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#### Module No.# 7.1 Lecture No. # 38 Analysis of Elastic Instability and Second –Order Effects

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Good afternoon to you. This is module 7, lecture 38. We are starting a new topic, a new module, a rather advanced topic. It is called analysis of Elastic instability and Second order effects..

So, we are at the home stretch. We have 3 more technical lectures on this topic. Quite frankly, this is normally not covered at this level, but, if you do not cover it now, you will probably never get to cover it.

So, I would like you to have the benefit of relaxing, sitting back and listening and trying to absorb some basic concepts in, in an advanced topic.

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Up till now, up till the 6 modules, we have really covered how to do from first principles, analysis of skeletal structures of any type, of any dimension, 1-dimensional, 2-dimensional, 3-dimensional. We now look at what happens, when you have non-linearity.

Till now, it was all linear elastic analysis. So, a glimpse of some aspects of non-linearity is covered in this topic, plus we look at an interesting phenomenon called elastic instability, otherwise popularly known as buckling.

So, you have studied a bit of buckling, Euler buckling, so.

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So, we will refresh all that. So, there are 4 topics we will cover in this module, an introduction, effects of axial force on flexural stiffness. We will cover these 2 topics in this session.

In the next session we will cover slope deflection method of analysis, modified to include second order effects and in the last lecture, we will cover matrix methods of analysis.

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So, this is covered in chapter, in the last chapter in the book on Advanced Structural Analysis.

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We will now look at the problem, from a design point of view.

See, what we are attempting do is to predict structural behavior, assuming certain properties and conveniently, in practice, we assume linear elastic behavior because it is very easy to, to do.

And, we can happily apply the principle of superposition. We can apply the effects of different loads, find the responses for each of the loads, we can apply factors to these loads and we can do superposition and the combined effects..

So, dead load we do, live load, wind load etcetera, earthquake load. But, the reality is, when the loads are large, are significant or when the structural elements are flexible, you get into non-linearity. And, non-linear analysis can be quite tricky and, we are not guaranteed of accurate solutions. But regardless of what you do, it is well known by experienced engineers that, if you satisfy certain requirements, minimum requirements, you will end up with a safe design.

So, what are those requirements from a design point of view? The structure should be able to carry its design loads safely, if 3 requirements are satisfied. Let us look at that.

Firstly, you must satisfy equilibrium. The calculated system of forces, it must be in equilibrium with the loads and reactions, throughout the structure, should not violate equilibrium.

Secondly, every component must have adequate strength to transmit the calculated force. So, you get something from analysis, you get a bending moment, your beam must have that bending moment capacity. That should not be exceeded, otherwise your equilibrium solution is not valid.

Secondly, even if you have non-linearity in the form of yielding here and there, you have to make sure that the structure remains integrated, that you have possibly the formation of plastic hinges, you have studied about plastic hinges.

And, you must have, in a plastic rotation capacity, so that, the hinges hang on, and that, that is the related to ductility. In reinforced concrete especially, it all depends on your design of those sections, so that, you have this plastic rotation capacity and you have adequate ductility for that hinge to stay on, resisting a certain ultimate moment and redistributing the load effects to other locations.

So, this is something, that has to be ensured and finally, you must have enough stiffness to keep the deflections small, because if you have very large deflections, the structure becomes un-serviceable. In an earthquake we do not mind for less important structures, because we want to prevent collapse and the structure may be badly damaged. We give up the structure, but, we hope that life is safe, but, for life line structures like, let us say hospitals and so on, we would not even want that. So, we would want functional behavior even under extreme events.

So, there stiffness is of paramount importance. But, we also should avoid instability. We have to avoid buckling instability and this can be elastic instability, it could be inelastic instability. So, it is, it is a very complex problem. We are going to have a glimpse of elastic instability.

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You must have studied, when you do trusses, you have this double bracing, right? Cross bracing and when the wind blows, one of those diagonals which goes into compression is expected to buckle and so, the other one which is in tension will, will operate. But, because of its elastic instability, it bounces back after the wind is gone, blown back, it is still effective. And, if the wind were to blow in the opposite direction, it will now go into tension and function. So, in trusses really, in pin jointed frames you have only component level instability.

So, you are very sure the length is known and then whatever little you have learnt of instability in, in local members is good enough. But when you talk of system, let us take a portal frame, rigid jointed portal frame and if, let us say, the column buckles, then it will, that rigid joint will ensure that the beam will also buckle, and the rest of the structure will also buckle. So, it is a very complicated analysis, you understand?

You have a global instability taking place. You cannot separate local with global. Local is possible only if both the ends are pin joint.

So, we are going to look at those kind of problems, in continuous beams and in frames, multi storied frames, where you have this possibility of buckling, which you must necessarily avoid, if you want to have a safe design, because you can have a disastrous failure of that structure.

So, what is the level of loading that will cause such an instability? And, how should the structural analysis be modified, when the stiffness is low.

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When the stiffness is low, you will have second order effects coming in. We will look into them, they are called p delta effects and usually when the deformations are large, you will also generally encounter material non-linearity, in addition to geometry, but, we will keep that aside for the time being, because it depends on the type of material.

Right now, we are looking at elastic behavior and we assume the material really does not matter. Once you have the E value of the material, we can go ahead with our analysis.

So, non-linear structural analysis is called for, when there is significant material nonlinearity or geometric nonlinearity. In this module, we limit considerations to aspects of geometric, geometric nonlinearity in frame elements and this is related to slenderness effects.

We need to account for the reduction in flexural stiffness in beam column elements due to axial compression. We briefly looked at this earlier. If I have a beam, the flexural stiffness is either E I by L, 3 E I by L, 4 E I by L, depending on the boundary condition.

But those values will reduce, if I have axial compression acting in that element, and those values will increase, if instead, I have axial tension. So, we are looking at that

behavior. It may not be of concern, if your level of axial loading is low in relation to the buckling strength.

So, I have suggested there, as long as your axial compression is less than 15 percent of your critical buckling load, perhaps you can ignore this in engineering factors.

But when the axial compression level is high, then it has a major effect in, in dictating your response. You will be underestimating your response if you ignore the interaction between axial stiffness and flexural stiffness, which we do in the normal frame analysis. Remember we write, E A by L and we put zeros and we write 12 E I by L cubed, all that will change, when there is a coupling between the axial compression and the bending. (Refer Slide Time: 12:30)



And so, we call such elements beam columns. We are going to look at those kinds of behavior. You see, when I have a beam column, and I say the flexural stiffness is 4 E I by L, if I apply a axial compression p, till now we have assumed it, that 4 E I by L will not change. It can change. It depends on p.

And if p is as high as the p critical, the buckling load capacity for that element, what do you think the flexural stiffness will do? It will degenerate to 0.

But if it is less than 15 percent of the critical buckling strength, then the degradation in 4 E I by L may not be very significant. So, you have to take a call, whether to do non-

linear analysis or not. So, definitely you should look into non-linearity, when the 15 percent is exceeded. Is it clear?

So, that is what we will look at and we will also look at modified slope deflection method and matrix method, which can do 2 things – one, tell us what is that critical buckling load in the frame, where it becomes unstable.

Number two, even if the loads are less than that, what is the correct solution, how much does my bending moments shear forces, deflections get amplified because of this reduction in stiffness caused by axial compression? Is it clear? And there may be instances, where you have axial tension, where your stiffness actually gets enhanced and your deflections becomes actually less.

So, let us do a quick review of something you already know only too well. If I apply an axial compression in a column like that, which is free on top and fixed at the base, do I get any lateral deflection? This is an ideal column which, which I will not, until I reach its critical buckling load. So, how do I find out that critical buckling load? Let us go through first principles, how do I find out?

Well, I assume it is, it is happened because, in practice it is going to buckle at some point and then what do I do?

Assume a shape.

Assume a shape and.

Find the moment at the.

Moment at any section and satisfy equilibrium related to the deflection. How do I relate bending moment to deflection?

Curvature.

Curvature is.

M by E I.

M by E I and curvature is d squared delta by d x squared ok.

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So, that is what you do. So, in a cantilever, the first mode of buckling is typically going to look like this and you could solve this equation. We will look at it in the next slide, but, you can work out the critical buckling load for a cantilever. If both the ends are simply supported, then you will have a different profile. Now, this is what can happen at that point, where P is touching on P critical, you hit what is classically called a bifurcation point.

Why is it called a bifurcation point? Because, at that loading, mathematically 2 equations of equilibrium, 2 states of equilibrium are possible. If you remember them, this is an Eigen value problem.

In the Eigen value problem, solution, you have 2 solutions, one is called a trivial solution, where delta is 0, the others are nontrivial solutions, right.

So, both are satisfied by the same equation. So, you have 2 possibilities, if delta is 0, that means it remains straight, and if delta is not 0, it means it buckles. Is it clear? That is the whole idea - that is the beauty of the Eigen value solution. Got it?

So, in practice, it prefers to buckle. It will not remain straight, and the solution that you get predicts actually no limit to the deflection, right. It will go on deflecting. The Eigen

solution, the Eigen function or the Eigen vector, if you are doing a matrix analysis, only tells you what is the shape, is it a sinusoidal shape etcetera.

It only tells you the shape. It does not tell you anything more than that, which means it tells you, let us say, the mid height of the column, what is the deflection in relation to the top. Those ratios are maintained, that is what we call a mode shape.

And the first mode shape looks like this. You can get another solution to the Eigen value problem at a higher mode, which will have a node and so on, but, that is unlikely to happen, because once it is failed in the first mode, there is no question of it failing in the second mode.

But in reality, it will not follow this horizontal line. Can you tell me why? It could follow this line, which means you have a post buckling strength or it could it will go down, where you usually, when you have, you know when large deflections are there, you will have material nonlinearity also coming in, it can go down.

So, why is this theoretical prediction of that horizontal line not valid in real practice? No mathematically, and the classic example given is, you have seen these pictures of Charlie Chaplin leaning against a cane, right. The cane, the more he leans against it, the more it buckles, but, it does not have an indefinite deflection.

It is related to the force he is applying, right. So, although he is buckled, it is buckled, the more he applies, so, there is a unique solution for every load is applied, which you will get from that yellow line, got it?

So, it makes sense that is called post buckling strength. Why does it happen? Why is it that mathematically you are not getting, you get an Eigen value problem in which that should not happen. It is an old question. Yes, the answer is, we are making an assumption, we are making an assumption, when we say curvature is E I into d squared delta by d x squared. That assumption is, that assumption is valid only when your deformations are small. When your deformations are large, you have to take the, the other form, remember, 1 plus and with the, raised to 3 by 2. If you do that, you do not get an Eigen value problem and if you do that, you will get that yellow curve.

So, these things are well known. So, that is called large deformation analysis which we, you do not normally study at, at an early level, but, in elastic stability you get a solution like that.

Now, if the 2 ends are simply supported or pinned like this, the buckle shape will look nicely symmetric and this is a standard case, which actually Euler explored and you know the, this is called the Euler load, Euler buckling load for the standard case, where the solution is pi squared E I by L square, you are familiar with this, right.

Look at that equation E I is on the numerator, which means if you had a member with large flexural rigidity, the buckling load will be much higher and the denominator, it is L is coming into the picture.

So, if the member is long, then its buckling load is going to be much less. So, these are intuitively obvious and you can prove for a cantilever, it is one fourth this value. You, you know this, for a cantilever it turns out to be, it is obviously less stiff than a simply supported case.

General approach for any set of four boundary conditions  $d^{2}\Delta(\mathbf{x})$  $d^2\Delta(\mathbf{x})$ de x = Asinµx + Bcosµx + Cx + D <u>le</u>: propped cantilever – boundary conditions: A(x = 0) = 0, whereby B + D = 0;  $\Delta'(x = o) = o$ , whereby  $\mu A + C = o$  $\Delta(x = L) = 0$ , whereby CL + D = 0; (x = L) = 0, whereby  $A \sin \mu L + B \cos \mu L = 0$  $\Rightarrow$  C = - $\mu$ A, D = -LC =  $\mu$ LA, A = -B/( $\mu$ L) tanµL=µL sinµL cosµL o D μL=4.493 0

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So, it turns out the buckling load is one fourth and if you really want to solve this from first principles, you have to write the differential equation of equilibrium. What you said, is one way of writing it, you know, the differential equations for equilibrium in a beam can be written as M is equal to, M by E I is equal to d squared delta by d x squared, that

is one way of writing. But a more powerful way, is going to the fourth derivative and writing it in this way and write if, if you do not have any lateral load on that beam, we are talking about pure buckling case, then you should put right hand side is not q of x, it is going to be 0, right..

So, this is the fourth order standard differential equation. How do you solve this? Well, we first do this substitution. We bring in a parameter mu which is square root of P by E I and you plug it into that equation and you get the mu squared term, you solve it, this is the fourth order equation, so, you have 4 constants, typically sin and cos and you just have to apply the boundary conditions, right. Let us take this propped cantilever case, ok, the third case we are going to look at. What would be the 4 boundary conditions you will apply? At x equal to 0, delta is 0, delta dash theta is 0, at x equal to L, delta is 0, you still need a fourth equation.

So, these 3 equations are called kinematic boundary conditions. They come from the fact that, the, you have fixity at the bottom end when x equal to 0 and you have a restraint against translation at the top end. You need one more boundary condition. Can you tell me? At x equal to L. You are right, the fourth boundary condition will come at x equal to L.

Can anyone tell me what that is? What is 0 at that end? Deflection, you have already written as your third equation.

Moment. Moment.

Curvature is, why is curvature 0? Because moment is 0. So, that is a tricky boundary condition, that is called a static boundary condition, that which...

So, when you run out of kinematic boundary conditions, search for static boundary condition. You will find either the shear force or the bending moment is either 0 or some known constant.

So, E I d squared delta by d x square is 0, which means second derivative of delta is 0. So, those are the 4 boundary conditions, if you apply them and you plug in the solutions, you can write it in a matrix form. When you solve this you get some constants for A, B, C and D and the last equation, when you solve will be an Eigen function, Eigen value problem, ok.

And typically, it will take some shape like this. So, tan mu L equal to mu L is a transcendental equation. You can have multiple solutions for these. The lowest value of mu L corresponds to your fundamental mode, and that is how you get mu L equal to 4.493 in this example.

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So, this example, that equation tan mu L equal to mu L is called the characteristic equation. You remember in, in, when you did Eigen value analysis, there was something called characteristic equation. And, the first mode solution in this case is, 4.493 and if you plug in the value of mu, you can get the critical buckling load, which we also call the Euler load and it turns out to be 2.047 pi squared E I by L squared, which can also be written as pi squared E I by 0.7 L the whole squared. So, what is this 0.7 L? Yes, that is your effective length. It is a length between the points of contra flexure, very good.

So, you can do that and you have a reference standard case. We will denote that as P E naught. P E is any Euler buckling load, for any boundary conditions in an ideal straight column and this is called the effective length. You are familiar with this so, that turns out to be 0.699 L or 0.7 L.

So, for any given boundary conditions, you can get, through such a mathematical analysis, where you have 4 boundary conditions, you can get the solution, a unique solution. It is an Eigen value solution, where you can find the critical buckling load P c r, you can express it as a factor times the Euler buckling load for the standard case, in this case that factor is 2.047.

You can also relate it to the effective length. That effective length k, will it be less than 1 or greater than 1? That effective length ratio or effective length factor k e...

When will it be less than 1, when will it be more than 1? Cantilever. How much will it be?

2.

2. So, can you generalize and tell me, under which conditions it will be less than 1 and under which conditions it will be, yes.

When distance go up...

When the ends are, no...

My ends can be fixed, but, let us say, I have a guided fix support, this can move. No. So, you have to be more clear, when will the... That is obvious. When will that threshold of 1 be crossed? It is ok, we will look at that, you, you do not have the answer.

So, similarly, for a braced column, for, for both ends pinned, you know that the, the mode shape is sinusoidal. This is a standard solution.

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If your both ends fixed, you know the effective length ratio is 0.5 and so, you get 4 times the standard case of a critical buckling load.

If you have a guided fixed support, it is very interesting, that is, the bottom is fixed, the top is fixed, but, the top is free to translate. When do you have a case like that? For example, you have, let us say, a water tank structure with, with a shaft, the solid mass on top reinforced concrete or a pile cap on piles.

So, you get stiffness there, you get joint stiffness there, but, it can move, nobody is holding that water tank, it can move. So, you, you can model it like that, ok.

So, in such cases you will find, you have to extend that line to get the, the second point of contra flexure. It turns out to be L. So, this case is called an un-braced case.

So, we use a word braced, when a braced column... How do you distinguish between a braced column and an un-braced column? Yes, Ashish, how do you distinguish between a braced column and an un-braced column?

Ok, a braced column is a column in which you do not have any chord rotation. That means the 2 ends of the column will not have any relative translation between them.

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In an un-braced column you have relative translation. So, the second case is un-braced, first case is braced. So, your generalized expression is given like this. You put L e squared at the denominator, pi squared E I by L e squared or you can write it as 1 by k e the whole square P E naught and you will find that k e is less than 1 for a braced column.

When you have both ends pinned, it is 1. When both ends are fixed, it is the lowest value of 0.5 and when you have an un-braced column, you can have any value between 1 and infinity. You will have 1, when you have a guided fixed support at the top, fixed at the bottom, in a cantilever it is going to be 2 and when do you get infinity? If both ends are pinned. It is unstable. So, that is the other...

Let us say, you have a beam. You have a multi storey building in a column and the column has pinned ends, which means the beams have 0 stiffness. Then that is asking for trouble, because it is going to sway like hell, it will be unstable.

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So, to show you a picture of a column in a, in a building, you need some elements in the building which will prevent, which will give the bracing effect. So, it usually comes when you give some shear walls etcetera.

So, the building as a whole is prevented from swaying. So, every column in that building will become braced, ok.

Please listen, whether a column is braced or un-braced is not necessarily dictated by that column or those beams connecting that column. It is dictated by the whole building, the rest of the building.

So, if I have, for example, shear walls, adequate number of shear walls, which are going to take most of the lateral load, they hold the building together. They do not allow much sway.

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So, because you have floor slabs connected to these shear walls, the framed elements also will not sway significantly and so, in such situations you model it as a braced column. But if you do not have any bracing elements like shear walls, then and if your columns are not very stiff, then you can have this case, where you have significant sway. In such cases, k e will always be greater than 1.

But do not have high values of k e, because then, you are asking for trouble. You will have lots of secondary, you will have lots of second order effects.



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So, when do you in have second order effects, primarily in un-braced frames. Now, our real interest is not so much column, as beam column right. So, we make our first entry into beam columns by looking at structures like this. Take a look at that, that is a channel shaped structure, in which I apply P eccentrically, ok.

The, that value of e eccentricity, it is called primary eccentricity, right. But the more I press there, the more it will bulge outwards. So, I get a delta, in addition to e. If the column is very stiff, that delta will be close to 0. If that column is slender, that delta is going to be significant, does it make sense?

Now, you can, this is a case where both ends are pinned. If you study this case, you take a free body and write the equilibrium equation, your moment at any section in the deform configuration. Please note, this is a big difference between second order analysis and first order analysis. First order analysis, we wrote all our equilibrium in the undeformed configuration. In the un-deformed configuration, delta will not come into the picture. You will have P into e only, that is first order analysis.

In second order analysis, we are writing P into delta plus e, because we are saying delta is no longer negligible, is it clear? That is the big difference. So, when you write this equation and you solve for this, you will find that you have an Eigen value problem here, which if you solve for, you get an expression. This is called a Secant formula, it is a famous formula and your deflected shape looks like this. Remember the first figure we drew, we had a bifurcation point right, ok.

We no longer have that bifurcation point. Actually, you do not have an Eigen value solution. This is not an Eigen value problem. You have a unique solution and what is interesting is that, all those lines, those lines you can see if you look carefully, your, if your column is stiff you are probably going to get this curve. If your column is slender you are going to move inward, you get it.

So, it is going to deflect more and more as you apply more and more axial load, but, there is no way you can apply an axial load higher than the critical buckling moment of a column without any eccentricity.

So, you will find all these lines theoretically, will asymptotically hit your critical buckling load, is it clear? This is the difference with respect to the earlier figure.

The earlier figure you had an ideal column. Now, you have a, same ideal column, but, we are deliberately putting a primary eccentricity and these are the curves it will take, is it clear? You have to know only this much.

Now, I want to ask you a question. Why is it that, that columns buckle, columns do not show that bifurcation buckling, even if I do not have primary eccentricity?

In fact, it starts showing the deflections very early itself, like this. I have a column, apply a load, actually if I test in the lab, it is not going to remain straight and then suddenly it going to do that. It is going to give you early warning signs that it is going to buckle, why? That is called a real column. Real columns give you warning signs before the instability takes place. Why? Because of what?

Because of initial imperfections. You can never make a 100 percent straight column. It is not possible. There is always is going to be some initially imperfection. That is what we are going to look at.



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So, if you have initial imperfections, then, this is not the solution you will get, because this is for an ideal column. You are beginning with a bent shape. Even that cane which Charlie Chaplin was leaning against, always had some bent in it in the beginning itself, right.

So, that, you cannot help it. So, you will have an initial imperfection. The order of their initial, impact, imperfection is something like span by 500 and so, and any material you are bound to get something. We do not know the profile of that initial imperfection, but, if you really want the worst case scenario, you can imitate the same fundamental buckling load profile. So, you, in this case, you can assume a sinusoidal initial imperfection, with the maximum value of e naught at mid height. So, I can write initial imperfection like this. So, the initial imperfection serves like a primary eccentricity.

Remember the previous slide, the only difference is in the primary eccentricity case, we showed e as constant along the full height. Here e is maximum at mid height or whatever is the real, reality and e is 0 at the supports and the analysis is similar. The behavior is similar.

You can solve this problem and that is a kind of behavior you will get. This is so, the real column takes a deflected shape like this. There is an initial imperfection, eccentricity of e naught and it follows that path and asymptotically it hits the P critical line, is it clear?

So, this is all general theory and the, you can work out the maximum total deflection which is e naught plus delta m and you get an interesting result. You get what is called an amplification factor, at least you should remember this. The amplification on the deflection that you get, which you compute elastically, is 1 divided by 1 minus P by P e, P is the load you apply, P e is the Euler buckling load for an ideal column.

So, this kind of amplification, you will encounter. Obviously, when P is equal to P e, what do you get? You get un-bounded deflections. That does not happen. The material is going to fail before that, but, does it make sense? In other words, I am giving you an equation for that line, an approximate equation. Omega is the amplification in your deflection, ok.

Now we come to our real topic, which is beam columns, ok.

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We looked at pure columns to understand what buckling is. We are now looking at beam columns and in most of our problems P is not very high, but, still, we want to know what is the maximum P it can take. So, we are looking at a beam which is subject to lateral loading, could be a constraint load, could be a u d l, could be end moments, could be anything, but, also having an axial load. How do we find out the behavior of such a beam? If p is 0, what is delta max? W L cubed by 48 E I.

Now, if you remember the problem, W L cubed by 48 E I. Do you think it will increase, when I put P? How much you think it will increase?

By that factor.

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By that factor which we looked at, that is a good guess, by omega. 1, that is a good guess. 1 divided by 1 minus P by P e is the increase I will get in deflection. Actually, you can prove it. So, let us do that.

So, you can write the equation, solve that equation, put the boundary conditions, write an expression for delta, find out the maximum value at the mid span. Let us normalize it with the static deflection, for, you know, a non-slender case, so, delta naught is W L cubed by 48 E I. So, if I take that ratio delta max by delta naught and I simplify, how do

I simplify a tan function? I expand it in, in a series. What is that series called? The Taylor series, but, in this case, it is a McLauren series, right. So, you expand it. Also, you expand mu L squared and when you make those substitutions, you can find out that, that series approximates to exactly 1 divided by 1 minus P by P e naught. So, that is a good guess.

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So, it really turns out to be what you guessed, that omega. So, that is a factor that it gets magnified, the deflection gets magnified by and if you are going to plot it, it will look like this, which means, I am plotting the magnification factor on deflection, which is omega on the y axis and I am plotting P by P e naught on the x axis.

When P is equal to P e naught, I get infinite deflection. When P is 0, I get that, that W L cubed by 48 E I. That means the factor is 1, otherwise I can find out. So, this make sense, ok.

Now, I am also interested in bending moments, because as a designer, it is not deflections that really matter to me, the how much does a moment get magnified.

Now, please understand. What is the bending moment at the mid span in a simply supported beam? W L by 4. Will that also increase, Ashish, will that also increase? When I have axial compression, will the moment also increase at mid span W L by 4?

Yes.

The deflection increases, by omega. Will the axial, will the bending moment also increase? It will increase, right.

So, that is, so, that is what you are going to look at. Why will it increase? Let us be very clear, this is a statically determinant beam. Statically determinant beam, your mid span moment will it not always be W L by 4? Whatever you do. Plus, plus P delta, is what you do in second order analysis, that is what you do.

In first order analysis, the answer will not change. Even if you had material non-linearity it will not change. W L by 4 remains W L by 4, even if that damn beam yields. It remains W L by 4.

But in the deformed configuration, if you would cut a section there, it will be W L by 4 plus P into delta, delta max in this case, is it clear? And, that second order moment will be significant, if delta max is significant and delta max will be significant when either the beam is too slender, which means it is, E I is small or its span is large or if P is close to the critical buckling load, got it.

So, that is second order analysis. So, it is also, this is also known as P delta effect because you are adding, literally adding a P delta moment to the moment that you have got in the un-deformed configuration. Does it make sense to you?

So, it is interesting from a design point of view. Where do you have such P delta moments? In which kind of structures? Commonly. I will give you a clue, we have an Ocean Engineering department. They should be learning all these. I hope they are. No, birthing structures. You must have seen jetties, where you know ships go and people get, get off.

So, those birthing structures which are decks, they are resting on piles which supposedly go deep and ideally should hit the sea bed. They are long, slender piles. They are slender. And they can... So, they are going to be vulnerable and let us say, there is an impact as a ship comes in. You know, you have ways of reducing the impact.

Then, these columns are subjected to lateral loads. Then you will surely get P delta moments. So, as an engineer you should know how to calculate P delta moments, right. That is where, all this knowledge comes into play.

Now, if you had a heavy load acting, then that delta is going to be large and so, you have to calculate correctly. Now, if the P is constant, let us say that load is not changing and if you plot delta max by delta naught use you can still do linear analysis.



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See, you look at this equation, look at this delta max equation. It is a linear function of W and it is a non-linear function of P. If P is constant, you can still do linear analysis. But if W is constant, then you have a non-linear analysis problem. Or if you have proportional loading, you know what is proportional loading? W and P keep increasing in a fixed ratio so, whether to do linear or non-linear analysis, depends on which is constant.

If you are lucky, P is constant. If your P is constant, you can happily continue with your linear elastic analysis by adding some amplification. But, if unfortunately, P is changing, either you have a constant W case or a constant W by P case. Then, you have to recognize you are dealing with a full blown non-linear analysis, which you cannot do so easily. This just for your information.

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Do not worry too much about this, but, you can generalize and say, look, if I have second order effects, my maximum deflection will increase by a factor omega delta, my maximum moment will also increase and that factor may not be the same as, the same factor that I have for deflection. So, I will call it omega M, is it clear.

And these things have been worked out for different boundary conditions, different load cases. You can find out what is the magnification factor, ok.

Now, we are getting into the last bit of this topic. Remember, I said introduction. Now, we said we will look at beam column stiffnesses, ok.

Why do we want to do this? Because, we really want to find out what is happening to our 4E I by L, 2 E I by L, 3 E I by L, that is our real interest. We are now in a position to do that, because in the previous case, we looked at a beam with a axial force P and some loading acting on it and we said, if it is concentrated, it is something, if it is u d l, it is something, we can work it out.

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Now, we do not put those loads, we apply an end moments, like this case. If I have an end moment M A and if I did not have any P, what is the rotation that I get at that end? The flexibility coefficient, it will be M L by 3 E I at this end, and at the other end 6 E I, right and your bending moment diagram would look like that.

If you had an axial compression, then the theta will get increased and we would not go through the proof of it, but, it can be, it will be increased by a function as shown here, and if the other end is fixed, then that 4 E I by L will also change, remember. If there was no P acting here, what would be this value? S is called a stiffness measure, it is 4 E I by L right, and r is the carry over factor. It will be half times 4 E I by L.

So, now can you understand these symbols r and S and S naught? They are our conventional stiffness measures. S naught is typically 3 E I by L. We want to know how much it changes, when you have P coming into play.

S is 4 E I by L. We want to know, how it gets affected when P comes into the picture. R is 0.5 or r S is 2. It is the same thing and we will not know how it gets affected by P.

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Response of a braced beam-column subject	to end moment $\mu = \sqrt{\frac{p}{m}}$
$\theta_{\lambda} = \frac{M_{\lambda}L}{3EI} \psi(\mu L)$ $\theta_{\mu} = \frac{M_{\lambda}L}{4} \phi(\mu L)$	$\Rightarrow \Delta^{n}(\mathbf{x}) + \mu^{2}\Delta(\mathbf{x}) + \frac{M_{A}}{EI}(1 - \mathbf{x}/L) = 0$
	$\Delta(\mathbf{x}) = \mathbf{A} \sin \mu \mathbf{x} + \mathbf{B} \cos \mu \mathbf{x} - \frac{\mathbf{M}_{A}}{P} \left(1 - \frac{\mathbf{x}}{L}\right)$
$M_{\lambda}/L$	$\Delta(\mathbf{x}=\mathbf{o})=\Delta(\mathbf{x}=L)=\mathbf{o}$
$EL\Delta^{\prime\prime}(x) + P\Delta(x) + M_{A}(1-x/L) = 0$	$\Delta(\mathbf{x}) = \frac{M_{A}}{P} \left[ \frac{\sin\mu(L-\mathbf{x})}{\sin\mu L} - \left(1 - \frac{\mathbf{x}}{L}\right) \right]$
$\theta_{A} = \Delta t(\mathbf{o}) = \frac{M_{A}}{P} \left[ \frac{-\mu}{\tan \mu L} + \frac{\mathbf{i}}{L} \right] = \frac{M_{A}L}{3EI} \Psi(\mu L)$	$\Psi(\mu L) = \frac{3}{\mu L} \left( \frac{\mathbf{a}}{\mu L} - \frac{\mathbf{a}}{\tan \mu L} \right)$
$\theta_{a} = -\Delta'(L) = \frac{M_{A}}{P} \left[ \frac{\mu}{\sin\mu L} - \frac{1}{L} \right] = \frac{M_{A}L}{6EI} \phi(\mu L)$	$\phi(\mu L) = \frac{6}{\mu L} \left( \frac{1}{\sin \mu L} - \frac{1}{\mu L} \right)$
NPTEL $S^{\circ} = M_{A}/\theta_{A} \longrightarrow S^{\circ} = \frac{3EI}{L} \frac{1}{\psi(\mu)}$	

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So, to cut a long story short, without going into the derivation, let us look at the results. So, these things can be worked out and you can prove this, the proof I will skip. It is an interesting proof. It is explained in, in great depth in the book, you can go through it. You can work these out, the carry over factor etcetera.

Let us look at the final pictures. So, they look like these. You know, if you have a spreadsheet, you can really write down, for any P value, for any mu L value, you can write down those factors.

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But, let us see what it looks like. It looks like this. On the x axis I have a parameter mu into L, which can be expanded as square root of P L squared by E I and on the y axis, I have that ratio, stiffness divided by E I by L and when one end is hinged, the answer is 3 E I by L typically, when there is no axial force acting, is it clear?

Now, look carefully. Look carefully. When I have no axial force acting, it is 3 E I by L. When I have some axial force acting, it reduces and, and it, it reduces to 0 in fact, when the level of axial force is what? You can see that, no, it is hitting 0. When will that be?

When you have, that is the limit. When you have the, asymptotically when it hits the pure buckling case for a simply supported beam, is it clear? So, listen carefully, if I have a simply supported beam, I apply an end moment here, M A, then, if I want to, or rather if I want a unit rotation, the moment that I need to apply here, is going to be 3 E I by L, if my axial force is 0.

If my axial force is a critical buckling load, then I can really play with it like plasticene, because it yields completely.

So, the stiffness drops to 0, is it clear? If I am applying P, which is much smaller, small value, then I really do not have to worry, because you see, it is kind of tangential at that initial value, can you see this?

So, the, I need to really worry, only if it crosses, the mu L crosses 1. So, I can really ignore. So, I take a call on, when I can ignore and when I can (( )), is it clear? But what happens when I apply a force? Mathematically I am getting a value, even when it crosses, crosses the value of pi, is it not? That value is pi there, 3.14. Why do I still get a solution? Mathematically you will get a solution, because it, you get a negative value of stiffness.

Ok, that is what the equation does. What is the negative value mean? It means your rotation and moment are in 2 different directions, which you will encounter, if you had a rotational spring there. It is very interesting, ok.

So, you can take some more load, till it hits the critical buckling load for this case, of a propped cantilever. So, this is just incidental. So, it can take some more values and that is the maximum it can take, is it clear? You do not have to worry about the negative portion. That positive portion is good enough, but, if you want the big picture, we get this also.

This is for the case of a simply supported condition. We will alnow look at the propped cantilever condition, where you need S and r and S starts with the value of 4 E I by L and r S starts with the value of 2 E I by L. As you apply axial compression, 4 E I by L will drop gradually and it is going to hit 0, when you have the critical buckling load, which is that propped cantilever case, remember the 2.045 pi squared E I by L squared and r picks up very, very slowly. R S is almost 2 all the time and then, if you keep applying more loads, you can get some more results, which are only of more theoretical interest than otherwise, is it clear.

So, we have these functions. These are called stability functions. S, r and S naught they are a function of a parameter called mu L. That is all you need, you can write, you can program it. So, if you give me P, if you give me L, if you give me E I, square root of P L squared by E I is mu L, that is my parameter for figuring out what is this stiffness measure. Stiffness measure will be related to 4 E I by L, 3 E I by L, 2 E I by L, is it clear.

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	Valu	Jes of	beam-o (under	axial c	ompre	ess me ssion)	asures	with µl		
$\mu L = \sqrt{\frac{PL}{E}}$	2 0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4-49 3
S° 3 EI/L		3 2.950	2.794	2.518	2.088	1.438	0.4082	-1.468	-6.51 8	-00
S EI/L	4	3.967	3.865	3.691	3.436	3.088	2.624	2.008	1.173	0
r	0.5	0.5063	0.5264	0.5637	0.6263	0.7310	0.9189	1.3157	2.5605	+00
	μ <b>L</b> = \	PL <sup>2</sup> EI	4.5	4	5	5.5	6		2π	
6	S EI/L		-0.0191	-1.9	807	-5.6726	-20.0	-20.634		
NPTEL	r		13.357	-2.5067		-1.3481	-1.0	-1.0396		

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So, to sum up, you can write a table, can make a table and generate these values and you have a different story when you have tension. Instead of axial compression, if you have tension, there is no question of any buckling happening and your stiffness is actually increased.

So, you also have equation. One way to check out is, when you solve the equation, you get hyperbolic function. Actually, you get the same function with a, instead of cos you

get cosh, c o s h, instead of sin you get sinh, you, you sinh here, tension is sinh, s i n h and you get tan h and so on, is it clear..



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So, this is just by the way. If you have chord rotation and we will finish with this, if you have chord rotation, then, if you have a simply supported case, do you think the chord rotation will have, cause any curvature? Pinned - pinned support settlement, do you think you will have any curvature? You do not get any curvature, but, will it affect, will you have any forces? Yes, you will, to satisfy equilibrium in the deformed configuration, you get now a vertical force.

So, what is your net axial force? It is not going to be P anymore, because you have vertical reactions, in the equally deformed configuration. So, it is going to be the resultant of P and P phi AB, phi is your chord rotation, right. So, if it is in compression it will look like this, if it is in tension it will look like that, just remember this ok.

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If you have a pinned-fixed condition, and you have a chord rotation, you have a curvature and you can prove that, you have, you have end moments, which are the same end moments as you got earlier, instead of 3 E I by L, you, right now, S naught, S naught is the replacement for the correction for 3 E I by L in which we include the effect of P, is it clear.

This is same as what we did earlier. Just remember, wherever you had 3 E I by L, you now start using S naught. Wherever you had, wherever you had 4 E I by L, you have S and the carryover will be r S.

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So, you have S into 1 plus r, instead of 6 EI by L squared, you will write S into 1 plus r. That is all you need to remember and the fundamentals are still the same. So, you can work out everything.

So, instead of 12 E I by L cubed, you will write S into 1 plus r divided by L into phi, but, you also have plus or minus phi AB, P phi AB in the deform configuration.



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We will end with this last relationship, which is an important point in, in, as an engineer. Let us say, I have a problem of buckling. I have a column. I want to increase its buckling strength. What should I do? Laterally support, ok.

So, shall I, if I have a pinned-pinned joint, shall I put a support in the middle? Will it solve the problem? So, will my, will my, will my critical buckling load get doubled? 4 times? Why will it go 4 times? Because of a L square. But that demands an infinitely stiff brace, right. K b should be infinity. No. No, you have to answer a question. Should k b be infinite or not? No.

I do not want buckling to take place. I do not want that joint c to move. So, k b should be infinite. Can you make an infinitely stiff brace? Not possible. So, be practical. No.

So, you can prove, you do not need to do this. You do not need to do this. You have to provide a minimum stiffness for that brace and that will guarantee this 4 E I. Let us check it out. Let us say, you provide a weak brace, low value of k b. How would you think it will buckle now?

If I put a hinge there, if I put a hinge there, then definitely it is going to buckle like this. Can, I, you find out the minimum value of k b needed from this picture? How do I get it? Yes, equilibrium, you are right. Yes, equilibrium. You will get a reaction there.

If I take the free body which is 2 P phi, when I solved that equation, that is a force in my spring. So, if I have a spring with a capacity greater than that, it will not happen.

So, that is how I get k b. So, practicing engineers have figured out a way of finding the minimum stiffness you need for a brace, but, this is when you cleverly put a hinge there, you had a pin joint. If I do not put a pin joint, then also will I get the same answer? Yes. Can you prove it? So, we look at that.

So, this is the answer. The minimum stiffness needed is 16 P E naught by L. You can prove it, which is, if you substitute the value of P naught, which is 2 pi squared E I by small I the whole square, small I is the, in this case, half the total length, right.

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So, how do you prove it? So, let us look at this. I am not putting a hinge in the middle. Today is a little long session, but, let us finish it.

We are almost there. We are going to score the goal now. We have a low stiffness, it is going to deflect like this, got it.

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You have high stiffness, it is going to deflect like that. The answer for this we know. This is going to be 4 times, right. I want it to do deflect like this and not like that. How do I do it? Well, I can prove it from first principles by solving this equation, that I, anything is possible.

I can write my equilibrium equations. I write my stiffness values, you, you understand? You can do an equilibrium study of this, solve the equilibrium equation.

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Here you have moments coming into the picture, which was not there when you had the hinge. And the moments are related to your stiffness measures. So, plug in those values

and if you substitute all those equations, solve the equation, it takes a little while. It is explained in the book. You will get in an interesting result. The interesting result is ...

You can find the critical value. That it turns out to be exactly the same solution. Because it is getting late, I am not explaining this to you, but, a quick look at this graph, this stiffness of the brace, when it is 0, I do not get any capacity. As I keep building up, it, at some point, it will buckle like an, you know it will buckle with a negative sine wave or it will buckle the other way. It will always buckle at the lowest load.

So, this is the mode shape that I want, the second one, k b minimum. Because it cannot take it, it will buckle at this rather than in the other... But, this is enough for me to achieve. I have 2 sine wave curves, reverse sine wave, I have got the, the stiffness that I need.

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So, for practicing engineers these kinds of fundamentals are very important and it is especially important, when you have a very long column. You can work out what the minimum bracing needed. It need not be infinite, but, you have to provide appropriate stiffness, to make a long column behave like a relatively short column, to increase its (( )).

This is the introduction. Tomorrow we will have a quick look at slope deflection method and finally, we look at matrix. Thank you for your patience.