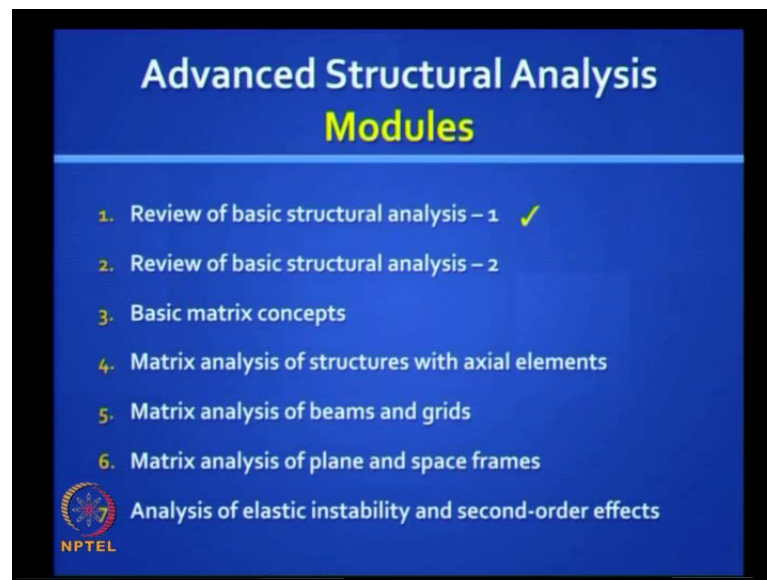


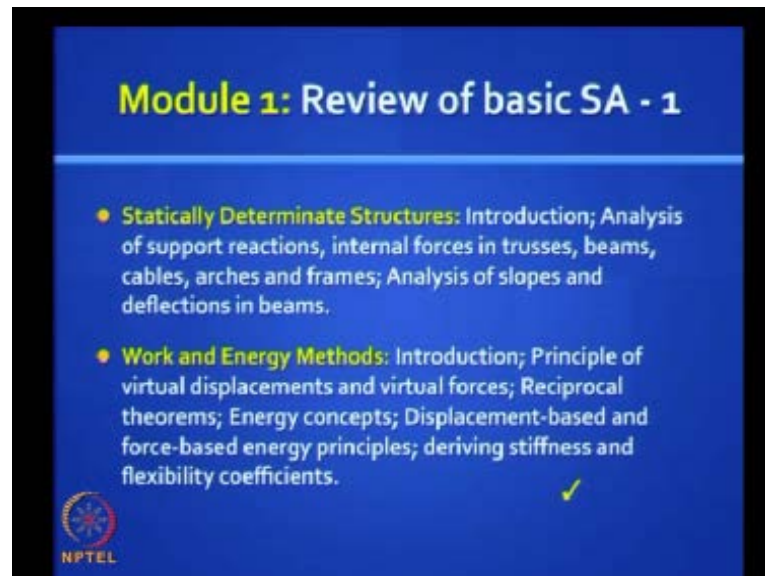
Advanced Structural Analysis
Prof. Devdas Menon
Department of Civil Engineering
Indian Institute of Technology, Madras
Module No. # 1.5
Lecture No. # 05

Review of Basic Structural Analysis-1

Good morning. We are now on to lecture number 5 on advanced structural analysis. We are still with the topic of basic structural analysis; we are doing a review.

(Refer Slide Time: 00:27)





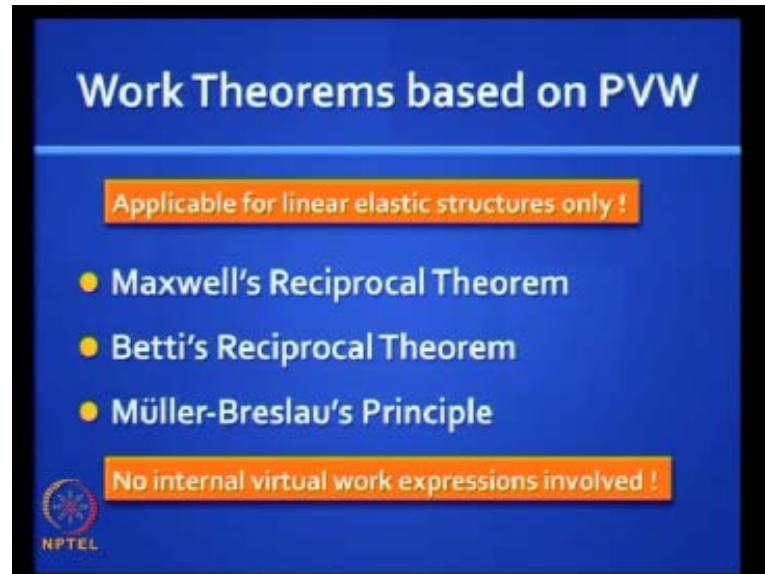
We had completed statically determinate structures. We had also looked at the work methods in the last class. We need to complete that portion and then we will do a review of energy methods in this session.

(Refer Slide Time: 00:45)



As you know, we are referring to this book on structural analysis and right now we are on to part III. The focus is on finding displacements in statically determinate structures, but we are also covering related areas. When you talk about work and energy methods, you can actually find the force response as well.

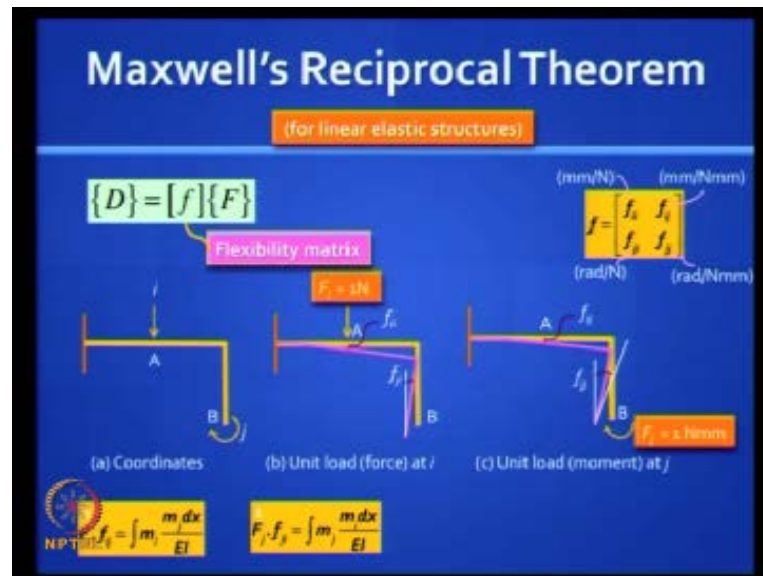
(Refer Slide Time: 01:11)



We ended the last class by discussing that there are three important theorems based on the principle of virtual work. Historically, these are very important theorems in the development of structural analysis. The Maxwell that we refer to here is the famous James Clark Maxwell who made many contributions to physics.

The reciprocal theorems, first contributed by Maxwell and later improved upon and generalized by Betti of Italy, actually gave the first major breakthrough in the analysis of statically indeterminate structures; they are beautiful theorems. The important point to note is that these theorems are applicable only for linear elastic structures. Unlike the principle of virtual work which requires you to compute both external virtual work and internal virtual work, here we skip the calculation of internal virtual work altogether.

(Refer Slide Time: 02:25)



Maxwell's Reciprocal Theorem in its modern form basically establishes that the flexibility matrix is always symmetric. In the language of flexibility matrices and coordinates, let us take any structure linearly elastic. Let us look at that cantilever frame and we mark two coordinates. Deliberately, we have taken two different types of degrees of freedom. What is indicated as i is a translation and j is a rotation. The conjugate forces would be a direct action force at i and f_j would be a moment.

Now, the flexibility matrix in this case would be a two-by-two matrix and you can write it like that. The four flexibility coefficients are f_{ii} , f_{ij} , f_{ji} and f_{jj} . **The physical meaning of these quantities are...** You can see that if I want to get the first column in this flexibility matrix, then I have to apply a unit load F_i equal to 1. Physically, that would be given by this picture (Refer Slide Time: 03:55). I apply a unit load at i and the deflection that I get at the coordinate i is referred to as f_{ii} and the rotation that I get at the coordinate B is referred to as f_{ji} . This follows from our discussion yesterday.

Similarly, if I want to get the second column in this flexibility matrix, I apply f_j equal to 1, which is actually a moment. The rotation I get at the location where I apply the moment is called f_{jj} and the deflection at this i location is f_{ij} . You would recall that f_{ij} is written in such a way that the indices i will refer to the effect location and j will refer to the cause location. So, following that format, it can be proved, and this is Maxwell's

reciprocal theorem, that f_{ij} is equal to f_{ji} . It is interesting because the units of f_{ij} and f_{ji} are completely different in this case.

If you look carefully, f_{ji} refers to a rotation; so, the units would be radians, but caused by a unit force and so radians per Newton in this case, whereas f_{ij} refers to a deflection caused by a unit moment; so, it is millimeter per newton millimeter. That is the essential difference, but both reduce to a dimensionless quantity divided by the force unit of Newton. Now, the proof in this case is very elementary; you can prove it using the principle of virtual work.

In the first instance, let us compute or let us establish the principle of virtual work looking at the force field in system I that is caused by f_i and the displacement field in system II that is caused by f_j . Then, do you agree that the external virtual work would be f_i into f_{ij} ? This is force and this is displacement (Refer Slide Time: 06:36). **The internal work is...** You know the units; m_i refers to the bending moment anywhere in that frame caused by f_i equal to 1 – that is the force field; the other one is the elemental rotation m_j into dx by EI .

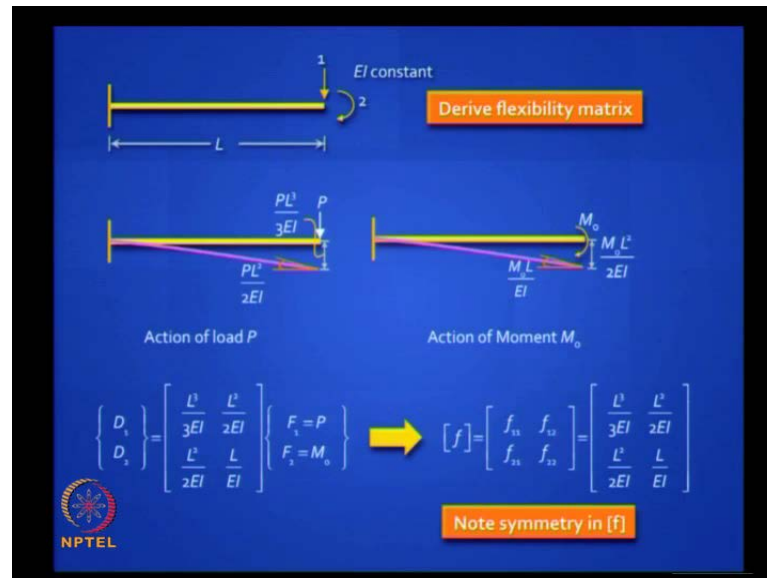
Now, we know this is true. It is true whether you have elastic behavior or not, in general; but to get the curvatures, we are using elastic behavior; that is why EI comes in the denominator, but we can also write another expression for the principle of virtual work, looking this time at the force field caused by f_j and the displacement field caused by f_i . If we were to do that, then you would write it in this manner; it is easy to understand.

Now, if you look carefully at these two equations, you will find that the quantity on the right-hand side, which is integral m_i into m_j into dx by EI is the same. There is a commutation between m_i and m_j , but it is the same quantity. Therefore, it follows that the quantity on the left-hand sides of the two expressions should be the same. Since you are applying a unit load f_i equal to f_j equal to 1, it follows that f_{ij} is equal to f_{ji} . This is a simple elegant proof of establishing the symmetry of this matrix.

Now, I can choose more than two coordinates. I can choose three coordinates; then, the size of the flexibility matrix would be 3 by 3, but it would still be symmetric in the sense f_{13} would be equal to f_{31} and f_{23} would be equal to f_{32} and so on. In summary, in a linear elastic structure, displacement f_{ij} at coordinate i due to a unit force or unit load acting at coordinate j is equal to the displacement f_{ji} at coordinate j due to a unit load acting at

coordinate i. The load could be a force or it could be a moment. Although we have proved this for a frame, you could prove it for a truss as well; there, the force would be an axial force if it is an internal force.

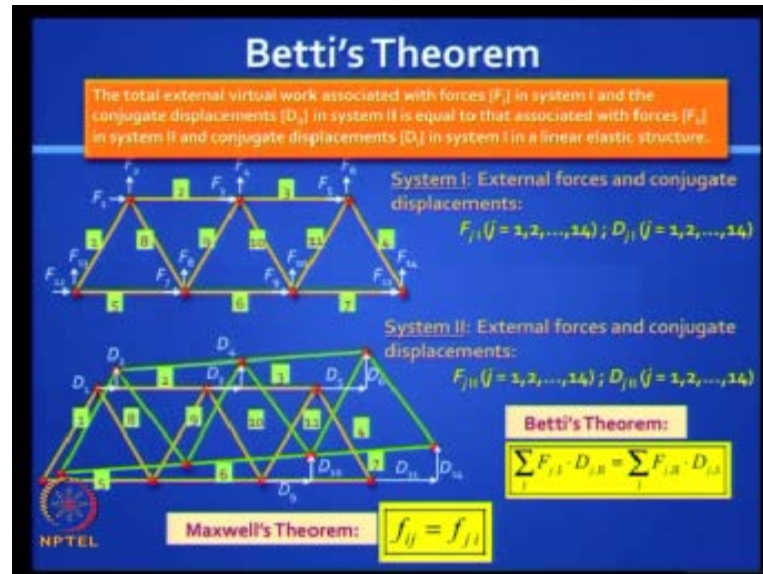
(Refer Slide Time: 09:00)



Let us just demonstrate this with a simple example of a cantilever beam. Let us look at the two coordinates referring to the deflection and the rotation at the free end. If you were to derive the flexibility matrix, you could do that by first applying a load P at the free end and then applying a moment M_0 again at the free end. You can use any method; you could even use the conjugate beam method to work out the deflection and the slopes; those values are very easy to calculate. The deflection caused by the load P is P into L cubed by $3EI$, the rotation is P into L square by $2EI$, the deflection due to the moment is M_0 into L square by $2EI$ and the slope is M_0 into L by EI .

If you were to write these expressions in a matrix form, this is how you would write it. You will notice that you can separate out P and M_0 into the load vector and the coefficient matrix turns out to be a property of the structure, independent of the loading. You can put in any value of M and M_0 and still that flexibility matrix has not changed. So, it is a very useful property of the structure. It is very evident that the matrix is symmetric and that is the proof.

(Refer Slide Time: 10:34)



Betti moved further and gave a more generalized expression of the reciprocal theorem. It is reciprocal in the sense that the effects are reciprocal. Here, Betti talked about the entire system; so, you look at the force field in system I and the conjugate displacements. You look at both the force field and displacements, but you need to look only at the joint forces and displacements; do not worry about the internal forces and deformations.

Let us see whether the same truss or the same structure but a completely independent force and displacement field. So, we will label this as system I, the top one, and the other one as system II. This is F_{ji} and D_{ji} (Refer Slide Time: 11:29); this is F_{ji} and D_{ji} . If you invoke the principle of virtual work, you can easily prove that if you multiply the joint forces in one system with the joint displacements in the other system, obviously there is no correlation between the two – there is no cause-effect relationship.

We call it external virtual work but it is really virtual; you are just multiplying a force with displacement disconnected with that force but at the same location in a clone of the same truss. That product is equal to the summation of the forces that you would get if you were to multiply the forces in system II with the conjugate displacement in system I. This is really reciprocal not only in terms of displacements but also in term to forces and this can be easily proved. You will find that the Maxwell's Theorem is a special case of Betti's Theorem when you have only one load acting and if you make it a unit load, you get the relationship f_{ij} is equal to f_{ji} .

Put together, the two reciprocal theorems are often referred to as the Maxwell–Betti Reciprocal Theorem. If you want a statement of the theorem, it would be the total external virtual work associated with forces F_I in system I and the conjugate displacements D_{II} in system II is equal to that associated with forces F_{II} in system II and the conjugate displacements D_I in system I in a linear elastic structure. The word conjugate has a special meaning; it comes from the word ‘conjugal’ which really refers to a marriage between a force and a displacement, but they really refer to the same coordinate location.

(Refer Slide Time: 13:47)

Applying Betti's Theorem to solve statically indeterminate beams

Statically indeterminate loaded structure (system I)

Virtual displacements: support settlement at B (system II)

NPTEL

$$P\delta_p - X\delta_x = 0$$

$$\Rightarrow X = P \frac{\delta_p}{\delta_x} = \frac{33}{16} P$$

$$\Delta(x) = \Delta_{max} \left[3\left(\frac{x}{L}\right) - 4\left(\frac{x}{L}\right)^4 \right] \text{ for } 0 \leq \frac{x}{L} \leq \frac{1}{2}$$

Let us now see a demonstration of Betti's Theorem: the first application of solving statically indeterminate structures. Here, you have a two-span continuous beam and that it is externally indeterminate. You have two equations of equilibrium to solve three vertical reaction unknowns; so, let us assume one of them to be the redundant. Here, we have chosen the middle reaction V_B as the redundant X . Now, we will call the first picture as system I and the second picture as system II.

System II is virtual. In this system, we allow a movement of that support x and you will see that the deflected shape would be like that of a simply supported beam supported at A and C; let us say those actions are R_A , R_B and R_C . We will refer to the deflection under the first load location as δ_p here and the deflection of the support settlement

here is δ_X (Refer Slide Time: 14:46). You have got displacements and forces in system II and you also have displacements and forces in system I.

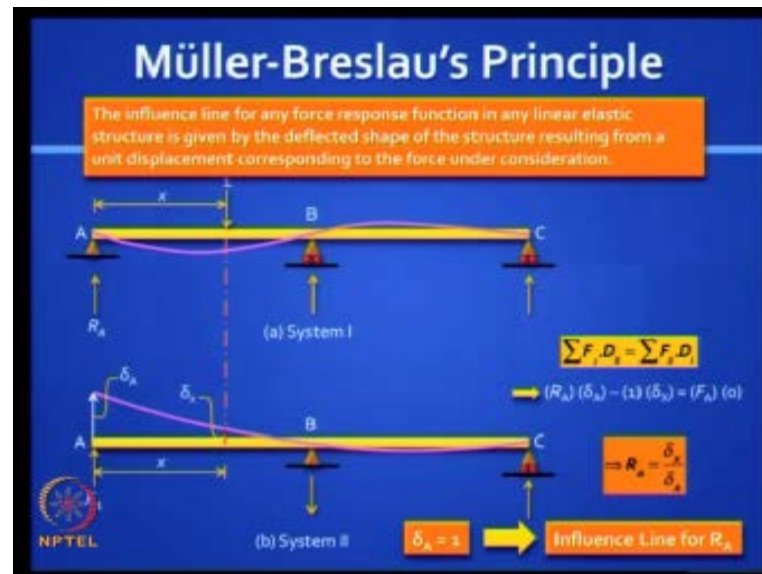
We now invoke Betti's Theorem and you will find that the total external virtual work that you get when you multiply forces in your first system with the displacements in the second system is just P into δ_P . Do you agree? It is because none of the other external forces have any support movements that are non-zero. Your left-hand side of the equation is P into δ_P , but you also have X doing work; X does negative work because δ_X is in the opposite direction.

We would say the total external work is P into δ_P minus X into δ_X . Now, you take the external virtual work caused by the forces in the second system with the displacement in the first system and you will find that that product is 0 because there are only three forces here R_A , R_B and R_C , but they refer to supports in system I and none of the supports are moving. So, it is a beautiful equation.

You end up discovering that the unknown X is nothing but P into δ_P by δ_X , which means if you take the ratio of this deflection (Refer Slide Time: 16:24) to this deflection and that ratio times P is the amount of force that you get in this reaction. That is a fraction of P that is passed on as a reaction in the middle support. Actually, in the early days, they did measurements to prove it; you can do this physically on a beam and measure the deflection and that is your reaction.

So, it is a simple, initial way of understanding indeterminate structures. If you really had an exact solution to those deflections, and you can derive those by any of the methods that you know (including direct integration, conjugate beam method, virtual work method), if you have an expression for δ_X and if you plug in the value of X equal to $\frac{1}{2}$ by 2 , then you can actually get the exact solution which in this case turns out to be $\frac{11}{16} P$. But even if you did not have the exact solution, you would get an approximate solution by just learning how to draw the deflection diagram reasonably accurately and scaling them.

(Refer Slide Time: 17:36)



I just wanted to show you something of historical value. Today, we do not use those old methods because we have very powerful methods of finding unknown redundants. The Müller-Breslau's principle is a powerful principle and it is credited to both Müller of Germany and Breslau, I think of England, who more or less at the same time came up with a powerful principle which helps us draw influence line diagrams.

As you know the influence line diagram is a diagram that is especially useful in finding a response quantity caused by the movement of unit load along the length of the beam. It is useful in bridge design; it is qualitatively useful in building designs as well. Here, I have a two-span continuous beam. I have positioned the unit load 1, which can occupy any position along the length of the beam; I wish to find, for example, the support reaction R_A . What do I do?

If I want to use Betti's Theorem, then in the second system I allow for a displacement where R_A is acting, which I can achieve by actually removing that support and deliberately applying a force F_A . You agree that we get a deflected shape roughly as shown in this figure. Now, if you invoke Betti's Theorem and you refer to this deflection (Refer Slide Time: 19:17) under the unit load 1 in the first system as δ_x , then you can show that this R_A into δ_A minus 1 into δ_x (minus because the two are in opposite directions) must be equal to 0.

So, it follows that R_A is equal to δ_X by δ_A and that simple equation has a powerful meaning. It basically says that if you want the influence line for the vertical reaction in this two-span continuous beam, if you want be influence line for R_A , all you need to do is to remove the restraint, remove that support and lift up the beam by a unit displacement; if you put δ_A equal to 1, then whatever deflected shape you get is the exact influence line diagram for the reaction. That is a very powerful principle.

Even if you were not to put a unit displacement, you give it some arbitrary displacement, then the shape that you get represents the influence line diagram to some scale. If you normalize the deflection so that the deflection is unity, then you get the exact influence line diagram. This can be proved for any response function; it could be applied to bending moments, shear forces and so on. It is a very beautiful principle called the Müller–Breslau's principle.

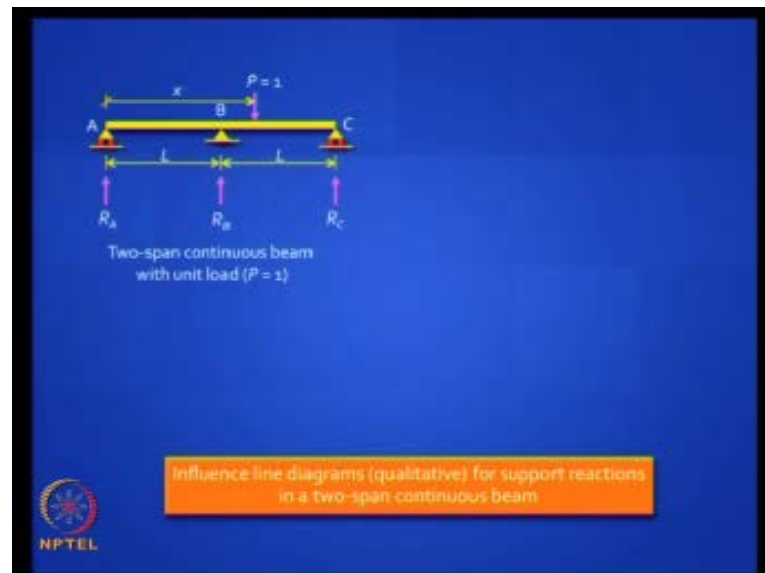
The statement would be the influence line for any force response function in any linear elastic structure is given by the deflected shape of the structure resulting from a unit displacement corresponding to the force under consideration. An interesting thing to note, and I want you to give me the answer for this, is that if the structure just rigid, if it is statically determinate, the influence lines are always straight lines. If they are indeterminate, as in this case, they are not straight; they get curved. Why is that so?

It is a good point to reflect. Why is the influence line diagram for a continuous bridge made up of curved lines, but for a simply supported bridge they are all straight lines? Statically indeterminate structures ((.)) displacement which is equivalent ((.)). There will be a bending moment ((.)) in the deflected shape and so the influence line ((.)). That is one way of arguing it out but perhaps a more simple way is to go back to the realization that the force field is related to the displacement field. Structures are statically indeterminate because they are over-rigid and they are over-constrained – more than the minimum required. If you remove those additional constraints, you get back the just-rigid structure. If you remove a constraint from a just-rigid structure, then it becomes a mechanism. Do you understand?

If you take a simply supported beam and remove one support and you move the structure, then you have a rigid body; it behaves like a mechanism. Is it clear? But if you had a two-span continuous beam and you removed the support, it is still statically

determinate, it still just rigid and it can still take load; that is the difference. Is it clear?
So, your answer is also correct; that is another way of looking at it.

(Refer Slide Time: 23:32)



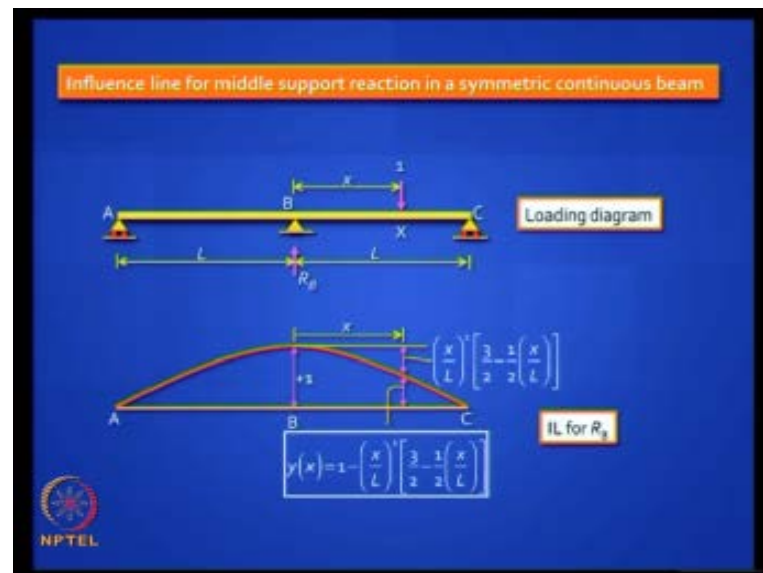
It is nice to qualitatively draw influence line diagrams; it is very easy. If you want the influence line for the support reaction of a propped cantilever, you just lift it up. You can just write the exact expression for y of x which is now possible because you have many techniques of doing that. Then, you actually have an equation for that influence line. I would like you to try drawing the shapes of the influence line diagrams qualitatively – just the shapes for R_A , R_B and R_C ; quickly. Just draw the shapes of the influence line diagrams. R_A we have already drawn.

Can you draw the influence line diagram shape for R_B and R_C ? Just the shape. The value will be unity at some location. You draw it and you compare it with the answer that I am going to show you. This is one diagram. This is the diagram we did in the previous problem. This is for R_A . Agreed? Why is the value negative in the sector BC? What does it mean? What is it mean to have a negative value? It is sagging. ((.)) The BC segment will be sagging. The beam segment in BC will be sagging. So, what does it mean for R_A ? ((.))

There is an uplift; you can visualize that the beam wants to lift up. You said the wrong thing – it is not sagging in BC that will cause the reaction, but it is the hogging in AB that wants the beam to lift off from the support A. So, you need to pull it down and that

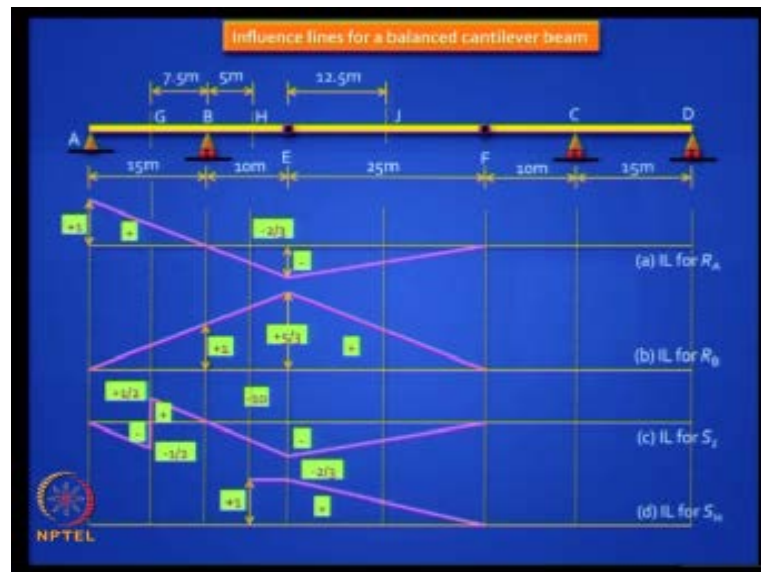
is why you have a negative reaction. Is it clear? Try drawing for R_B . Will you have anything negative? No, because you just lift it up; so R_B is very straightforward. What about R_C ? Similar to R_A . It is quite easy. If you want the exact equations, so you have techniques of doing that.

(Refer Slide Time: 26:19)



The exact equation, for example, is something you can generate. There are handbooks available where they do not give you the exact equations; they are useful in bridge design where you get the ordinates at different locations, you divide the span into ten segments and you get those locations ((.)) sometimes used in practice.

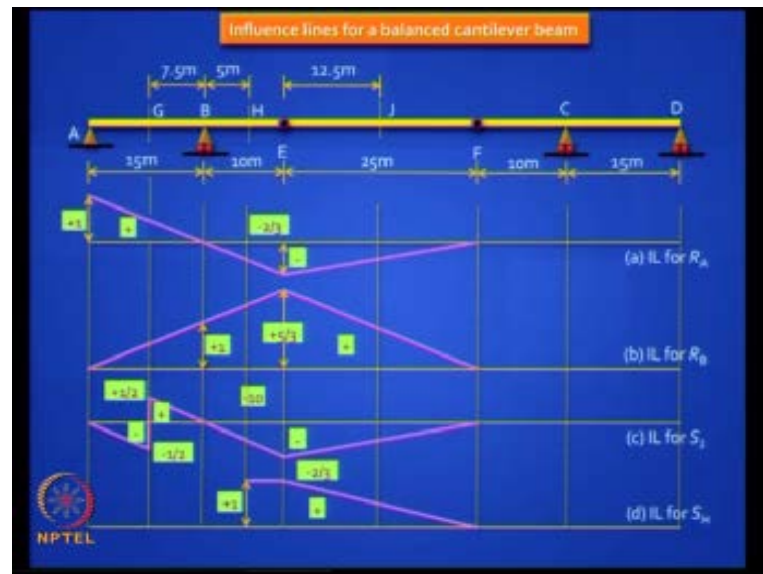
(Refer Slide Time: 26:44)



Look at this beam. It is called the balanced cantilever beam. I think we looked at it last year. Will you get curves or will you get straight lines, if you were to draw? In this case, you get straight lines, but you can still use the Müller–Breslau's principle. I have marked here locations G, H and J. Can you try drawing the influence lines for the bending moments at let us begin with J, then G and then H?

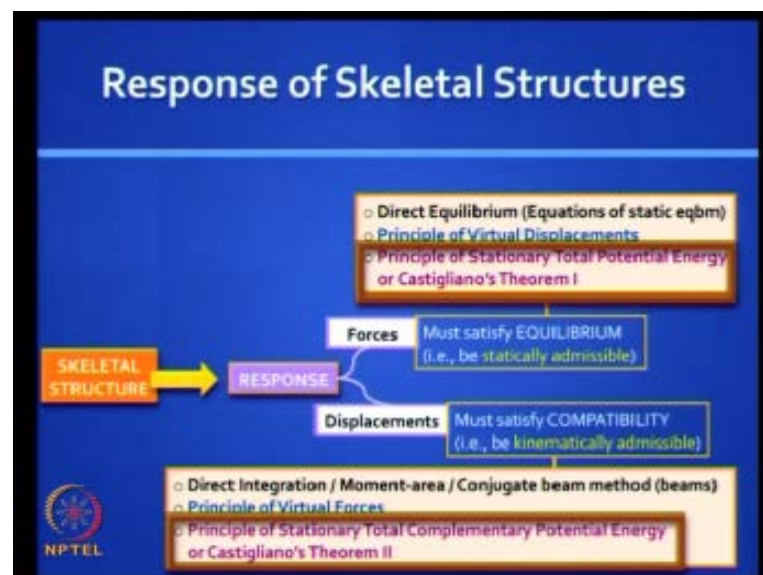
What you need to do is to... To find the influence line for M_J , what should you do? You put an internal hinge there and move it either up or down. If you move it up, then you will get positive values which is what you should get for sagging moments. You should follow a sign convention. You can also try drawing and getting the support reaction at A and B. Can you try that? The influence line for support reaction at A and B and so on? It is very easy; you remove the reaction at A and you lift it up. How will this move? We have done this earlier. You get a series of lines; they are very easy to do; they are all straight lines. Once you catch one value, you can catch the rest of it. We have done this earlier.

(Refer Slide Time: 28:17)



I would like you to recall what you we did. With this, we wrap up work virtual work and work theorems and we go straightaway to the next topic which is energy methods.

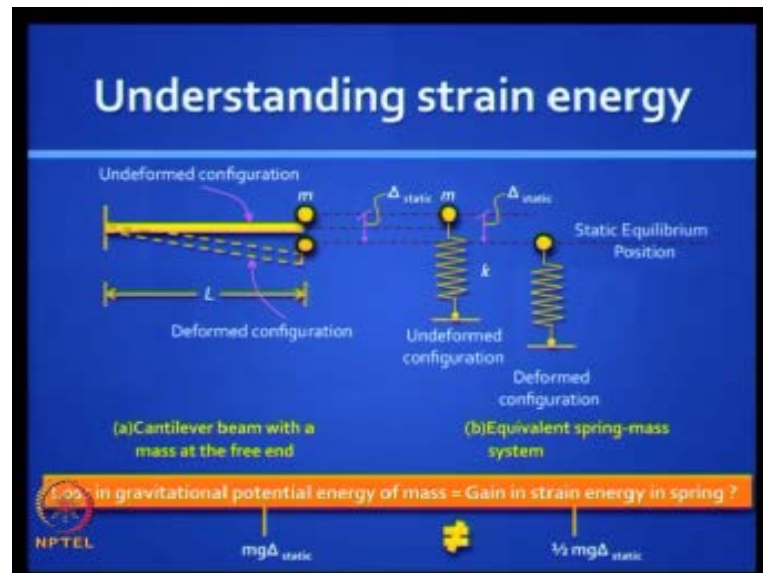
(Refer Slide Time: 28:34)



Specifically, we look at energy methods that help us solve problems of finding unknown forces that need to satisfy equilibrium and also energy methods that will help us find out unknown displacements. Mostly, we will be using Castigliano's Theorems I and II which you have heard of, but it is good to know the origin of these theorems. They come from the principle of stationary total potential energy and the principle of stationary total

complementary potential energy. They have very wide applications. Energy methods are not usually very well understood by students; it takes time to really understand.

(Refer Slide Time: 29:19)



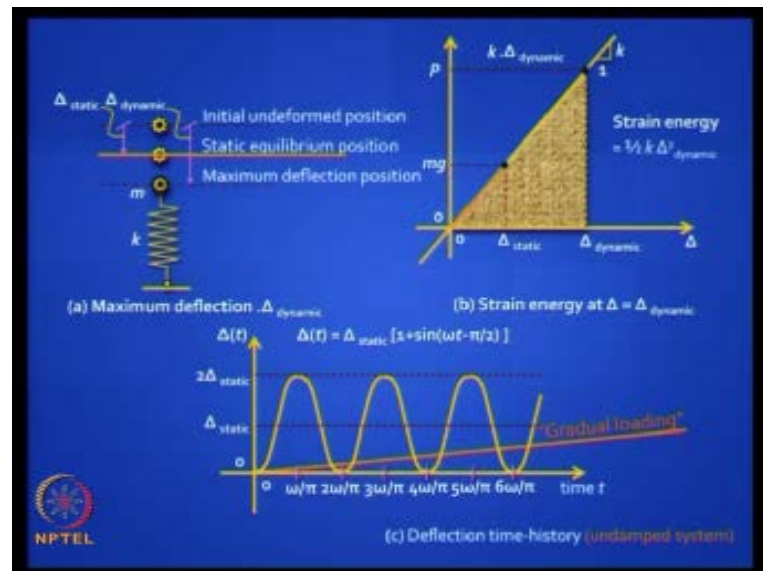
Let us do a quick overview. You recall I had asked you this question earlier. If you have a cantilever beam, for example, and if you have a mass at the free end, you can replace the beam with a spring; both have elastic behavior. Let us say the static deflection is Δ_{static} . In the undeformed configuration, you have potential energy in the mass but no elastic energy in the spring. Both are forms of potential energy; in the deformed position, the mass has lost some potential energy and the spring has gained some elastic potential energy – that is the elastic potential energy; the other one is gravitational potential.

Are these two quantities equal? Are you convinced that the answer is no? We have heard about the principle of conservation of energy and they should be equal; the loss in potential energy in the mass should be equal to the gain in elastic energy in the spring. Am I right or wrong? But you would find that when you do the calculation.... If it would be an ideal system, then it would have some velocity. That is the correct answer.

You really do not have a static equilibrium position. It is like a pendulum – the pendulum moves back and forth unless you stop it. If you stop it.... It will stop its own due to various dissipative forces due to friction and that energy is lost. The same thing happens here. If you work out the calculations, the loss in gravitational potential energy is m into

g into Δ_{static} and the gain in strain energy in the elastic spring is half into m into g into Δ_{static} ; so, there is a half which gets lost. That is the energy dissipation that you get.

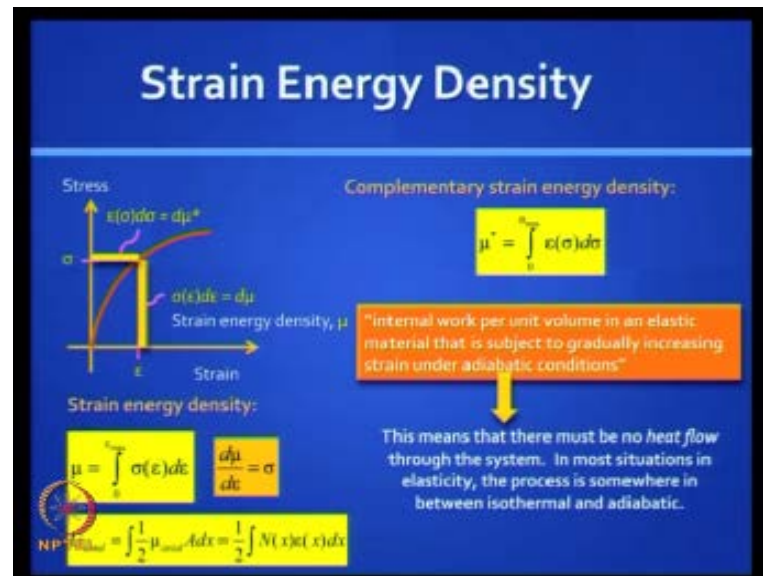
(Refer Slide Time: 31:31)



If you really had in ideal system, it will not lose any energy; it will actually gain maximum velocity at the so-called static equilibrium position and it will move down further. But then, you are violating static equilibrium and so the spring will move back, will move back and forth, it will keep tracing the path up and down on that P-delta curve. You can prove that the dynamic displacement or the total displacement that you get will be two times that of the static displacement in an ideal system. But, our real systems are not ideal; there is going to be dissipation of energy. If you really want to model it, you should put a dash part which takes care of the dissipation energy.

We make one big assumption. The graph that you see here shows simple harmonic motion, but in reality we know that things come to a stop and they come to a stop fairly quickly. Instead of loading it suddenly, we simulate in our minds the concept of gradual loading under adiabatic conditions, which is called static loading. That means in our minds, let us assume that it takes a long time to gradually load the beam – so slowly that there are no accelerations in the beam and we reach the same static deflection.

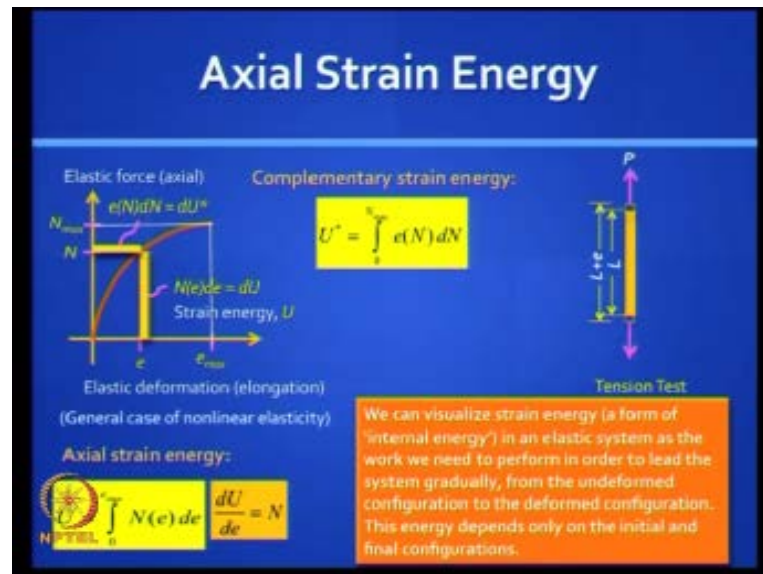
(Refer Slide Time: 33:12)



There is a nice concept called strain energy density in a material; that is referred to by the symbol μ . It is really integral σ into $d\epsilon$. It is a measure of strain energy per unit volume in an elastic material subject to gradually increasing strain under adiabatic conditions; there is no heat transfer assumed. This means that there must be no heat flow through the system. In most situations in elasticity, however, the reality is somewhere in between isothermal, which means constant temperature, and adiabatic.

With this assumption, if you integrate the strain energy through the volume of a bar that is subject to axial tension, then you get the classic equations of half... Half comes if it is linear; half integral axial force times the strain integrated over the length of the bar. There is another term which is called complementary strain energy density. Here, you visualize the incrementing of energy coming from an increment in the force or the stress rather than in the displacement. You know you can do loading either displacement-controlled or force-controlled. If you do it force-controlled, then the product which is also energy would take this form and we refer to it as μ^* . It is the other complement of the rectangle which contains the total energy. Sometimes, it has a physical meaning but often, it does not.

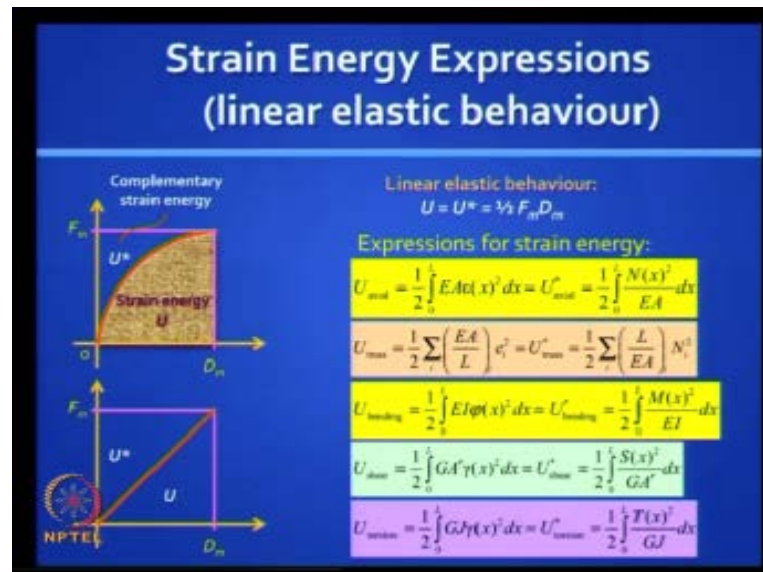
(Refer Slide Time: 35:03)



Let us not worry about strain energy density; let us talk about absolute strain energy. Let us talk of a nonlinear elastic system where you can prove that the total strain energy is the area under the curve when you have gradual loading. Ideally, you can visualize strain energy as a form of internal energy in an elastic system as the work we need to perform in order to lead the system gradually from the undeformed configuration to the deformed configuration.

The energy depends only on the initial and final configuration. In a truly conservative system, this is true; it is called a point function. When you return, the path of that curve is completely retraced and that energy is totally recoverable. The complementary strain energy is a complement to this energy. That means if you take the area of the rectangle and you subtract this energy, what you get is U^* .

(Refer Slide Time: 36:10)



Now, let us talk about linear elastic behavior. In fact, that is what we will deal with most of the time unless you have nonlinearity. The beauty in linear elastic behavior is these two terms are interchangeable. U is equal to U^* is equal to half the maximum force times displacement. What we need in structural analysis is how to compute strain energy and complementary strain energy in beams, frames, trusses, torsional members and so on. You have axial strain energy; this is the expression.

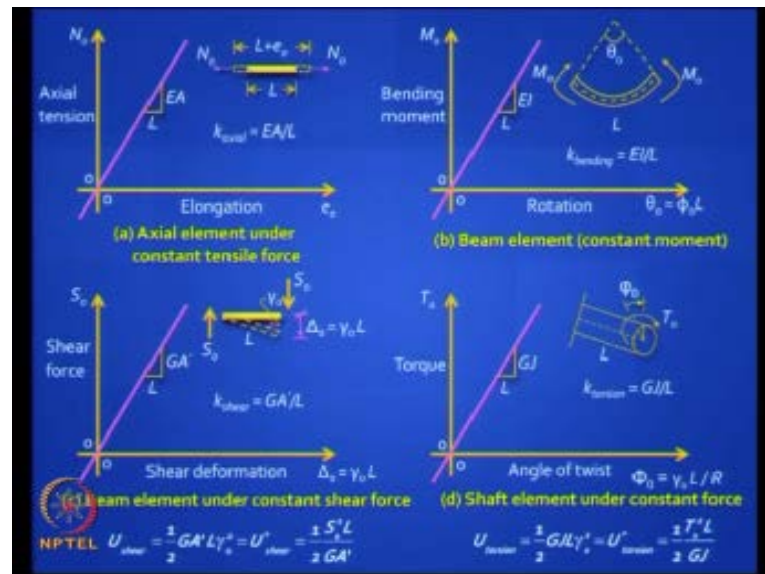
You have two forms. In one form, you express it in terms of the displacement or the strain; in the other form, you express it in terms of the force. Both are equal: U and U^* . This is the expression for a truss which is made up of many members (Refer Slide Time: 37:10). You just have to add up the strain energies in all the elements; you are familiar with this; the factor half comes.

This is the strain energy in a beam or in a frame where you are concerned only with flexural strain energy U and U^* . This is the strain energy caused by shear, which is usually neglected (Refer Slide Time: 37:33). We will see later that it is not always negligible. You have the torsional strain energy. In all these expressions, you will find some rigidity terms appearing. For axial strain energy, the rigidity term is EA . It is called axial rigidity; it is a product of the elastic modulus and the cross-sectional area.

For the bending term, you have EI , which is flexural rigidity. It is a measure of the stiffness; it is the product of the elastic modulus and the second moment of area. For

shear rigidity, you have $G A$ dash – that gives us the shear strain energy; you have $G J$ for a prismatic circular section – J is the polar moment of inertia and G is the shear modulus. For non-prismatic sections and for non-circular sections, you do not have a case of pure torsion; you have warping coming into place; you have some distortion coming into place.

(Refer Slide Time: 38:43)



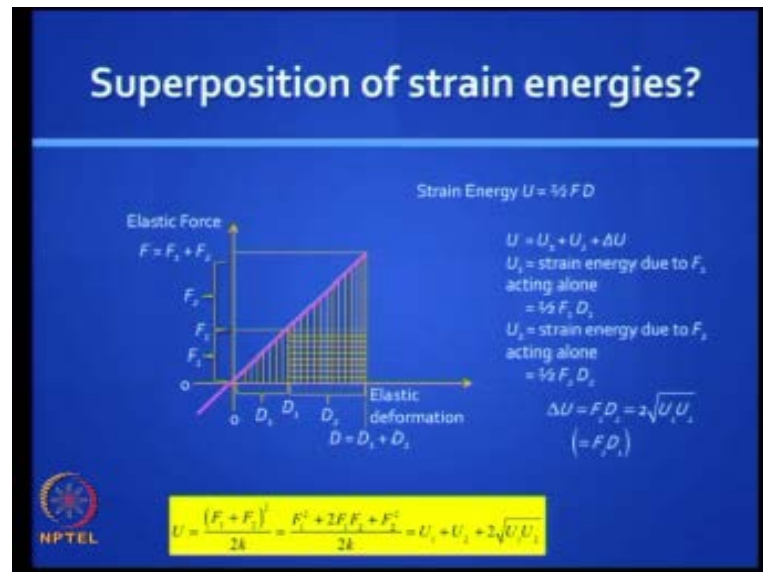
If you have axial force, the corresponding displacement deformation parameter is elongation; you are familiar with the straight line behavior. This is what you get in a truss member. In a beam, let us say, a beam subjected to constant bending, the radius of curvature is constant, so the angle subtended is θ_0 ; θ_0 is curvature into L and so the conjugate of bending moment is rotation and the slope of that line is EI by L .

EI by L is a measure of flexural stiffness; in the first picture, EA by L is measure of axial stiffness. If you do not have uniform moment, then it is EI by L into some multiple – 2 or 3 or 4 or 5. Similarly, here, if you talk about shear strain energy and let us say you have a beam subjected to constant shear force, then the shear deformation is Δ_0 , which is the shear strain γ and is constant for that segment, into the length of the element. The slope of this, which is a measure of shear stiffness, is G into A dash by L .

Where are these terms used? You know about shear walls in buildings. If you have a very tall building and the wall is narrow in length, then it will behave like a beam; so, you have mostly flexural strain energy. If you have a short squat shear wall, it resists

lateral load, then the predominant energy would be shear strain energy. If you have a wall which is of intermediate size, then you have both these stiffnesses coming into play. If you want to be exact, you should not ignore the shear deformations and include their effect. Similarly, the deformation that you would relate with torsion would be the angle of twist and the slope of that line; here is a shaft subjected to pure torque is G into J by L .

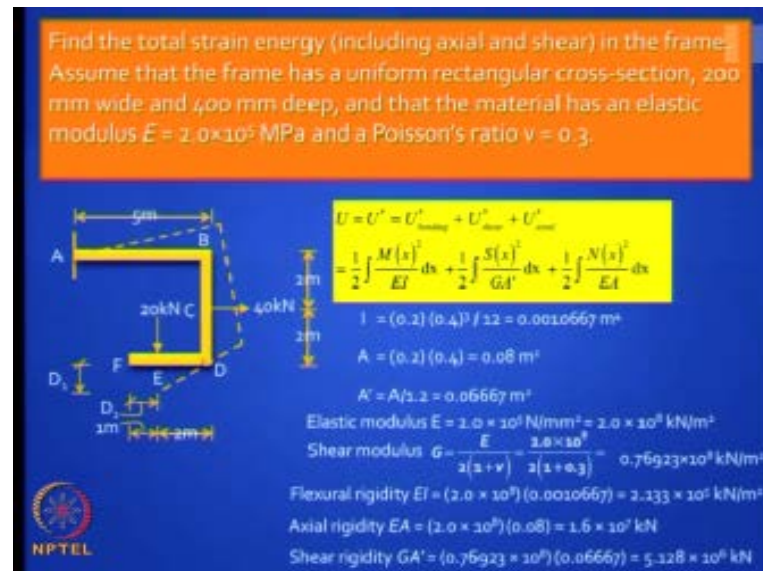
(Refer Slide Time: 41:09)



Next question. If I have a beam or any elastic body and if I apply some load and it has a strain energy U_1 , then I apply independently some other load and it has a strain energy U_2 and if I apply both these loads simultaneously, would the total length strain energy be U_1 plus U_2 ? ((.)) Why not? On the graph, we can ((.)). You can understand this when you draw a graph; you are right. You will find that the two triangular areas caused by F_1 and F_2 are what you would get independently as U_1 and U_2 , but you have to take the total area under the curve, which includes this rectangular segment. What does that area represent – that rectangular area? The first system that does work on the second system ((.)).

You first applied a load F_1 and the system underwent a deformation D_1 ; then, you applied F_2 and F_1 is also going to move, but not caused by itself but caused by another agency F_2 ; that movement does not have a cause-effect relationship; so, there is no half coming in; it is a solid quantity. That is the portion that we tend to miss out; you can

(Refer Slide Time: 42:50)



The standard problem that you need to know is how to compute strain energy in any truss, in any frame and in any beam. Very quickly, we will go through this example; you have done this earlier. Let us say you have a frame – a cantilever frame; it has got this loading. We have earlier calculated D_1 and D_2 and now we want to find out what is the strain energy in this system.

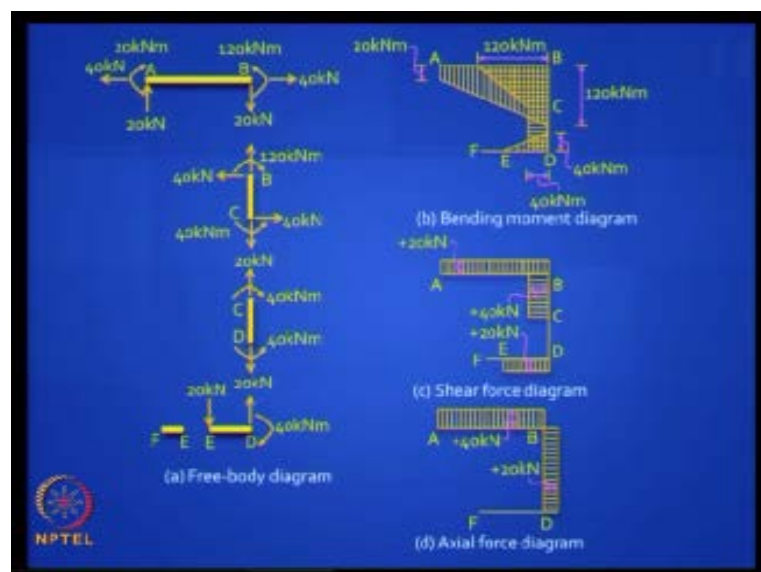
Let us do it exactly; that means let us compute all the strain energy components that are possible. A frame will have, in general, bending strain energy, shear strain energy and axial strain energy. Please note that we are actually computing the complementary strain energy and we are invoking the principle that that is numerically equal to the strain energy. So, the formulas are half integral M^2 into dx by EI plus half integral S^2 into dx by GA plus half integral N^2 into EA by dx , representing the bending shear and axial strain energies.

One question I need to ask you: why do we have $G A$ dash? What is this A dash? A dash is the reduced area of $((.))$. Why do we not take the full area of the rectangular section? Total area is $((.))$ subjected to total shear stress, maximum shear stress; we have to take the average $((.))$. That is right. Under flexure, you have nonuniform shear stress; it is not like direct shear where you have uniform stress. If you have direct shear, you could use

GA, but you have nonuniform shear stress; you can prove that you have to get an equivalent area and it is usually a reduced area; it is that area divided by 1.2, for example, for a rectangular section and that factor is called a form factor.

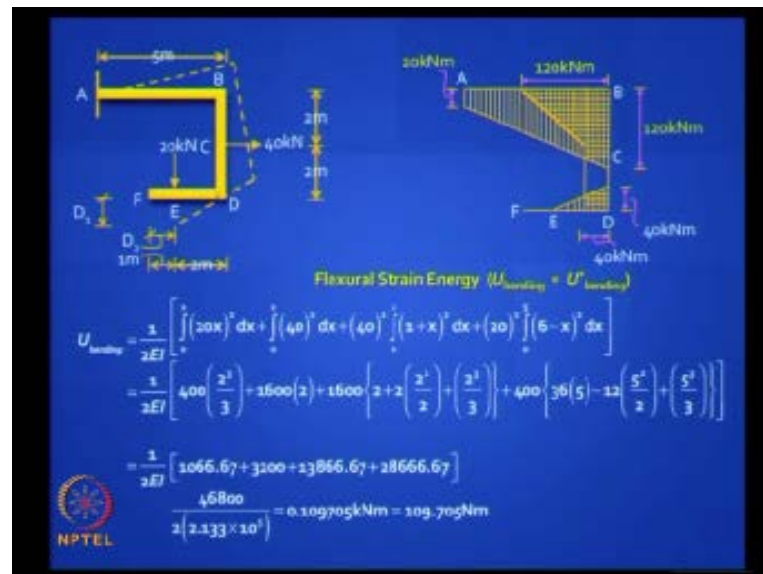
Let us quickly do the calculations. You have a rectangular section; it is easy to calculate I , the second moment of area; it is B into D cubed by 12. The area is B into D and the reduced area, the shear area, is A divided by 1.2 for a rectangular section. The elastic modulus, E , is given to you and the shear modulus is E divided by 2 into 1 plus ν , which is Poisson's ratio; so, you have those two material properties. Then, you can find the three rigidities just by multiplying; E into I will give you flexural rigidity; E into A will give you axial rigidity; and $G A$ dash will give you shear rigidity. We will just quickly go through the concept; you have done this earlier.

(Refer Slide Time: 45:41)



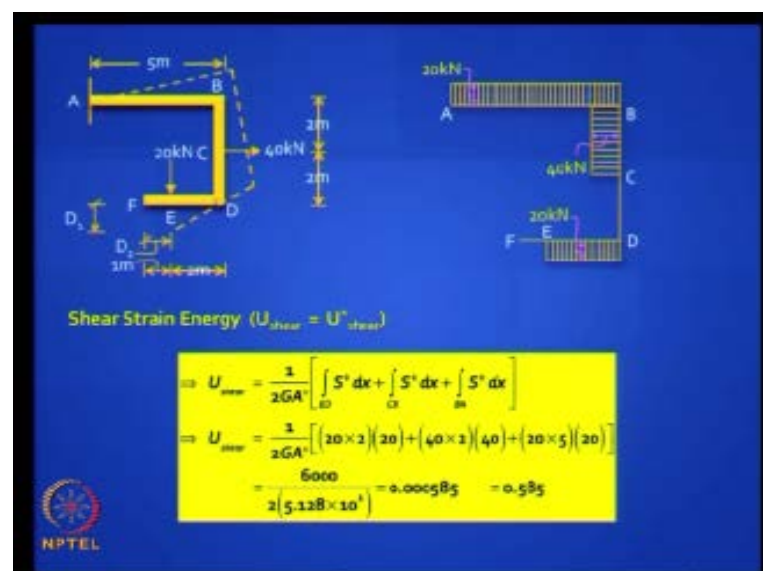
Your first job is to analyze this frame. You can easily draw the free body diagrams and then draw the bending moment, shear force, and axial force diagrams; we have done this earlier. You can use these three diagrams, one at a time, to actually compute the energy quantities.

(Refer Slide Time: 46:01)



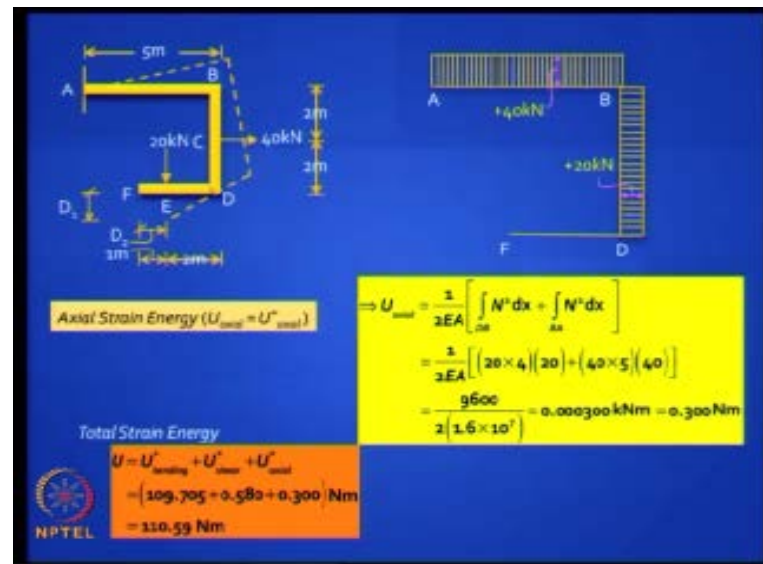
Let us take flexural energy. This is your bending moment diagram and we are going to prove that the flexural energy is the predominant energy. You can do this integration. It is integral $M^2 dx$ by EI ; you have to do this in parts because the bending moment diagram is changing; you can do it. This is a hard and long way of doing it. Is there an easier way of calculating? Volume integral. Yes, you can use volume integral; you can use the area multiplication method and it is much faster.

(Refer Slide Time: 46:37)



Let us take the second case which is shear strain energy. This is very easy to do by the area multiplication method; you just have to multiply this diagram with itself.

(Refer Slide Time: 46:45)



Then, you also have the axial strain energy. You have the three quantities and when you add them up, you will find something very interesting; that is, the bending strain energy here is 109.705 newton meter, shear strain energy is only 0.58 and axial strain energy is only 0.30; the total is 110.6 (Refer Slide Time: 47:09). You can clearly see that you can ignore shear deformations and axial deformations. It is like assuming that the shear rigidity is infinite and the actual rigidity is infinite; that is the reason why this assumption that we frequently make in structural analysis when we deal with frames is justified. But, there are exceptions and if you want to be accurate, you can include those effects as well. In matrix methods, we will do problems where we can do this more exactly.

(Refer Slide Time: 47:42)

"Strain Energy = External Work"

Strain Energy = External Work

$$U = \frac{1}{2} \sum_i F_{i,ext} D_{i,ext} \quad (\text{Proof: Principle of Virtual Work}) \quad \sum_i F_{i,ext} D_{i,ext} = \sum_i F_{i,int} D_{i,int}$$

(if there is initial strain energy) $U - U_{initial} = \frac{1}{2} \sum_i F_{i,ext} D_{i,ext} + \sum_i F_{i,intial} D_{i,intial}$

Betti's Theorem

Apply system 'I' loading first, then system 'II' loading

$$U = \frac{1}{2} \sum_i F_{i,I} D_{i,I} + \sum_i F_{i,I} D_{i,II} + \frac{1}{2} \sum_i F_{i,II} D_{i,II}$$

Apply system 'II' loading first, then system 'I' loading

$$U = \frac{1}{2} \sum_i F_{i,II} D_{i,II} + \sum_i F_{i,II} D_{i,I} + \frac{1}{2} \sum_i F_{i,I} D_{i,I}$$

NPTEL $U = \frac{1}{2} \sum_i (F_{i,I} + F_{i,II}) (D_{i,I} + D_{i,II})$ $\rightarrow \sum_i F_{i,I} D_{i,II} = \sum_i F_{i,II} D_{i,I}$

Now, one of the theorems which is sometimes very useful to apply is this concept called the external work done. When you lower a system gradually ((.)) from 0 to its maximum value will be equal to the total internal work done. Remember: if it is a linear elastic system, the factor half will apply also to the external work. This is different from virtual work; this is real work. So, if you invoke this theorem, you can prove it very easily using the principle of virtual work because the principle of virtual work just says the total external virtual work is equal to total internal virtual work. Here, you are referring to the force field and displacement field in the same structure with a cause-effect relationship.

If you multiply both sides of the equation by half, then you will notice that what you get on the right-hand side, the internal virtual work times half is nothing but elastic strain energy. Is it clear? That is why the strain energy is exactly equal to the external virtual work. But if there is some initial strain energy locked into a system and often you have that for example, when you load rolled steel beams, you have some residual stress locked into the beam, then you need to subtract that part because it is only the incremental change in energy that you are measuring when you are calculating.

You can actually prove Betti's Theorem using this formulation. If you apply one system first and then the other system, you change the sequence and you write the two expressions and you compare the two expressions, you can end up proving Betti's

Theorem; so, this is an alternative proof. You can prove Betti's Theorem either following the path of the principle of virtual work or invoking this energy theorem.

(Refer Slide Time: 49:49)

Maxwell's Reciprocal Theorem

Apply load F_1 first, then apply F_2
 Apply load F_2 first, then apply F_1



$$U = \frac{1}{2}(F_1)(f_{11}) + (F_2)(f_{21}) + \frac{1}{2}(F_2)(f_{22})$$

$$U = \frac{1}{2}(F_2)(f_{22}) + (F_1)(f_{12}) + \frac{1}{2}(F_1)(f_{11})$$

$$(F_1)(f_{21}) = (F_2)(f_{12})$$

$$f_{12} = f_{21}$$

$$\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$U = \frac{1}{2}(F_1 D_1 + F_2 D_2)$$

NPTEL

Similarly, you can prove Maxwell's Reciprocal Theorem; the proof is clear. We will stop here. Thank you.

(Refer Slide Time: 49:56)

Review of Basic Structural Analysis - 1

End of Lecture 2 - Module 1.2

Next : Lecture 3 – Module 1.3

"Review of Basic Structural Analysis - 1"

NPTEL