we have worked the background for doing this. Is it clear?

Now, if you had just one load, the vertical reactions are easy to calculate; the horizontal reaction is what you get in that expression divided by 2. Instead of 5 by 4, it becomes 5 by 8 times that same expression (Refer Slide Time: 37:13). What happens if you had one concentrated load at the crown? All you need to do is to substitute a by L equal to half, 1 by 2, and this is worth remembering. This is a standard formula for a single concentrated load on a symmetric arch acting at the crown. All these are special cases of the eccentric case. It is 25 by 128 into WL by h.

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Now, let us put together all that we have learned and apply it to this problem. Just go through the motions of solving this problem. You have two concentrated loads: one is at the crown and the other is eccentric. You have a UDL but not all the way – till one half. Can you write an expression for capital H with whatever we have derived? You do not need to invoke the theorem of least work and solve for M naught; you can very well use the formulas we have derived; that is what engineers do.

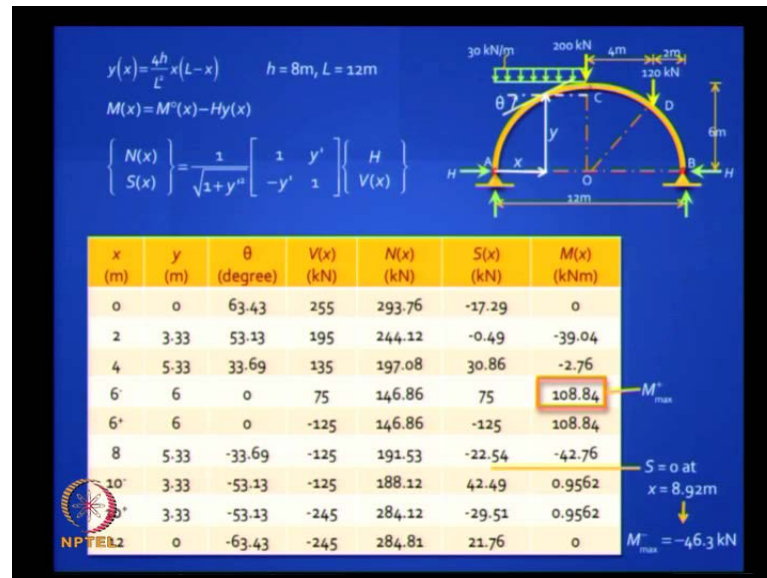
As students, you should know how to derive but you do not need to do that; today, you have handbooks and so on; you need to apply as well; so, let us apply here. If you apply that, first you have to find the support reactions which are pretty straightforward. You should also have the ability to check the reactions. Let us take the UDL. The total load of the UDL is 30 into 6, that is, 180 kilonewton. How much of it will go to the left support? Three-fourths of it; one-fourth will go to the right support. Take the

concentrated load at the crown 200 kilonewton (half, half – 100, 100) and take that 120; 2 by 12 will go to the left support and 10 by 12 of that will go to the right. It is simple stuff; you can find it; you can take moment equilibrium and solve it. Clear?

For the expression for horizontal force, you can just use what we have derived. **Due to the UDL...** Do you remember the formula? q_0 into L squared by 8 h into half; that is easy to remember. You just have to plug in those values. q_0 is 30 kilonewton per meter, L is 12 and small h is 6 meters. Then, take the eccentric load first. We have derived this: 5 by 8 into W – I am calling it W_1 here. Here, a is how much? 2 meters. a is the smaller value, please note. Then, due to the symmetric concentrated load at the crown, 25 by 128. Can I move ahead? This is easy to do; it is solved in the book. You get some answer. You should not take too much time over this. The assignment problem is actually a simpler problem. Clear?

Now, what do you do? You have found the support reactions. What do you do? You draw the free body; we have to go through this. From a practicing engineer's point of view, you need to have a feel for the bending moment diagram – at least get the shape right. Then, you need to know only the maximum and minimum values; that is all that we expect from you. You do not need to get all the values; if you want to, you can. Similarly, if you want to get the shear force and axial force, you can get them from the vertical component of the force at any section and the horizontal; the horizontal is constant throughout the arch.

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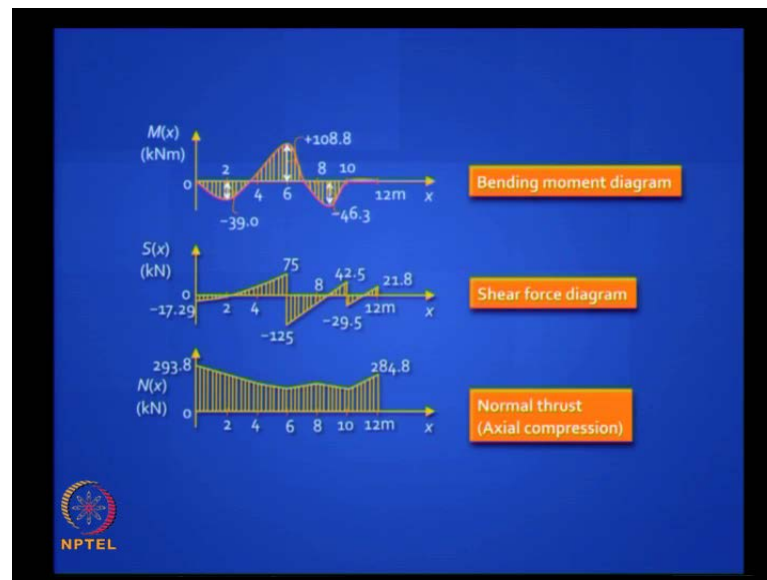


Do you remember this equation? We have derived this. You have an expression for y; if you differentiate this, you get an expression for y dash; you have an expression for moment. What you can do is just tabulate over a few points on the arch; wherever you have a concentrated load acting, do not miss out that point. We have done that – 0 meters, 2 meters, 4, 6 minus and 6 plus show that there could be a singularity there because of that load (6 is the middle of the arch), then 8 and 10. Every 2 meters I have done something. It is easy to calculate all this; it is straightforward; you just have to substitute this in the equation; we need to plug in.

We have done this for three-hinged arches. This is not something worth doing in the class here but definitely worth doing for an assignment so that you are familiar with it. Please go through this problem; see that you are getting the results and then you plot. Look for the peak values. We have got the maximum moment as 108.84 and that is exactly at the crown. It makes sense; that is where you will get it.

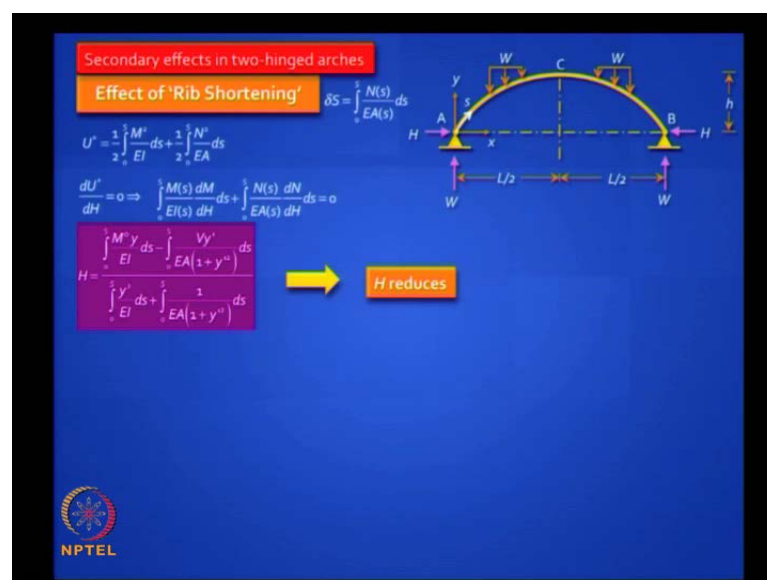
What about the maximum negative moment? This was checked at 8 meters but where do you think it will be maximum? Will it be this value? Will it be here? You need to check it out. How do you find out where to get the maximum moment, especially from the shear force ((.)) – where the shear force is changing sign? You will find that it is changing sign at 8.92; when you work that out, you get a slightly higher value than 42.76 – 46.3. These are things you can do and we have done this earlier; that is all.

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You can plot the bending moment diagram, the shear force diagram and normal thrust diagram. The rough shape is all that we want; we want a rough shape and we want to know the maximum hogging moment, the maximum sagging moment and the location. Is it clear? I am not going through this exercise in detail because you have already done this for a three-hinged arch and it is similar. Is it clear? We are done with arches.

(Refer Slide Time: 43:20)



It is also important to note that you get some secondary effects which are normally ignored in a first-order analysis. What do you think are the secondary effects that a

designer should be concerned about when you do two-hinged arches? In cables also we looked at those secondary effects. First, change in the shape. What do you think will happen to the arch? Will it increase in length or reduce in length?

The cable will increase in length and the arch will reduce in length because of the axial compression. That is sometimes called rib shortening, rib because the arch looks like the rib in your skeleton. That is the first effect that might be worth considering when you have slender arches and when the axial force in the arch is very high. The axial force in an arch will be high when the rise of the arch is small because you know that the beneficial effect of the horizontal thrust is not there.

Let us look at it conceptually. If the arch was a straight member with a constant axial force, the elastic shortening will be NL by EA ; you know that; EA is the axial rigidity. In this case, you have a varying axial force and so you will have to integrate along the curved length of the arch. Is it clear? That is the rib shortening. How will this affect the horizontal thrust in the arch? Will it increase compared to the situation where you ignored it? Or will it reduce? How to find out?

Let us say I want to do an exact analysis including rib shortening. What should I do? Go back to the first principles. How did we solve for capital H ? Which method did we use? We used the energy method; we did not explicitly calculate any deflection. In that method, how do you include rib shortening? We did not use any work done in that; we used complementary strain energy. We have to add an additional term. A term due to what? Complementary strain energy is equal to strain energy assuming linear elastic behavior. What is that term we need to add?

In a frame, you can have many components to strain energy. The main component is flexural strain energy (bending strain energy) but you also have two components from axial strain energy and shear strain energy, which we usually treat as negligible. Here, which is the strain energy you have to include? Axial. That is all you do. Do not forget your fundamentals; you just have to include that second component of axial strain energy; then, you invoke this theorem (Refer Slide Time: 46:34).

When you invoke this theorem, you get an additional term, where you account for the actual truss; when you substitute in your capital H – you can prove this, the final form takes this shape. Look at this carefully and tell me if you were to include the axial force,

will your capital H increase or decrease? The numerator has decreased and the denominator has ((.)). What does it mean? It will decrease. I am not asking you to calculate but least you should know that it can be done.

(Refer Slide Time: 47:16)

Secondary effects in two-hinged arches

Effect of 'Rib Shortening'

$$\delta S = \int_0^s \frac{N(s)}{EA(s)} ds$$

$$U^* = \frac{1}{2} \int_0^s \frac{M^2}{EI} ds + \frac{1}{2} \int_0^s \frac{N^2}{EA} ds$$

$$\frac{dU^*}{dH} = 0 \Rightarrow \int_0^s \frac{M(s)}{EI(s)} \frac{dM}{dH} ds + \int_0^s \frac{N(s)}{EA(s)} \frac{dN}{dH} ds = 0$$

$$H = \frac{\int_0^s \frac{M^2}{EI} ds - \int_0^s \frac{V y'}{EA(1+y'^2)} ds}{\int_0^s \frac{y'^2}{EI} ds + \int_0^s \frac{1}{EA(1+y'^2)} ds}$$

Effect of Temperature Rise, δT

$$\frac{dU^*}{dH} - \alpha L (\delta T) = 0 \quad \frac{dU^*}{dH} = \int_0^s \frac{M}{EI} \frac{dM}{dH} ds = H \int_0^s \frac{y'}{EI} ds$$

$$M = -Hy$$

$$H = \frac{\alpha L (\delta T)}{\int_0^s \frac{y'}{EI} ds}$$

Effect of Support Slip

$$H = \frac{-\delta_{\text{slip}}}{\int_0^s \frac{y'}{EI} ds}$$

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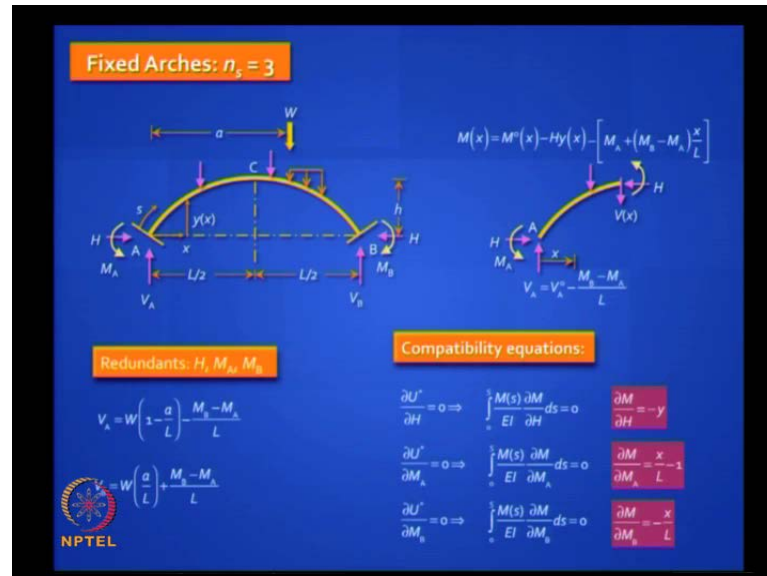
The second effect that we should worry about, because it is an indeterminate structure, is what would happen if you had a temperature rise or reduction in temperature. That can also be worked out. Here, you can use the consistent deformation principle. Treat it as a roller support, a simply supported beam, and let the heating take place. What do you think will happen to the arch? It will expand in length and so the roller will move to some distance. Then, you have to push it back and that force which you need to push it back is capital H.

You can work this out; you can easily prove that capital H takes this form (Refer Slide Time: 48:00). It is related to the stiffness of the arch. H increases and the last thing that you have to worry about is important. Supposing your support slips; we have conveniently assume that the two supports remain in place but when the horizontal thrust is very large, it will move.

In bow string girder arches, what prevents the movement is the tie member at the base, but the tie is not infinitely stiff; it can move. Is it clear? That movement also you can call it a slip. If it moves, will capital H increase or decrease? It will decrease. If it moves all

the way, it will go to 0. It is a function; you can prove it; it is a function of the **axial stiffness** that you get.

(Refer Slide Time: 48:54)



Finally, if you want to deal with fixed arches, you have three unknowns now. You cannot take advantage of symmetry in loading now; you have three unknowns and so it is convenient to take the two bending moments M_A and M_B and the horizontal reaction. Earlier, we had only the horizontal reaction; you need three equations now. What are the three equations? Vertical reactions can be expressed in terms of those unknowns. The three equations are, remember, $\delta U^* / \delta x_1 = 0$ (in this case x_1 is H), x_2 equal to 0, M_A and M_B . Is it clear?

You can write an expression for the bending moment at any section in the arch in terms of the free bending moment and the effect caused by all these values which you can write an expression for (Refer Slide Time: 49:49). You take the derivative of this with respect to these. You can simplify and they are easy to work out. I want you to go through this carefully; you can write these.

(Refer Slide Time: 50:03)

Symmetric Loading on Fixed Arches: $n_s = 2$

$M(x) = M''(x) - Hy(x) - M_f$

$\frac{\partial M}{\partial H} = -y$ $\frac{\partial M}{\partial M_f} = -1$

Compatibility equations:

$$\frac{\partial U^*}{\partial H} = 0 \Rightarrow - \int_0^{L/2} \frac{M'' y}{EI} ds + H \int_0^{L/2} \frac{y^2}{EI} ds + M_f \int_0^{L/2} \frac{y}{EI} ds = 0$$

$$\frac{\partial U^*}{\partial M_f} = 0 \Rightarrow - \int_0^{L/2} \frac{M''}{EI} ds + H \int_0^{L/2} \frac{y}{EI} ds + M_f \int_0^{L/2} \frac{1}{EI} ds = 0$$

Parabolic Arch:

$y(x) = \frac{4h}{L^2} x(L-x)$

$NPTEL \Rightarrow I_0 \sqrt{1+y'^2}$

Flexibility matrix:

$$\begin{bmatrix} \int_0^{L/2} \frac{y^2}{EI} ds & \int_0^{L/2} \frac{y}{EI} ds \\ \int_0^{L/2} \frac{y}{EI} ds & \int_0^{L/2} \frac{1}{EI} ds \end{bmatrix} \begin{Bmatrix} H \\ M_f \end{Bmatrix} = \begin{Bmatrix} \int_0^{L/2} \frac{M'' y}{EI} ds \\ \int_0^{L/2} \frac{M''}{EI} ds \end{Bmatrix}$$

Flexibility matrix:

$$\begin{bmatrix} \frac{4h^3 L}{3EI} & \frac{hL}{3} \\ \frac{hL}{3} & \frac{L}{2EI} \end{bmatrix}$$

If you do this and if you take the case of a symmetric arch, then your two moments will be equal and opposite; you have only two unknowns; it simplifies to this form (Refer Slide Time: 50:16). Remember earlier when you had only the unknown horizontal force, you had only integral y squared by EI ; now, you have this. This turns out to be a flexibility matrix. Your two unknowns are H and M_f . There are some solved problems in the book. These are properties of the arch which you can calculate by integration. We will stop with this; you can go through this later. Thank you very much.

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Review of Basic Structural Analysis - 2

End of Lecture 8 - Module 2.2

Next : Lecture 9 - Module 2.3

"Review of Basic Structural Analysis - 2"

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