Advance Structural Analysis

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Module No. # 1.6

Lecture No. # 06

Review of Basic Structural Analysis-1

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Good morning. We are on to lecture number 6 in the first module, in this course on Advanced Structural Analysis. We are reviewing the basic structural analysis, part 1.

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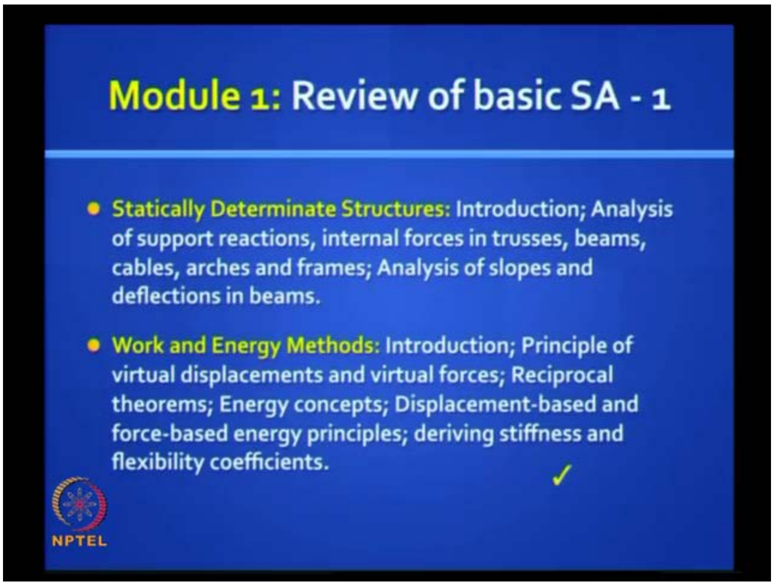

We are at the concluding section of this part, where we are reviewing work and energy methods. In fact, work method we finished; energy methods we had started in the last session.

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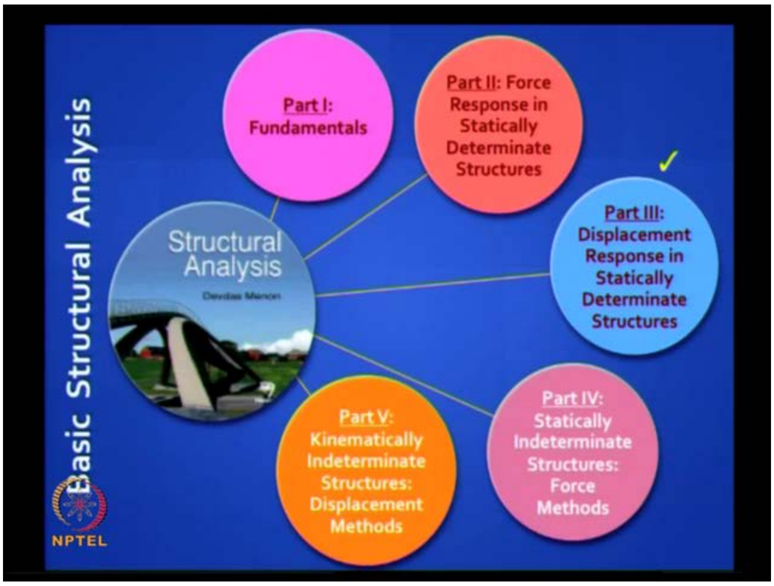

We are referring to this book on Structural Analysis and this portion is covered in part III of that book.

If you recall, we ended the last session with the theorem, a popular theorem in energy methods, which says - the external work on a structure is equal to the internal strain energy. The use of that theorem is limited to finding unknown displacements. Why is it limited? It is not versatile. What is their limitation in the use of that theorem? In all energy methods, there should be a cause effect. That is implicit in most of the energy methods. The limitation is that you can find a displacement only under a load location; there must be only one load. You understand? We will demonstrate that with this problem, looking at shear deformation.

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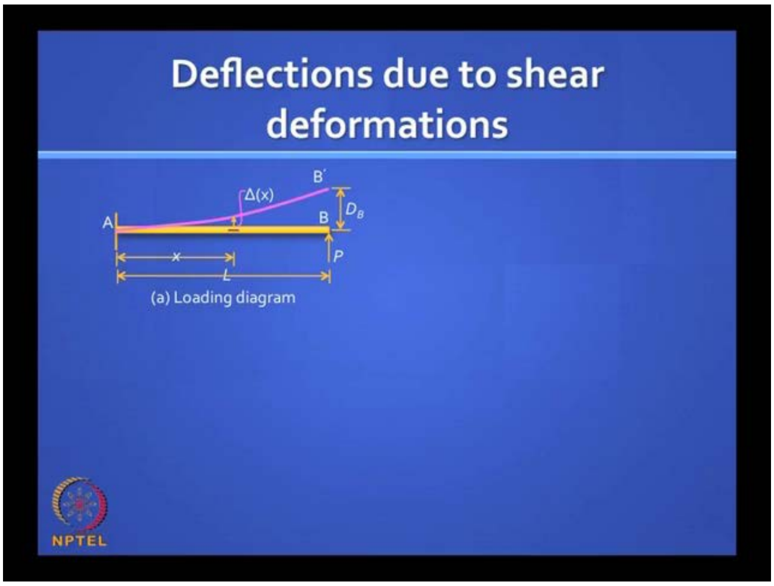

Let us look at this cantilever beam, subjected to a concentrated load P at the free end B, and let that deflection beam D_B. What is the answer for D_B?

P L cube by 3Ei.

You can prove it by conjugate beam. P L cube by 3Ei, but that is not a complete answer. Why not?

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Because it does not include shear deformation. So, we will see closely at how the shear deformation effects?

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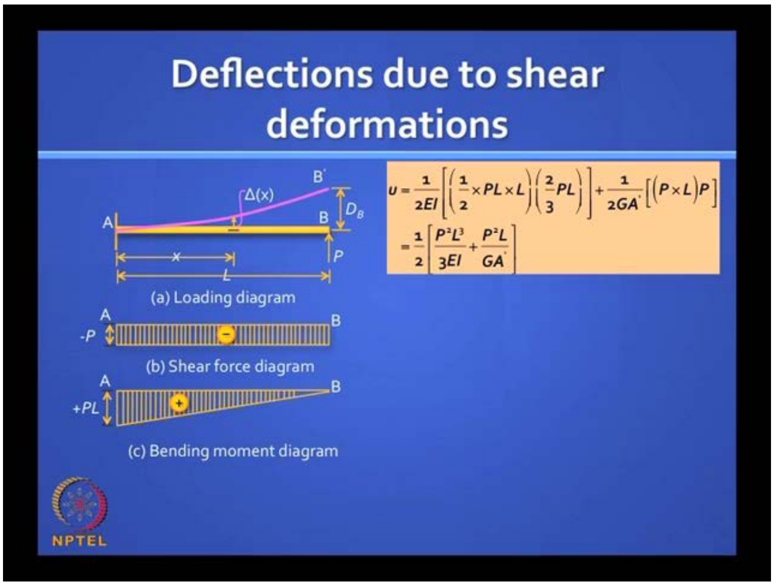

We can see that the shear force diagram, shows a constant shear force P and the bending moment diagram. In this case, it is a sagging bending moment diagram, linearly varying. If you want to find the strain energy in this system, it is easy. You can use that formula, but now we will include shear strain energy as well.

So, if you look carefully at this part, where we multiply the bending moment diagram by itself, it is the part that comes from flexural strain energy. This part (Refer Slide Time: 03:07) is the part that comes from shear strain energy. So, the total energy would be given by this expression, where you have both flexural rigidity and shear rigidity – GA dash; this we can equate to what? This is strain energy in the system. Total external work - the real work, which is equal to?

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No.

P times by D_B is virtual work. What is the full expression for real work?

Half P $D_{B.}$ Because there is a cause effect relationship, we assume gradual loading and linear elastic behavior. This is how we invoke the theorem of…, the theorem which says - the external work is equal to the strain energy. Mind you, if in that cantilever beam, I put a uniformly distributed load, I would not be able to find the deflection using this theorem because then there will many displacements.

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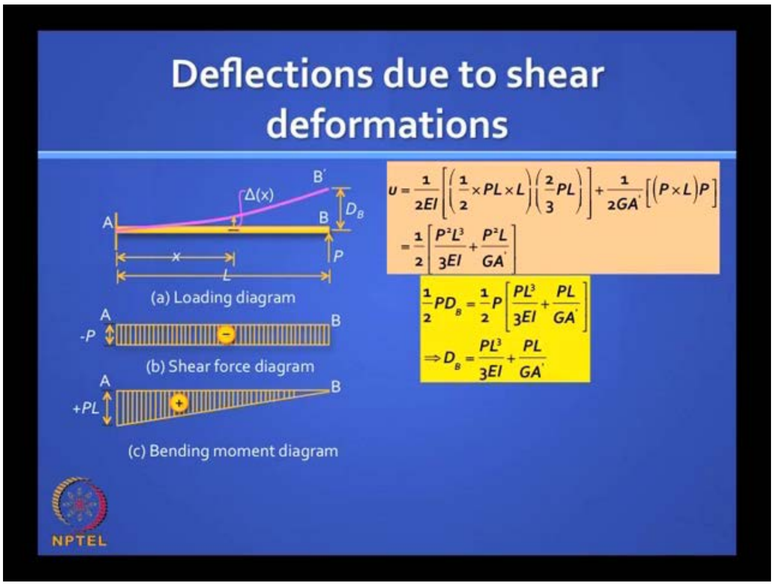

Look at this: you will find that, the deflection now is PL cube by 3EI plus an additional factor PL by GA dash which reflects the deflection that comes from shear.

An interesting point to note here is - you find that the deflection that comes from flexure varies with the cube of the length, whereas is the deflection that comes from shear is linearly dependent. This is because the shear force is constant and the bending moment varies linearly.

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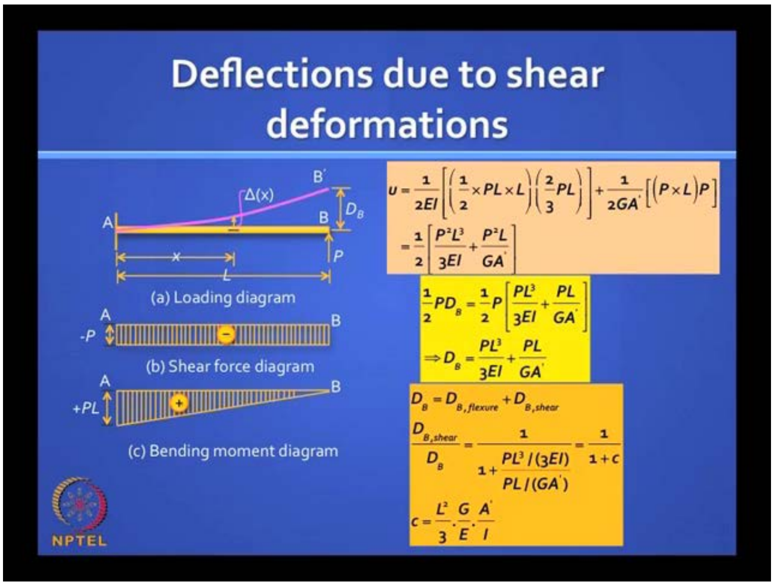

Let us try to see what is the relative contribution of shear deflection? We can take the ratio of deflection due to shear divided by total deflection and you can get this expression, which can be written in this form - D_B shear by D_B is equal to this.

Let us define a parameter C, which is L square by 3 into G by E. G by E is a ratio that depends on Poisson's Ratio and A dash by I.

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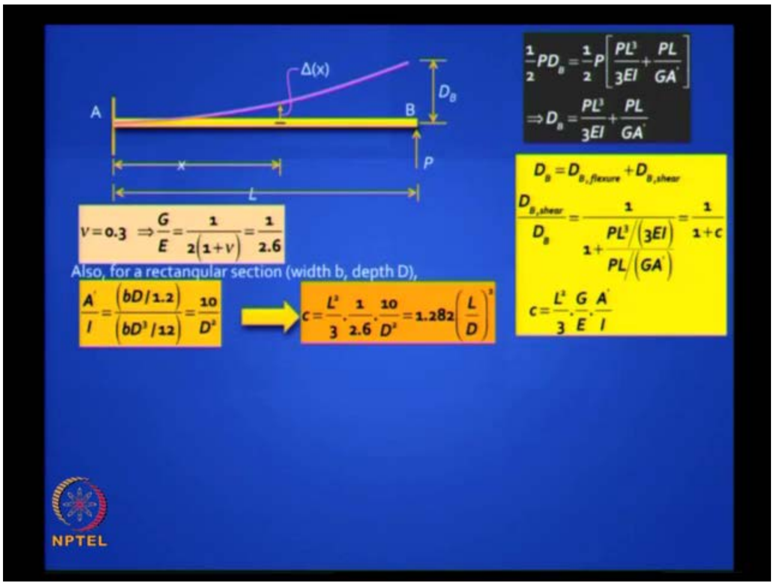

Now let us take a simple example of a rectangular section. For a rectangular section of material like steel, where Poisson's Ratio is 0.3 - that expression for C will reduce to something like 1.282 into L by D, the whole square. So, you will find that L by D span to overall depth ratio actually dictates the contribution of shear deformations to the overall deflection, to the overall energy.

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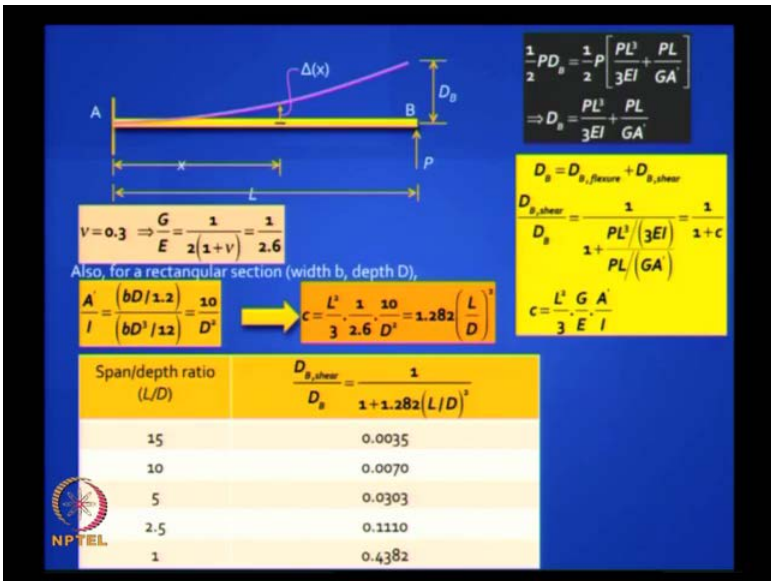

Span/depth ratio (L/D)	$\frac{D_{B,shear}}{D_B}=\frac{1}{1+1.282(L/D)^2}$
15	0.0035
10	0.0070
5	0.0303
2.5	0.1110
1	0.4382

If you want to look at some practical examples: if the span by depth ratio is high, as it normally is, which is what makes the beam a skeletal element, line element, you will find that the ratio of shear deflection to total deflection is low; negligibly low; 0.0035 is very low; it is 0.35 percent. If the span by depth ratio is 10, it is double that - 0.7 percent; if it is 5, which is an intermediate beam, it is 3 percent. If it is 2.5, it is 11.1 percent. But if it is 1, which is the span is equal to depth, it is as high as 43.82 percent.

This is the reason, when you deal with short squat shear walls, you really have to include the shear's difference. But, if you have tall slender shear walls, then, you can treat it as a flexural element. For normal beams, you can ignore shear deformations and you would not have an error more than 1 percent, but for deep beams, you can have very high errors.

We now come to energy methods proper.

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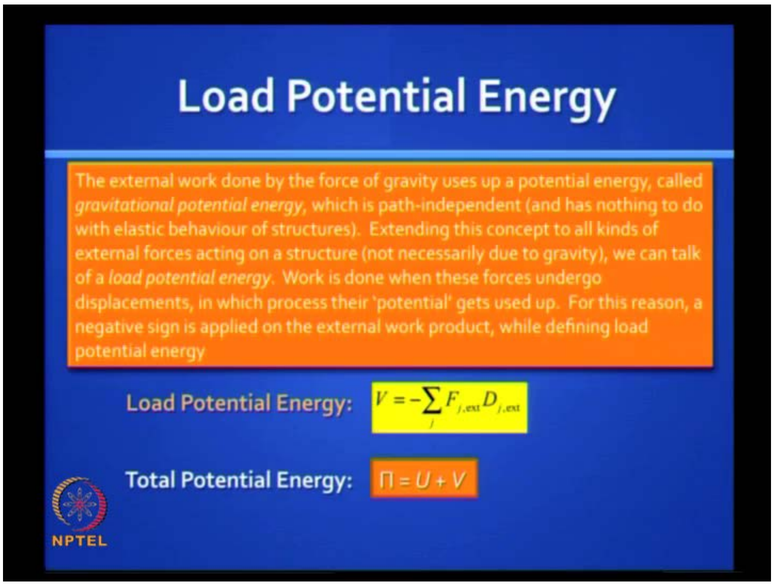

There is a term invoked in energy methods called Load Potential Energy. That is a concept which you need to understand. It is given here that, the external work done by the force of gravity uses a potential energy, called gravitational potential energy, which is path-independent and has nothing to do with the elastic behavior of structures. Every body subjected to the force of gravity has the potential to do work - that is gravitational potential energy.

We can extend this concept to all kinds of external forces acting on a structure, not necessarily due to gravity. So, we can talk of load potential energy. Work is done when these forces undergo displacements, in which process their potential gets used up. For this reason, because the energy gets used up, we associate a negative sign with this kind of work product. It is nothing but the external work product with the negative sign. So, the definition is very clear.

Load potential energy: that means we are converting the work done by the external forces, but it is not a real work. If it were to be real work, you would attach a constant like half. This is a virtual work and we give it a negative sign and label load potential energy.

Now, if you have a system, a structure, where you have external forces and you have internal energy. You can add up all the energy, sum it up and call it total potential energy, defined as Pi, Capital letter Pi equal to capital U plus capital V. U is internal energy, strain energy, which is always recoverable and V is load potential energy. Mind you, U is always positive; U can never be negative. And V is negative. Do not we actually talk about the changes in capital Pi? Whether it changes or not, we will see in the next slide.

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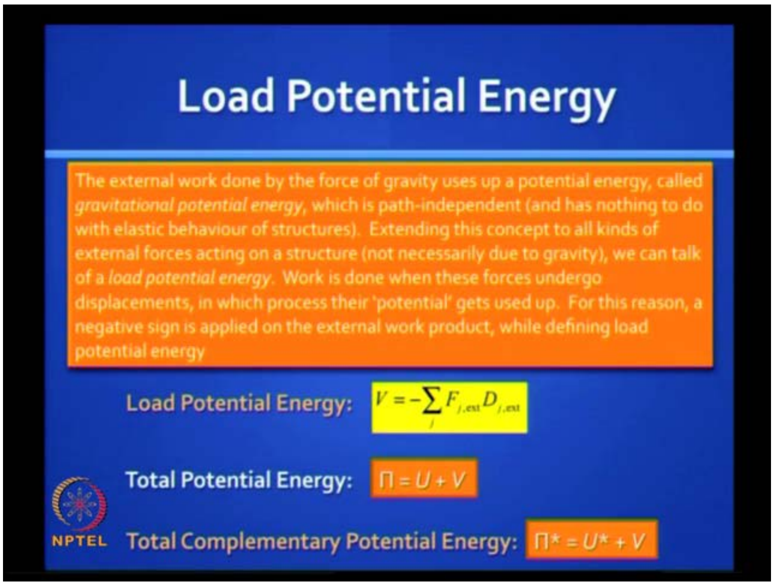

We can also define a complementary total potential energy, where we put a star - an asterisk. So, Pi star will be U star plus V. If you are dealing with the linear elastic system, U will be equal to U star and Pi will be equal to Pi star.

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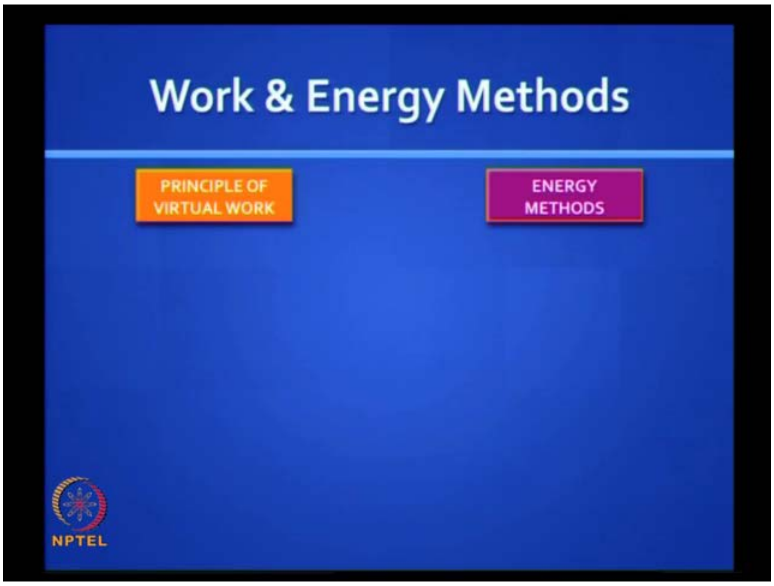

Now, we have two broad sets of theorems: one - related to total potential energy and another - related to complementary total potential energy. In the same way, as using the principle of virtual work, you have two broad principles: principle of virtual displacements and principle of virtual forces. So, you find that there is a strong correlation between these principles.

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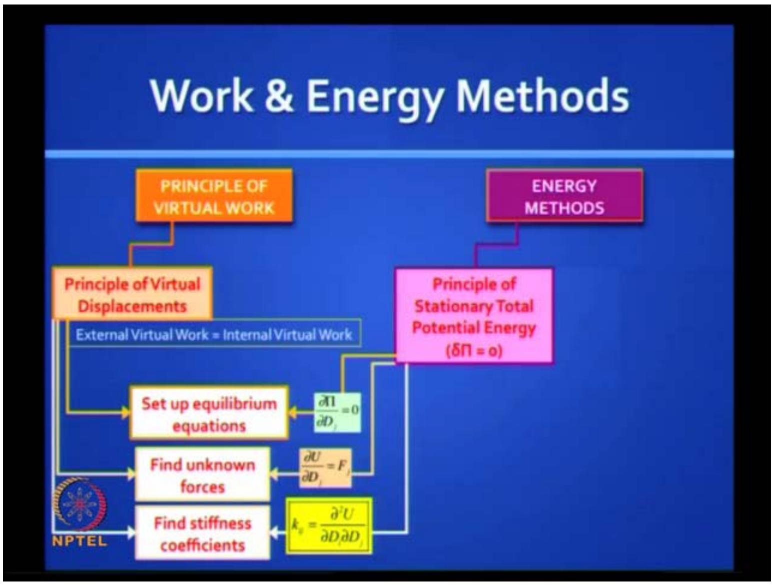

The principle of virtual displacements is linked to the principle of stationary total potential energy. We will come to this in a moment.

These sets of relationships where we look at the displacement field are called energy methods based on the displacement field. We imagine that displacement field is modified ever so slightly. You give a perturbation to the displacement field and then you see what happens. You will get another set of principles when you disturb the force field without disturbing the equilibrium in that field; those principles are related to complementary potential energy. You have a one is to one relationship with the principle of virtual forces.

This is a kind of a big map, an integral map, where you see all the energy methods. We will take a quick overview of these principles.

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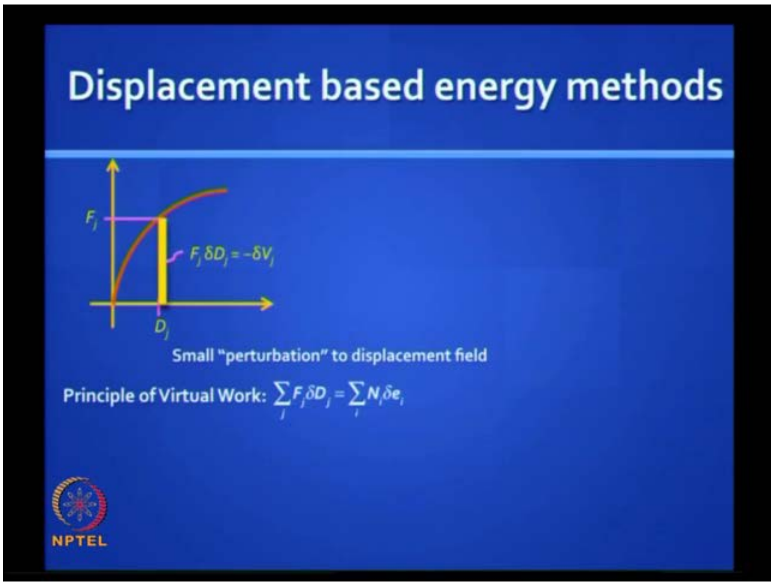

We will first look at the displacement based energy methods. So, let us take, for example, a truss. To generalize, let us assume that all the elements are elastic, but not necessarily linearly elastic. So, if the elements exhibit non-linear elastic material behavior, then the loads, the deflections that happen at the joint locations are called external component of force field and displacement field, will also show corresponding non-linear behavior.

Let us take any coordinate j in the truss. It could be a vertical coordinate or horizontal coordinate. Let us just see how that relationship between F_j and D_j changes, as you increase the loads from 0 to the maximum value.

Let us say that, at some point, it is stabilized and you have equilibrium and we imagine that we give a small perturbation to the displacement field, still maintaining compatibility. Let us say, we change D_1 by a very small quantity, say, 0.1 percent; either positive or negative. Similarly, D_2 I do by some other percentage, D_3 and so on. So, I do this in my mind; it is all imaginary. We are going to invoke variational principles to prove this theorem. So, you will see - if you look at the j th coordinate, if I give a small perturbation delta D_j, then there is a small change in the load potential energy. Because F_j is not changing, you are not disturbing the loads. So, you will find that the external work or virtual work as F_j delta D_j and you will give it a negative sign as it becomes the increment of the variation in the load potential energy.

Now, if you sum up this over all the joint locations, you get the total load potential energy variation. Is it clear? And this must be equal to the corresponding change - internal virtual work.

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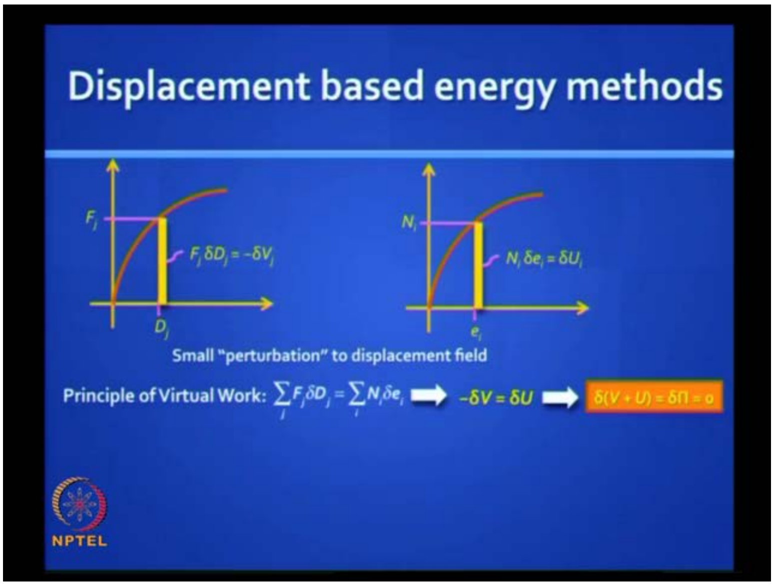

The principle of virtual work says sigma F_j delta D_j must be equal to sigma N_i delta e_i, where N_i e_i graph - the non-linear picture, may look like this and that incremental strain energy is delta U_i. First, we look at the principle of virtual work. It says sigma F_j delta D_j is equal to sigma N_i delta e_i.

Now, we bring in the concept of load potential energy. We see the term on the left hand side is minus delta V and the term on the right side is delta U. When you bring them both on the same side, you get delta V plus U, which we have defined as delta pi.

What do you conclude from this small proof? You get a theorem. What does the theorem say?

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It is a statement of equilibrium.

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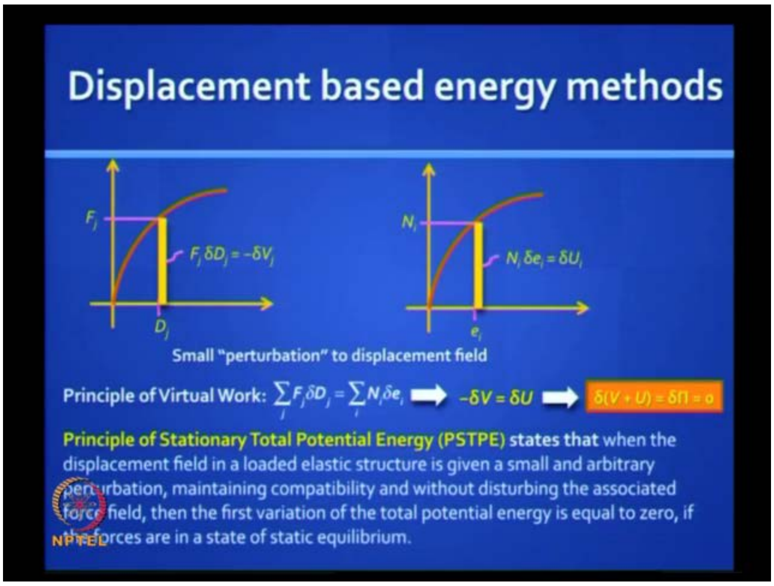

The Principle of Stationary Total Potential Energy (PSTPE) states that when the displacement field in a loaded elastic structure is given a small and arbitrary perturbation, maintaining compatibility and without disturbing the associated force field, then the first variation delta pi, first variation of the total potential energy is equal to 0, if the forces are in a state of static equilibrium.

This reminds you of which work principle? Goes back - to Bernoulli. This was his idea of principle of virtual work, more correctly called, the principle of virtual displacements.

It is a same thing expected in an energy form. It basically establishes equilibrium in the force field and you can find an unknown force component in that force field.

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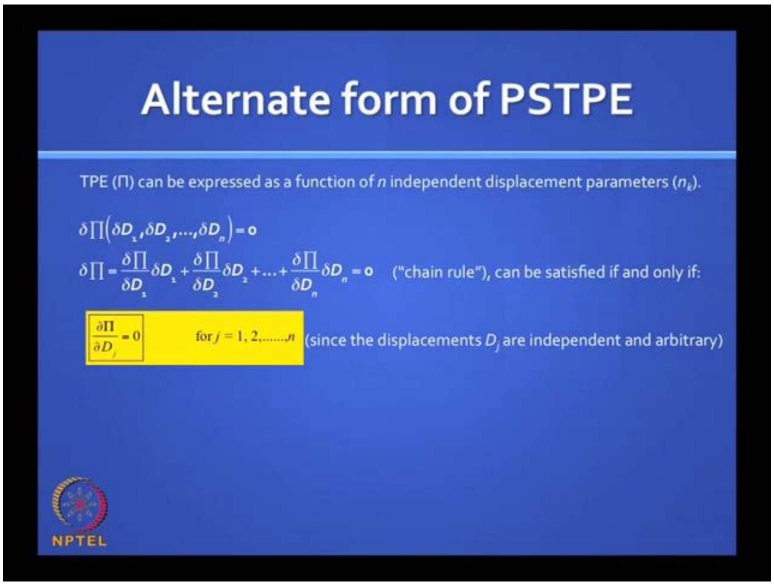

Now, the alternative form of the same principle is more popular. Let us say, you have some independent displacement parameters, I will give an example, which usually is equal to the degree of freedom in that system. Let us call them D_1, D_2, D_3, etc. Then you can also write Pi. You can always write U and V in terms of these independent displacements. You can write an expression, delta Pi as a function of delta D_1, delta D_2, etcetera equal to 0. If they are really independent, then you can invoke this chain rule.

Chain rule says that delta Pi can be written as delta Pi by delta D_1 into delta $D_{1,}$ plus this, plus that (Refer Slide Time: 17:56) equal to 0. What does the chain rule say? If this condition satisfies, each one of these terms should be equal to 0 because delta $D_{1,}$ etcetera are independent and arbitrary. If you do that, that one equation multiplies into large number of equations, as many as there are degrees of freedom and you shift from variational calculus formulation to a differential partial differential equation. So, dou Pi by dou D_j is equal to 0, for j equal to 1 to n. So, this is a better form for engineers to work with.

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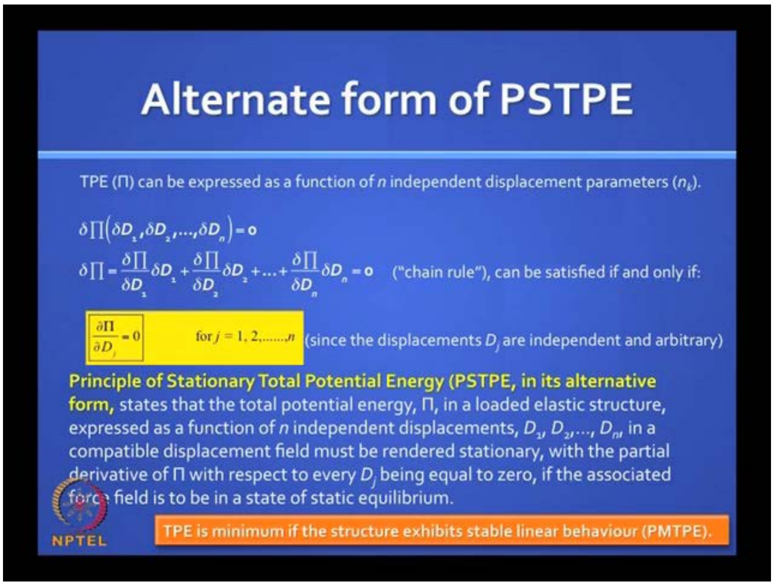

In this form, that same theorem says that when Pi is expressed as a function of independent displacements D_1, D_2, etcetera in a compatible displacement field, it must be rendered stationary, with the partial derivative of Pi with respect to every D_j being equal to 0. This happens if the associated force field is to be in a state of static equilibrium.

Now, the word stationary in calculus refers to a point of inflection. It could also be a maximum point or it could also be a minimum point of the function, for which you are taking the partial derivative. It can be proved, if you have a linear elastic stable structure, then, the stationary point refers to a point of minimum energy. So, in that form, it is more popularly known as the Principle of Minimum Total Potential Energy. It is PMTPE.

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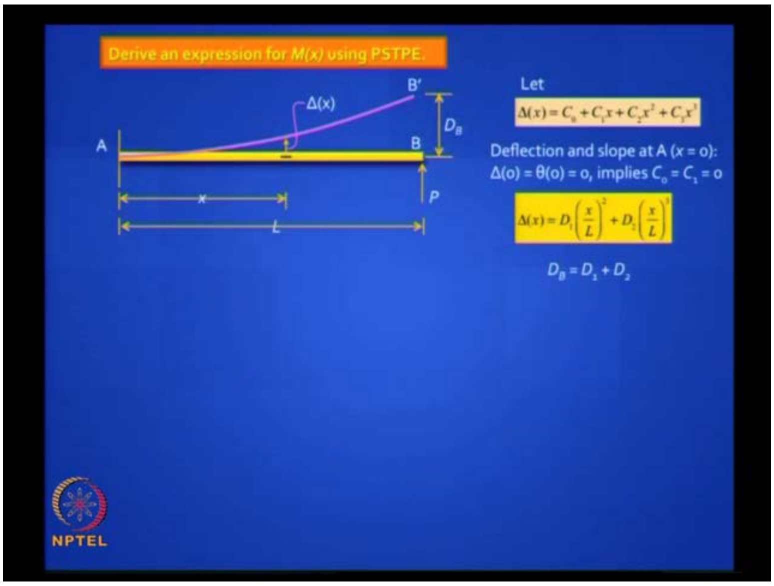

Let us demonstrate how you do this. Take that same cantilever beam problem. Let us derive an expression for bending moment, not using the conventional direct equilibrium, but we go the long winded way through energy formulation, assuming a displacement profile and figuring out what could be an expression for bending moment.

Now, let us read as exactly as possible. We know very well that the deflection function should be a cubic function. Why should be it a cubic function?

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It is because you are dealing with a concentrated load and hence linearly varying bending moment, and if the bending moment varies linearly, curvature varies linearly, whereby the slope will vary quadratically and the deflection would vary cubically.

Let us take a polynomial cubic equation: C_0 plus C_1 x plus C_2 x square plus C_3 x cube. If we invoke the boundary condition, you have kinematic boundary conditions at x equal to 0; the deflection and slope are zero. And that equation will simplify to this equation (Refer Slide Time: 21:12), which is clean cubic equation having two components: one - involving a square term and other - involving cube term at consonants D_1 and D_2, which are now the independent parameters, we were looking for. So, we can say, the deflection at the free end is D_1 plus D_2, because when you put x equal to L, that equation degenerates to this one.

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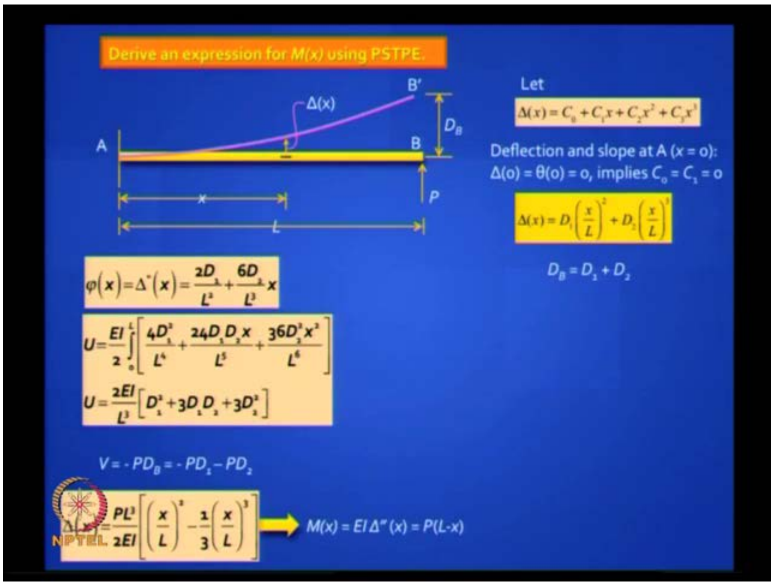

You can also derive an expression for curvature by taking the second derivative of delta x. Using this expression for curvature, you can get an expression for strain energy.

We have done this earlier; so, I am not going in depth. You have second derivative, which is assumed to be equal to the curvature. From curvature, you can get strain energy. When you integrate this, I am going fast over this, you get that expression.

You look at the total potential energy, which is V equal to minus PD_B. You substitute D_B as D_1 plus D_2.

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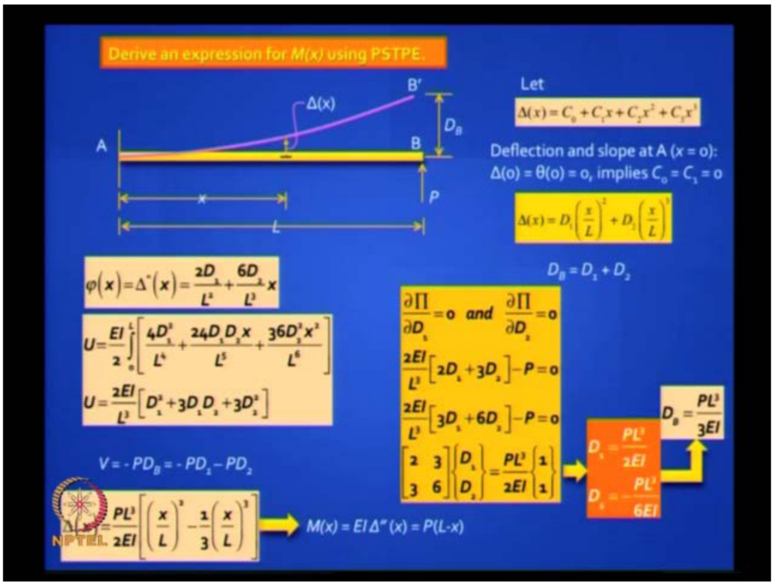

In the next step, you can invoke the principle of minimum potential energy and take the two equations -- dou Pi by dou D_1 equal to 0 and dou Pi by dou D_2 equal to 0. You can solve them simultaneously and get exact values of D_1 and D_2. You get back the expression that we had derived earlier for deflection, including…of course, there we had shear deformations, but without shear deformations, you get L cube by 3EI.

Once you have the expression for delta x, because If you look here, in this expression, D_1 and D_2 are unknown. Once we invoke this theorem, we get the values of D_1 and D_2. If you plug-in these values into that equation, you will get the exact equation for the deflection function. You know that, from this you can get the curvature and from the curvature by multiplying with EI you can get the bending moment. That bending moment equation, P into L minus x is 100 percent correct. You know that. You can check it through equilibrium.

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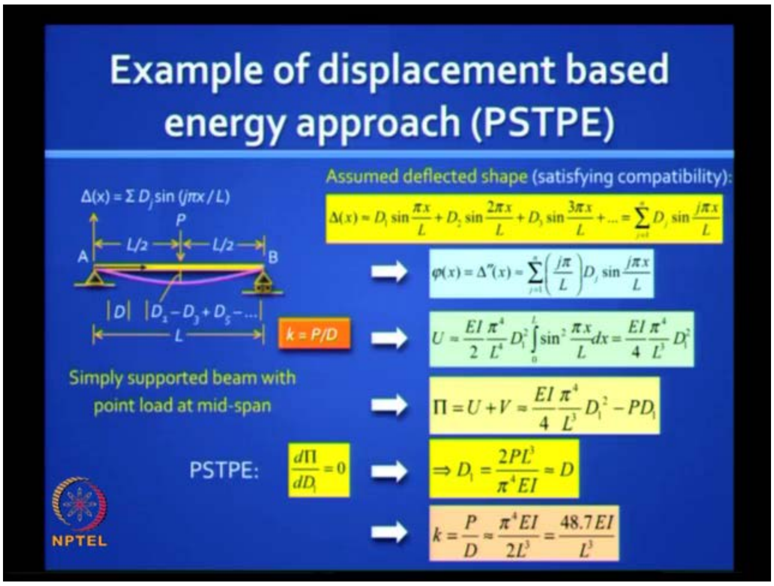

This is an alternative way of establishing equilibrium. Not particularly useful in this example, which you could have solved more easily using direct equilibrium.

Another common example is to find, assume a deflection shape which satisfies compatibility to some extent. Take a simply supported beam, assume a series function, which satisfies compatibility, let us say, a sin-o-swaddle series. Let the mid span deflection be D. So, D can be expressed as a function of many independent other functions, D_1, D_2, D_3, etcetera and the definition of stiffness would be K equal to P by D, where P is the concentrated load acting at the midspan.

If you repeat this exercise - find the curvature, find the strain energy, write the expression of total potential energy and if you take one term, for example, assume that the higher order terms are not important, you invoke this principle and get an expression for deflection, and thereby for stiffness, which is quite close to the exact expression. What is the exact expression? 48EI by L cube. You got a reasonable good expression. These are ways of using this principle.

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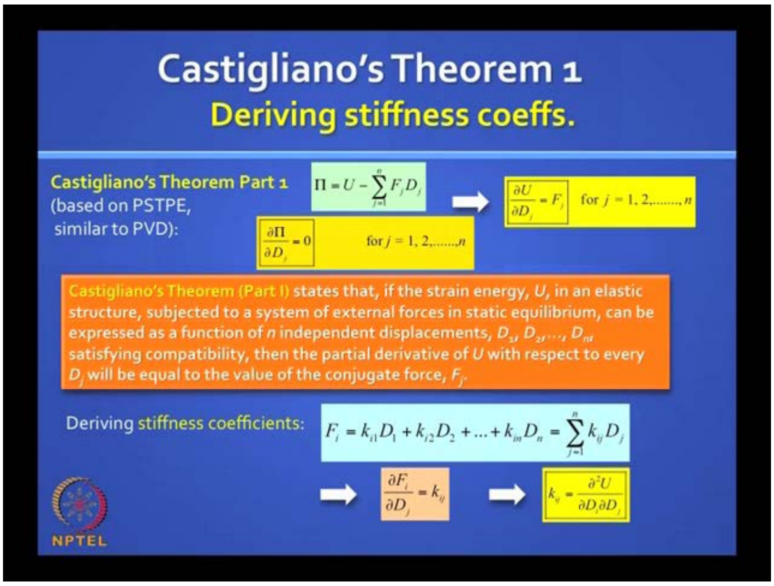

The more popular use of this is Castigliano's Theorem, which can be derived from this principle. So, Pi can be written as U plus V. V is written as minus sigma F_j D_j. When you invoke the principle of minimum total potential energy, you will find that this will reduce to dou U by dou D_j, which is equal to F_j. In this form, it is known as Castigliano's Theorem - Part I. It is similar to a principle of virtual displacement.

This theorem states that strain energy U in an elastic structure, mind you, we are not saying it should be linearly elastic, subjected to a system of external forces in static equilibrium, can be expressed as a function of independent displacements D_1 to D_n, satisfying compatibility, then the partial derivative of U with respect to every D_j will be equal to the value of the conjugate force F_j.

It is actually same as the earlier principle, expressed in another form. You can use this to actually derive stiffness coefficient. You can prove that K_{ij} is the second mixed partial derivative of the strain energy with respect to D_i and D_j. This proof follows from… So, you get from Castigliano's theorem, which starts with this expression - dou U by dou D_j is equal to F_j. You can derive an expression for K_{ij}, which is dou square U by doi D_i dou D_j.

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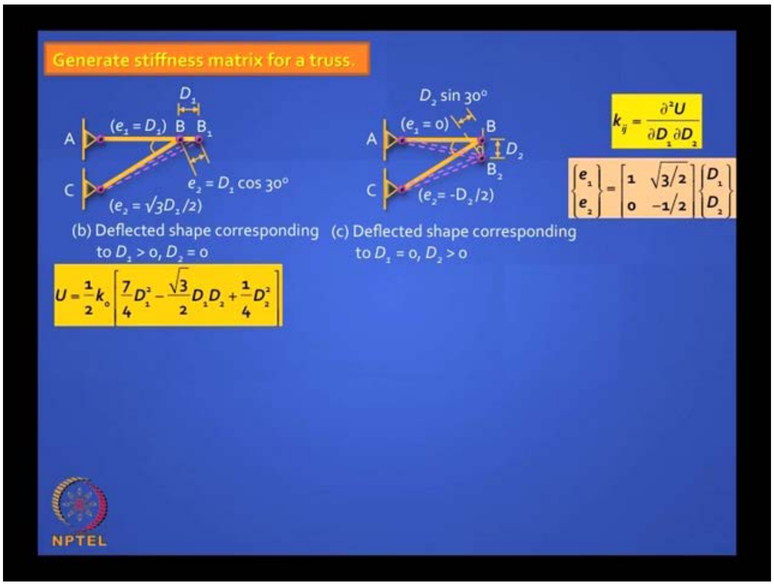

Let us demonstrate this with a simple example. Here are two bar truss. It has two degrees of freedom. Joint D can move horizontally, it can also move vertically, and the elongations e_1 and e_2 can be expressed in terms of D_1 and D_2. We can easily prove this. e_1 is equal to D_2 and e_2 is equal to D_2 cos 30 degree.

The first diagram is what you do when you allow only D_1 to occur and D_2 is restrained. The second shape is when you allow D_2 to occur with D_1 is restrained. You can work out the relationship between bar elongations and the deflections. You can write them in a form: e_1 and e_2 are related to D_1 and D_2 in that form. So, if someone gives D_1 and D_2, you get e_1 and e_2.

You can write an expression for strain energy. How do you write the strain energy expression? What is the strain energy for a spring element? Half K into elongations square. You have elongations here. Let us say, both elements have the same stiffness K_0. Half K_0 into e_1 square, plus half K_0 into e_2 square. e_1 is equal to D_1 plus root 3 by $2D_2$

and e_2 is equal to minus half D_2. You plug in those values, you can write an expression for U, in terms of D_1 and D_2. I am going fast. You can verify this.

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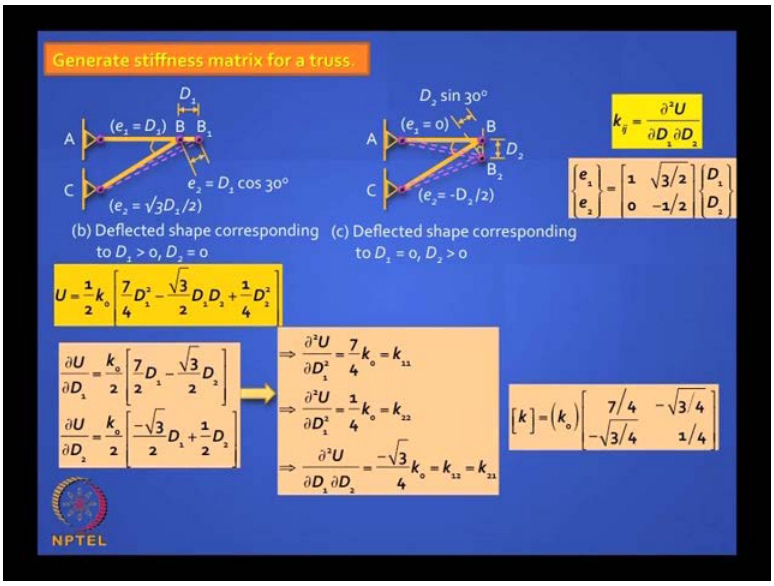

Now, if you take partial derivative of D_1 and D_2, you will get two expressions. If you go to the fundamental definition of stiffness matrix, stiffness coefficient k_{ij} dou square dou D_1 dou D_2, you can derive values of K_{11}, K_{22}, etcetera in this manner. This is a hard way of doing it. We will be studying matrix methods, where you do not need to do all this. You can generate it automatically, but this is the original background to the derivation.

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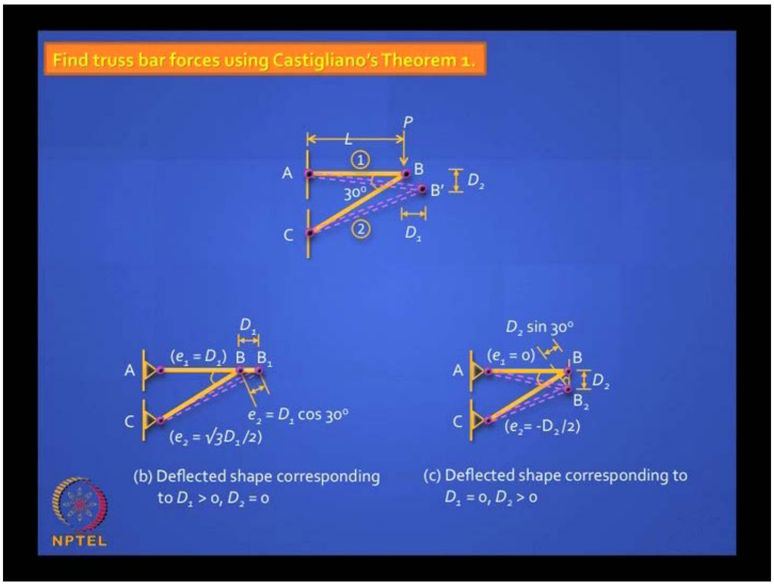

You can also use it to find unknown bar forces.

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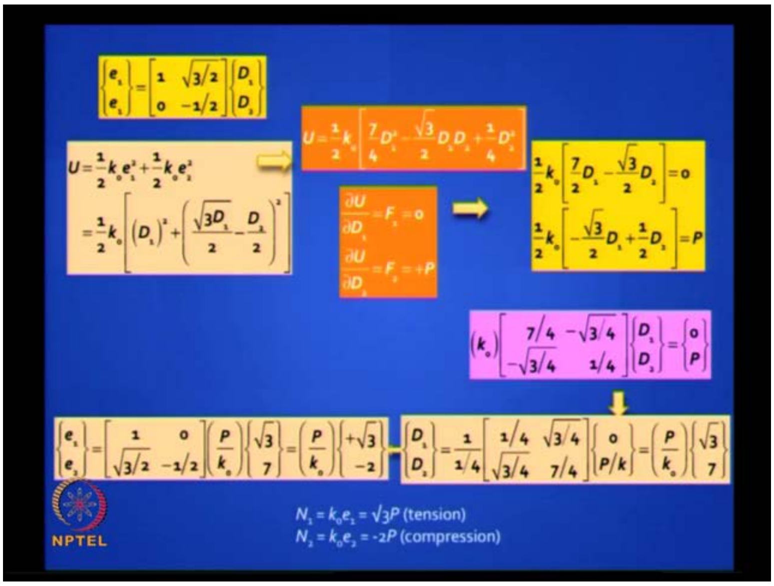

For manual use, usually, we would find unknown forces directly. You would not be using Castigliano's first Theorem, because it is more difficult. The real use of these theorems, is to find unknown displacements, for which you need to shift.

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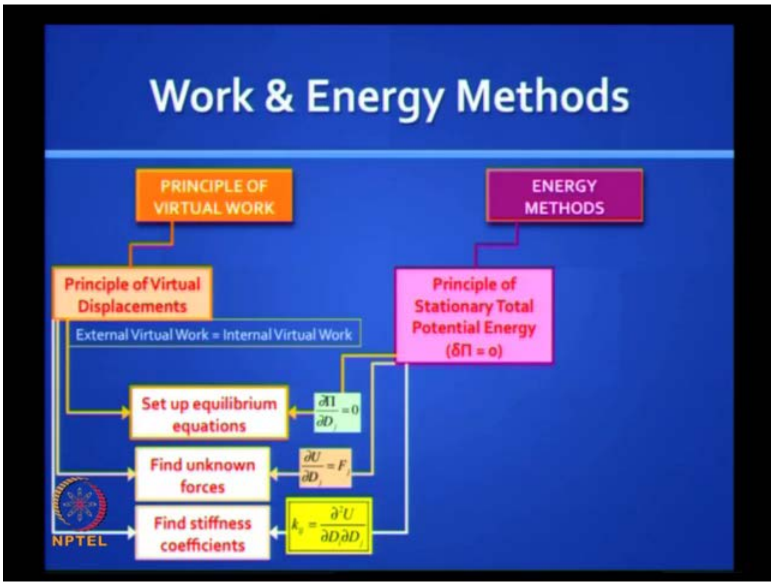

From this set of theorems, you have noticed that we finished the use of displacement based theorems. You can use it to set up equilibrium equation**s** dou Pi by dou D_j equal to 0. You can use it to find unknown forces at Castigliano's first theorem dou U by dou D_j

equal to F_j. You can also use it to find stiffness coefficient K_{ij}. These are powerful uses, at least theory-wise you should be familiar with these terms.

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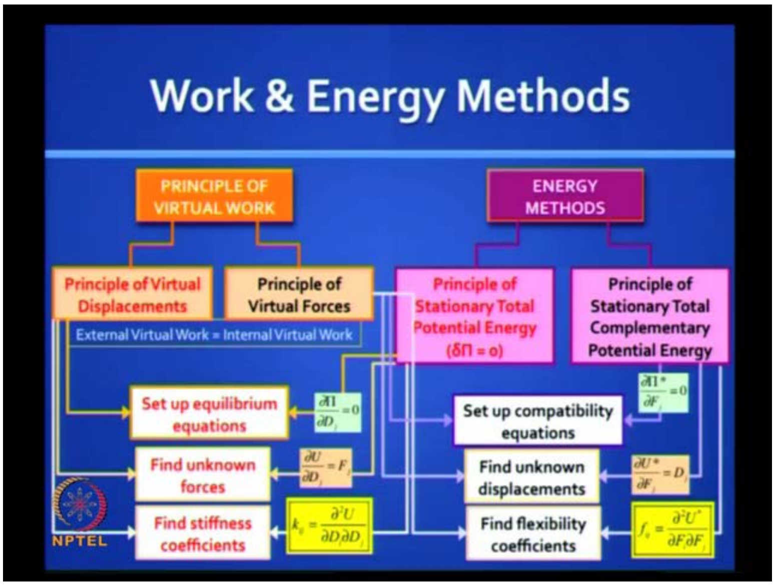

We look at the force field, which is related to the principle of virtual forces. You have corresponding principles: corresponding to delta Pi equal to 0, you have delta Pi star equal to 0. So, you would call that theorem - the principle of stationary total complementary potential energy.

You see a parallel. You use this to find some unknown displacements in that displacement field. So, instead of setting of equilibrium equations, you now set up compatibility equation. Instead of finding unknown forces, you find unknown displacements. Instead of finding stiffness coefficient, you find flexibility coefficient.

You are doing a similar exercise and you will see a beautiful symmetry in these relationships. You can do the same thing using work methods, without getting into energy and that would be a use of the principle of virtual forces.

Can you see this map? With practice, you will be familiar with these different approaches.

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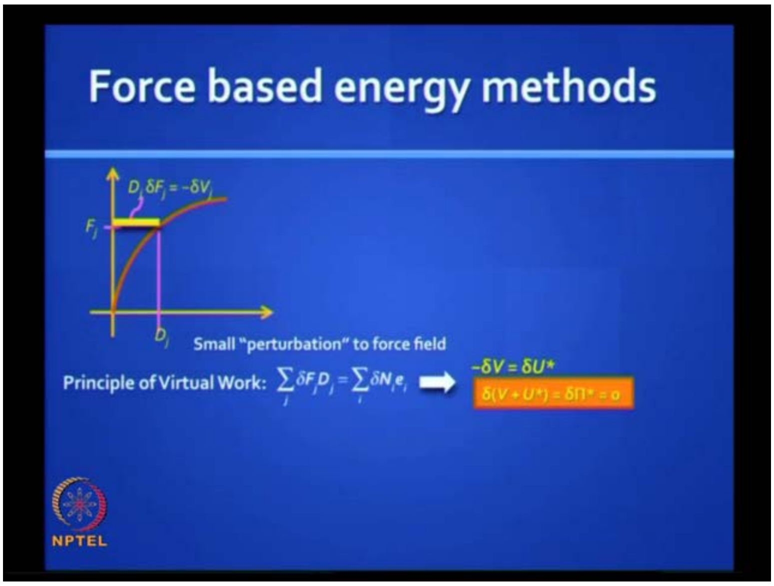

Now, I will go fast. You are familiar with the way of deriving this theorem. Here, you go back to the truss and instead of disturbing the displacement field, you disturb the force field. You increment those forces positively or negatively by a very small component and invoke the principle of virtual work. You will find delta V plus delta U star, which is delta Pi star, is equal to 0. You can prove this in the same way. There is no need to explain it further. So, you have the external incremental change in work and the change in strain energy.

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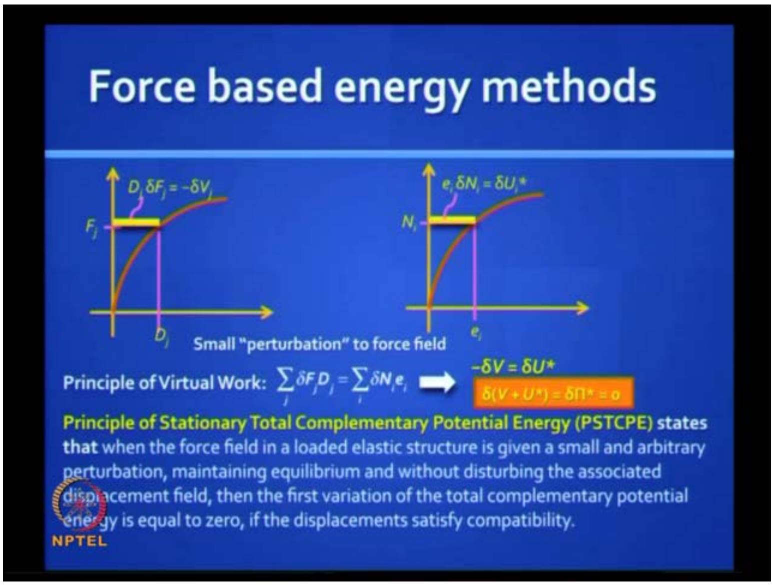

The principle in this form states that when the force field in a loaded elastic structure is given a small and arbitrary perturbation, maintaining equilibrium and without disturbing the associated displacement field, then the first variation of the total complementary potential energy is equal to 0, if the displacements satisfy compatibility.

If you write this equation side by side with the earlier equation, which is a principle of stationary total potential energy, you will find many similarities and differences. The format is same, but there you are disturbing the displacement field; here, you are disturbing the force field. When you are disturbing the displacement field there, you are not changing the forces. When you are disturbing the force field here, you are not changing the displacement. That is important to note.

It is something you do in your mind. It is arbitrary and a very small value. The cause that you refer to here is perturbation and the effect is variation. Those are the terms used.

Now in the first one, when you disturb the displacement field, you are maintaining compatibility. Here, when you are disturbing the force field, you are maintaining equilibrium. When you find the first variation delta Pi here, delta Pi equal to 0 is a statement that establishes equilibrium in the force field, although you disturb the displacement field. Whereas, here, delta Pi star equal to 0 is a statement of compatibility which you get, although you disturb the force field. So, there is symmetry in these relationships

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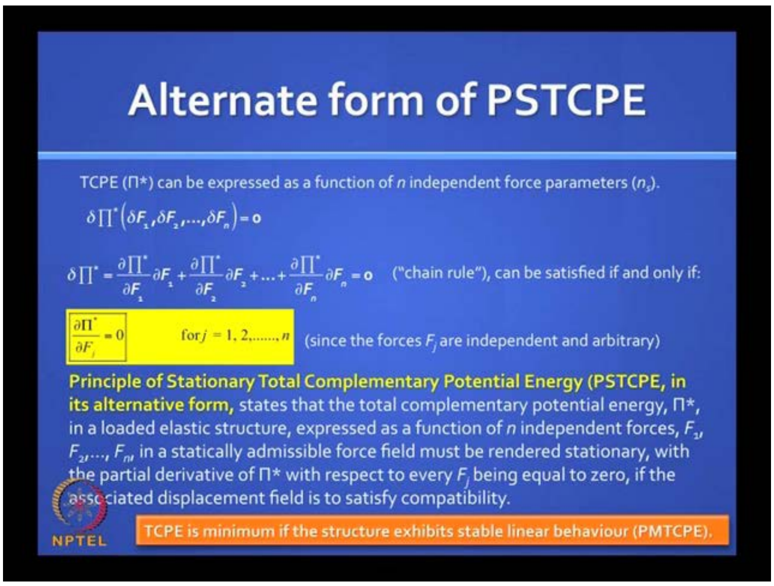

Expression of compatibility - you can go through the same procedure and can find the alternative form, which reduces to dou Pi star by dou F_j equal to 0. Earlier, it was dou Pi by dou D_j equal to 0. That one equation, that is, the first variation of P_i star equal to 0, now multiplies into n number of equations, depending on the number of independent forces that you get.

So in this alternative form, it states that the total complementary potential energy Pi star in a loaded elastic structure, expressed as the function of n independent forces F_1 to F_n in a statically admissible force field. That is, force field, which satisfies the equilibrium must be rendered stationary, with the partial derivative of Pi star with respective to every F_j being equal to 0, if the associated displacement field is to satisfy compatibility.

Here again, if you are dealing with stable linear elastic structure, the condition of stationarity reduces to a condition of minimal value of the function, which is Pi star.

In this form, it is called PMTCPE, that is, Principle of Minimum Total Complementary Potential Energy. Mouthful of words, but it is a concept that you need to remember.

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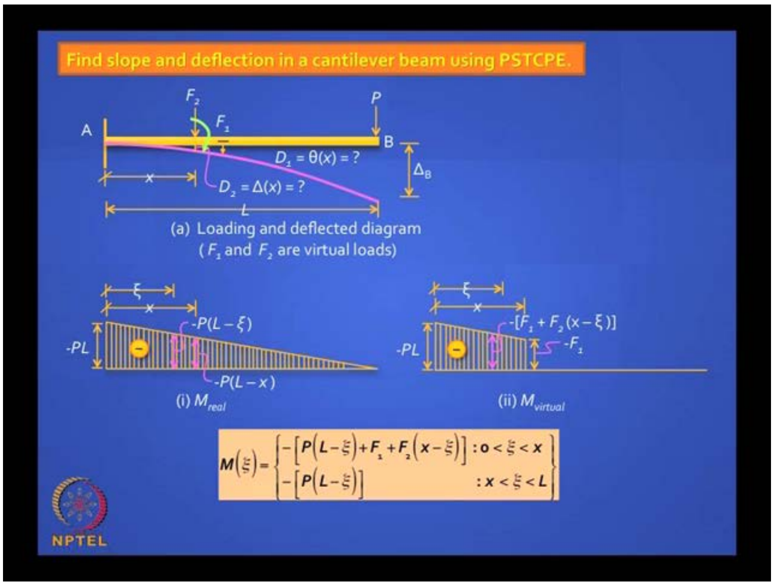

Let us have a look at one simple demonstration. Let us take a cantilever beam, but this time, it is different. You have many loads acting. You have one load. Let us take load P and you want to find the slope and deflection at some arbitrary location x.

In this theorem, if you want to find D_1 and D_2, D_1 is the slope at x and D_2 is the deflection at x. You have to introduce imaginary corresponding conjugate forces, F_1 and F_2. Later, put them equal to 0, because that is how you generate the equation. You are familiar with Castigliano's Theorem. So, write down the bending moment expression. You can separate out the real one caused by P, which is a straight line, and the one caused by imaginary F_1 and F_2, which is also a straight line but not starting. It is exactly as shown. You can write down the value.

At any location, zi has two forms. You have to break it up into two parts: one up to x, and one beyond x. Beyond x, you will find that F_1 and F_2 do not have a role to play because there is no bending moment cause by F_1 and F_2. So, you have got this expression for bending moment at any location zi.

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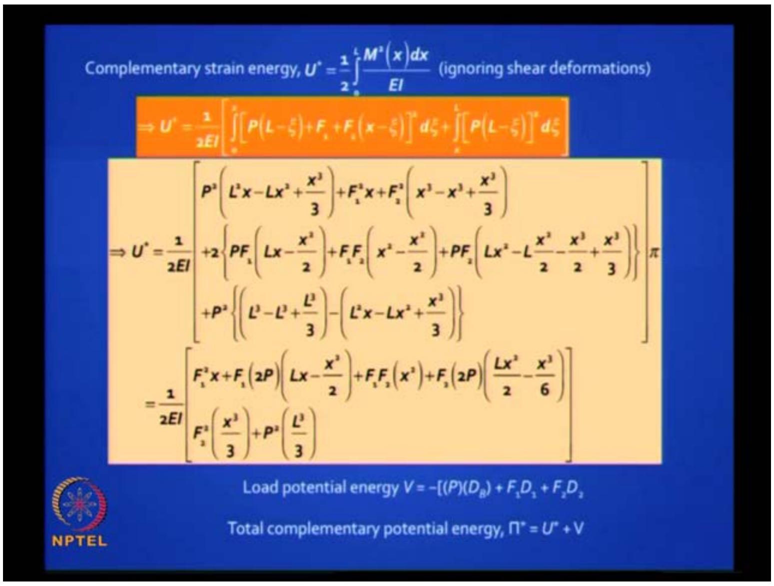

You can expand this by ignoring shear deformation. Write an expression for U star can be generated easily. This is U star -- the full form of U star. This U star is a function of not only p, but F_1 and F_2.

Now, invoke the theorem, load potential energy Pi star. What is V? V is minus P into the total deflection under the load plus the deflections caused by the imaginary F_1 and F_2.

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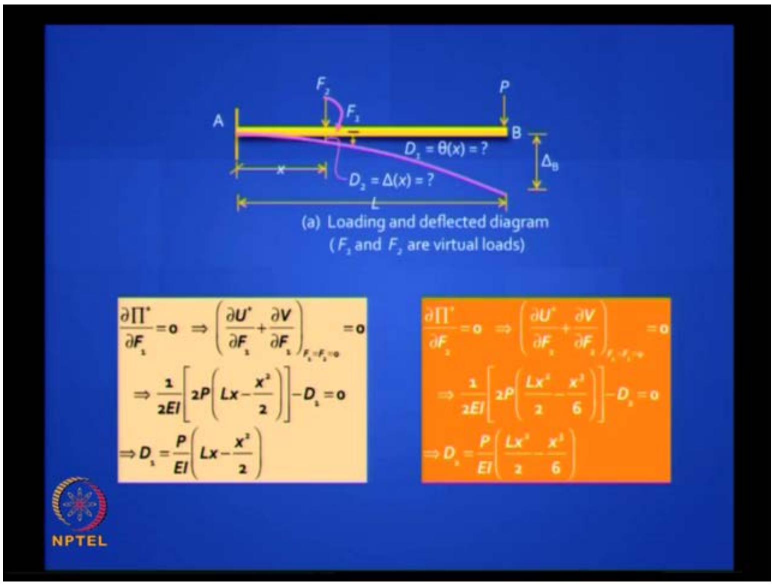

You will get that expression and invoke the theorem - dou Pi star by dou F_1 equal to 0. When you invoke this expression and take the final form, you should insert the values of F_1 and F_2 equal to 0. Similarly, take the second equation - dou Pi star by dou F_2 equal to 0. You get the values of D_1 and D_2. You get the slope and deflection using this energy formulation.

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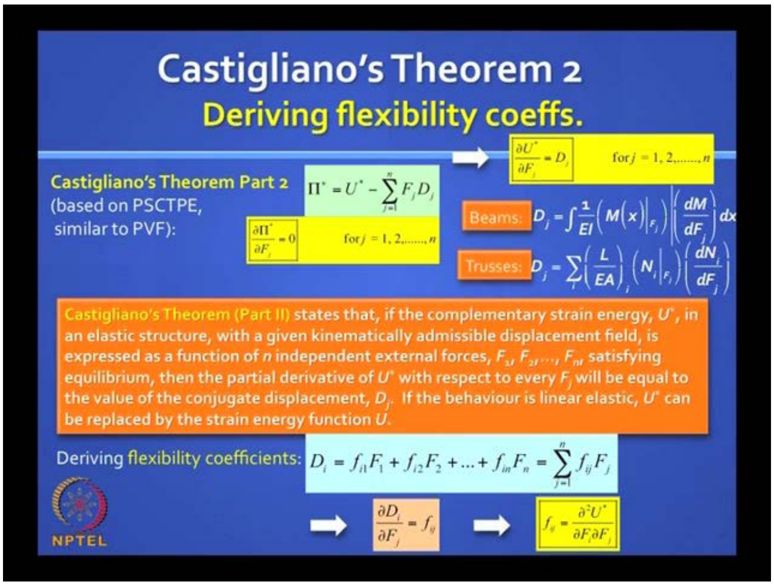

You can do the same thing using Castigliano's Theorem, part II. Here again, you can prove using a similar procedure, dou U star by dou F_j equal to D_j, for 1 to n. Here, you can take advantage of the fact that U is equal to U star for linear elastic behavior, and for beams, you get that expression. Can you say something about these expressions? Have you encountered these expressions earlier, in a different form? For beams, it takes this form.

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If you recall the unit load method which is the principle of virtual forces, there also we have 1 into D_j equal to these quantities. This is the internal work. Now, the similarity is everything is same, except this expression, that is, **"**What is dM by dF_j? It is a small m_j that we refer to that. It is the bending moment caused by F_j equal to 1. So, if you look at it carefully, they are all the same. There are only different ways of approaching the same problem. We are doing the same kind of integration. You can do area multiplication, whether it is a beam or a truss.

In summary, this theorem, in its part II says, if the complementary strain energy used, in an elastic structure with the given kinematically admissible displacement field, is expressed, the function of n independent external forces F_1 to F n, satisfying equilibrium, then the partial derivative of U star, with respective to every F_j, will be equal to the value of conjugate displacement D_j. If the behavior is linear elastic, U star can be replaced by the strain energy function u.

Again, you can expand D in terms of flexibility coefficients. Like in the earlier case, you can use energy methods to get an expression for flexibility coefficient. F_{ij} is a mixed partial derivative of U star, with respect to F_i and F_j. The similarity is now complete with stiffness coefficient.

There is a special application of this theorem. It is known as theorem of least work to solve statically indeterminate structures. Let us say, you have a continuous beam and you want to choose the redundant reactions as your redundants. So, x_1 to x_j dou U star to dou x_j actually denotes what? By Castigliano's Theorem, it denotes that displacements at those support locations. Those supports do not move; so, the displacements are zero. You can also interpret this as the minimization of strain energy. We will discuss it shortly.

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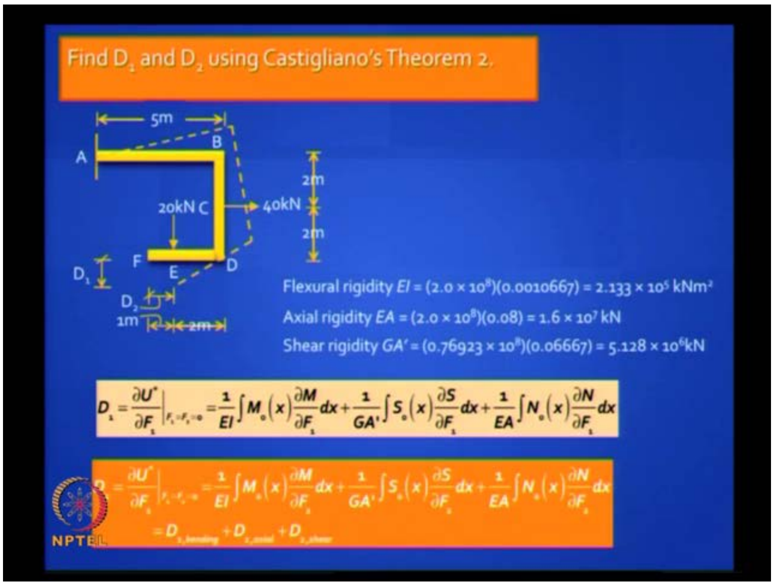

That is called the theorem of least work. Let us demonstrate this with a problem. You remember we did this problem, finding D_1 and D_2 using unit load method. Here, let us make it more complex by including actual deformation and shear deformations. I will go through it fast. You can find it using Castigliano's Theorem.

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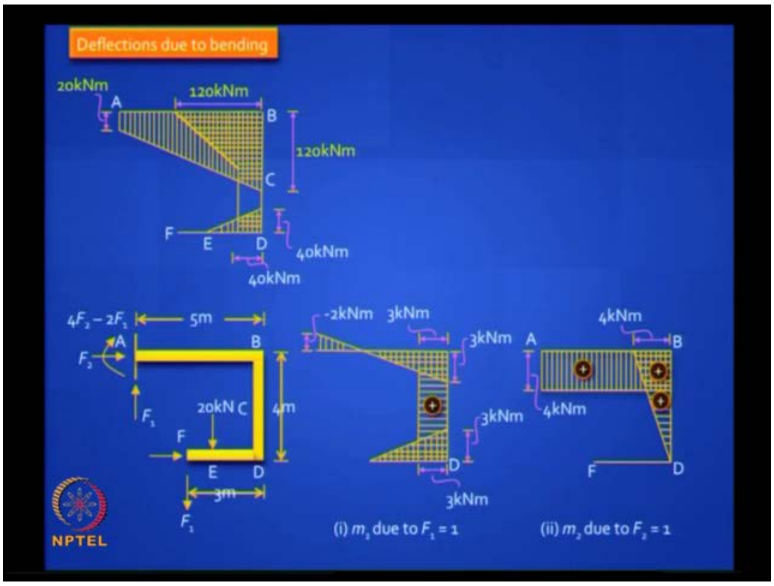

First, find the deflections due to bending. You have a bending moment diagram. To find D_1, you apply F_1 equal to 1. You will get the unit load bending moment diagram m_1. To find D_2, you find m_2. These are nothing but dou U dou capital M divided by dou F_1 and dou F_2.

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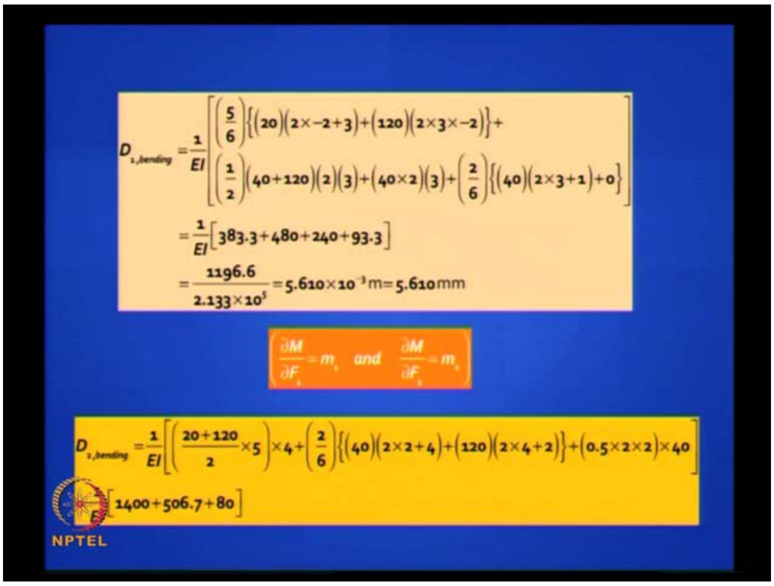

You can work out the same method. It resembles a unit load method and you can find that the deflection caused by bending is 5.61 at point one and if you want at D_2, it is something. You can also find deflection caused by shear using these similar expressions.

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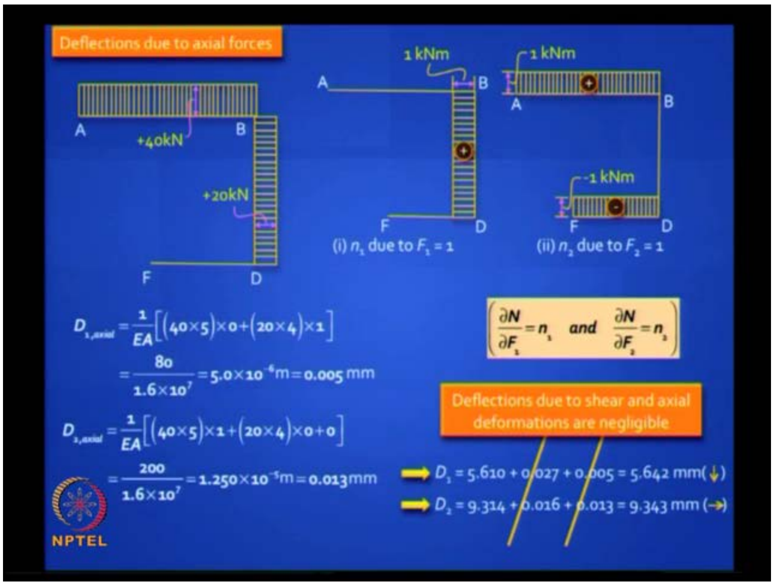

At the end of day, we are interested in total values. You will find that, the total deflection D_1 and D_2 is 5.64 and 9.34 mm downward and to the right. Those additional terms that we have cut here are terms at come from shear and axial; they are actually negligible. This is again another proof why we can ignore shear deformation and actual deformations in normal frames which are well proportion.

Earlier, we have said that the energy caused by shear and energy caused by actual forces, is negligible. Now we are saying it is not just the energy, but also the deflection. So, do not bother; make your life easier. When you see a frame, worry only about bending-flexural strain energy. When you see a beam, do that unless the beam is deep. When you see a truss, only action strain energy.

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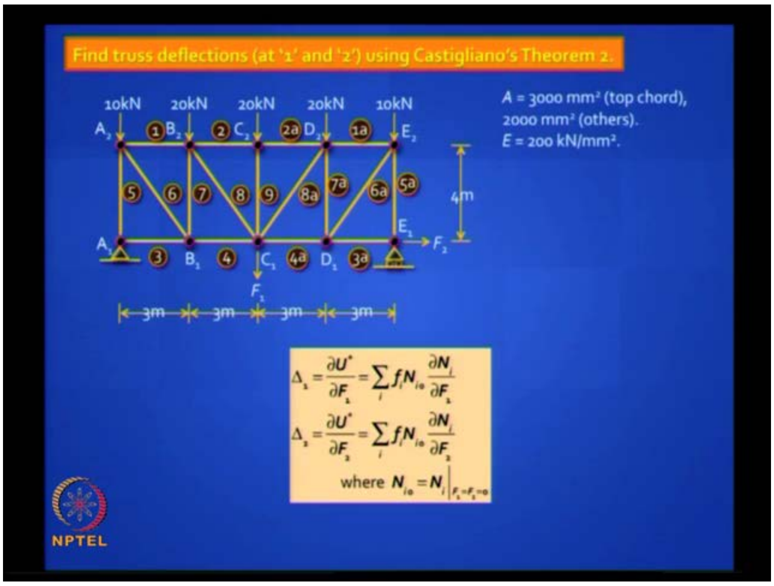

You can use it to find truss deflections. Same method but little complicated. You have many members here. You are given the areas of cross section of each member, given model of the velocity, you can use Castigliano's Theorem to invoke the deflection. There are many points of your interest.

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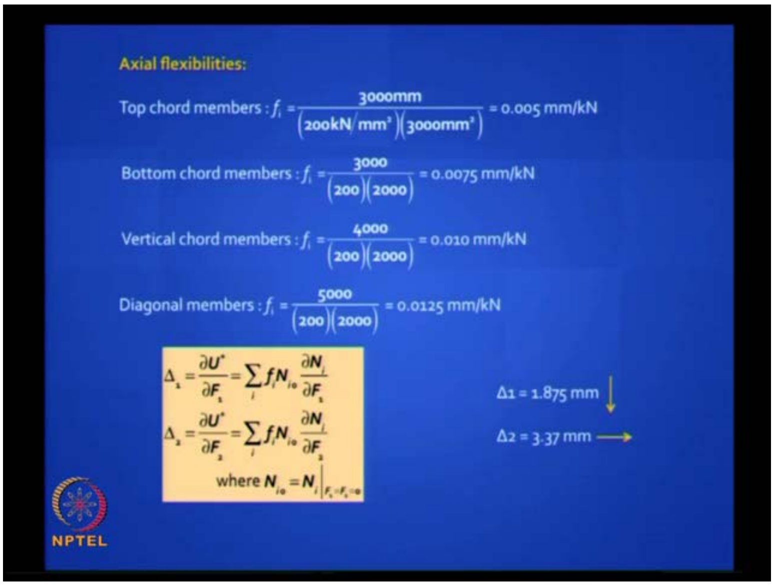

First you need the actual flexibilities of all the members. Then invoke this equation and we can prove this. These are exercises that you need to do. We are just reviewing something that you already learnt.

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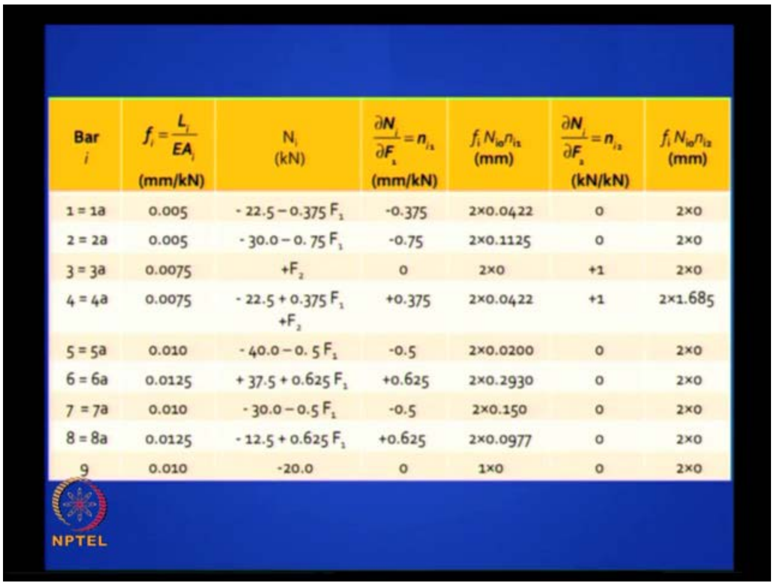

Bar i	$f_i = \frac{L_i}{EA_i}$ (mm/kN)	N_i (kN)	$\frac{\partial N_i}{\partial F_1} = n_{i1}$ (mm/kN)	$f_i N_{io} n_{i1}$ (mm)	$\frac{\partial N_i}{\partial F_2} = n_{i2}$ (kN/kN)	$f_i N_{io} n_{i2}$ (mm)
1 = 1a	0.005	$-22.5 - 0.375 F_1$	-0.375	2×0.0422	0	2×0
2 = 2a	0.005	$-30.0 - 0.75 F_1$	-0.75	2×0.1125	0	2×0
3 = 3a	0.0075	$+F_2$	0	2×0	+1	2×0
4 = 4a	0.0075	$-22.5 + 0.375 F_1 + F_2$	+0.375	2×0.0422	+1	2×1.685
5 = 5a	0.010	$-40.0 - 0.5 F_1$	-0.5	2×0.0200	0	2×0
6 = 6a	0.0125	$+37.5 + 0.625 F_1$	+0.625	2×0.2930	0	2×0
7 = 7a	0.010	$-30.0 - 0.5 F_1$	-0.5	2×0.150	0	2×0
8 = 8a	0.0125	$-12.5 + 0.625 F_1$	+0.625	2×0.0977	0	2×0
9	0.010	-20.0	0	1×0	0	2×0

You can do it in neat tabular format and get the answers.

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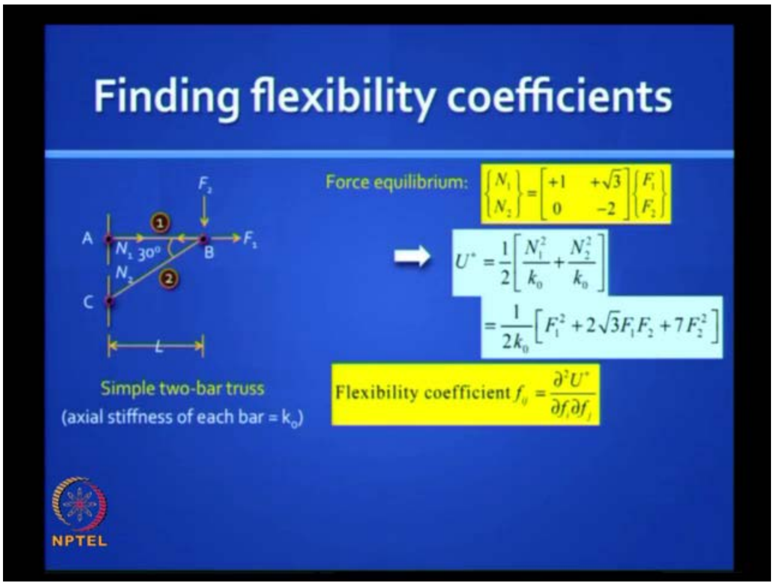

You can also find the flexibility coefficients, just the way we found the stiffness coefficients. It is similar, once you get the hang of it. Earlier what did we do? We wrote a relationship between e_1 e_2 and D_1 D_2. Now, we write a relationship using equilibrium between N_1 N_2 and F_1 F_2. We will study in matrix methods that this coefficient matrix you get here is actually the transpose of the other matrix, which we got in displacement method. We will give a formal proof later.

Now you write an expression for complementary strain energy. You must remember that Castigliano's Theorem simplified everything because it got rid of load potential energy and it got rid of total potential energy. So, you will have only strain energy and complementary strain energy. Many students studying the structure analysis remember only that; it is good to remember the background, which includes load potential energy terms.

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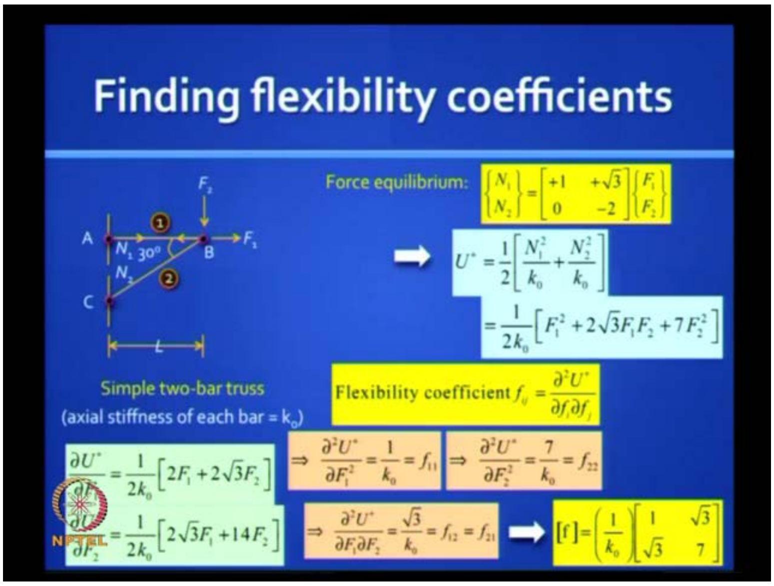

So, you can invoke it in this example. Exactly similar operation and you can find an expression for flexibility matrix. If you go back to the same problem we did earlier, you will find that. This matrix is related to K matrix. How? One is the inverse of the other. You can prove it.

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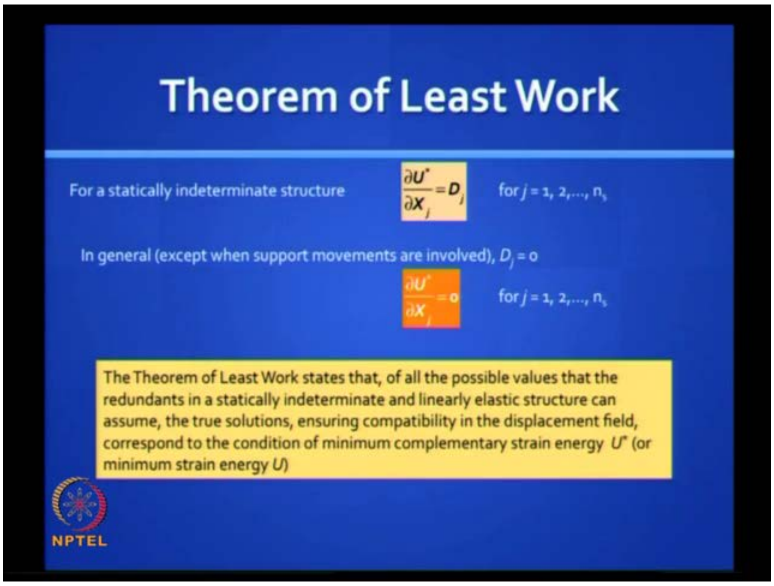

We will end with this theorem of least work, which I have already explained. For a statically indeterminate structure, dou U star by dou X_j is equal to D_j. In general, when support movements are involved, D_j is zero. This is even true for internal indeterminacy.

You can give this an interpretation. When you say dou U star by dou X_j is equal to 0, you can do it as a minimization of complementary strain energy.

So, the expression goes this way - The theorem of least work states that, of all the possible values that the redundants in a statically indeterminate and linearly elastic structure can assume, the true solutions, ensuring compatibility in the displacements field, correspond to the conditions of minimum complementary strain energy U star, or, because it is linear elastic, you can say, minimum strain energy U because both these terms are equal. Simple demonstration which I asked you in earlier class.

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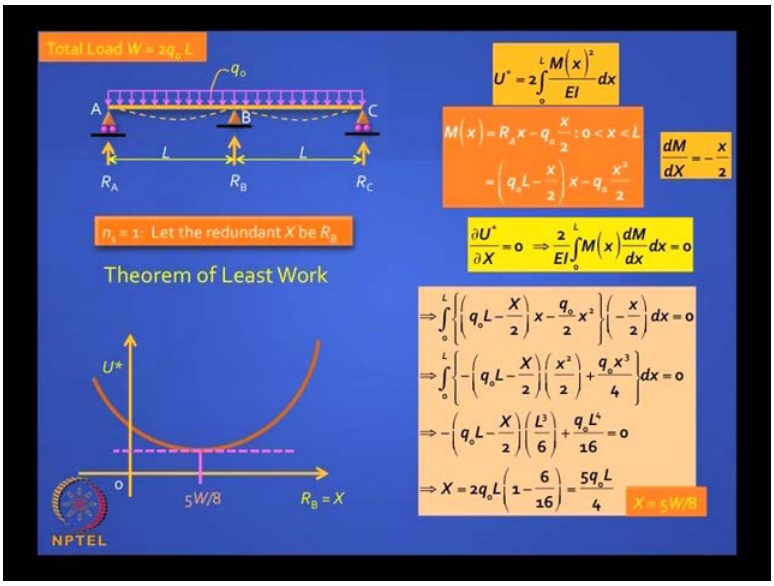

Let us say you want to solve this problem. The support reaction in the middle is treated as the unknown x. You remember I said every student in this class can give his own value of x. You can compute strain energy and if you actually compute strain energy, which means you have to integrate the bending moment diagram, take the square of it, and so on. You will find that, for any value of x, you can get complementary strain energy. You can even assume foolish values of x, which go negative. You know it is not going to be negative. You will find that if you plot x, every value of x will give you another reaction which is statically admissible. You will find that you will get some strain energy, but the strain energy value will be high. The exact solution - the correct solution is one for which the strain energy is minimum.

You can prove this. You can write an expression for M, bending moment at any location. Write an expression for U star and take the derivative. And you can prove that there are series of steps with which you can get the final answer.

So, with this, we have completed review of structural analysis I. In the next sessions, we will cover part-II, in which, first half will cover force methods, including theorem of least work, and the second half will cover displacement methods, which you have not yet studied.

Thank you.