

Advanced Structural Analysis

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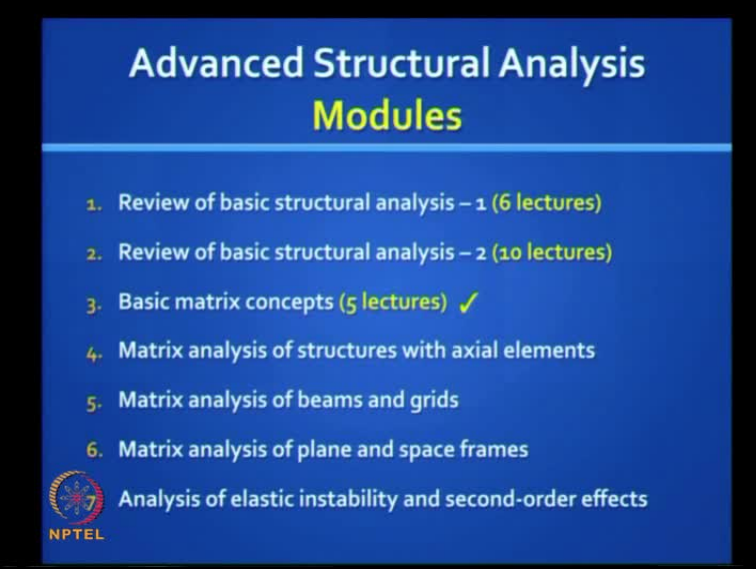
Indian Institute of Technology, Madras

Module No. # 3.5

Lecture No. # 21


Basic Matrix Concepts

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Advanced Structural Analysis
Modules


1. Review of basic structural analysis – 1 (6 lectures)
2. Review of basic structural analysis – 2 (10 lectures)
3. Basic matrix concepts (5 lectures) ✓
4. Matrix analysis of structures with axial elements
5. Matrix analysis of beams and grids
6. Matrix analysis of plane and space frames
7. Analysis of elastic instability and second-order effects

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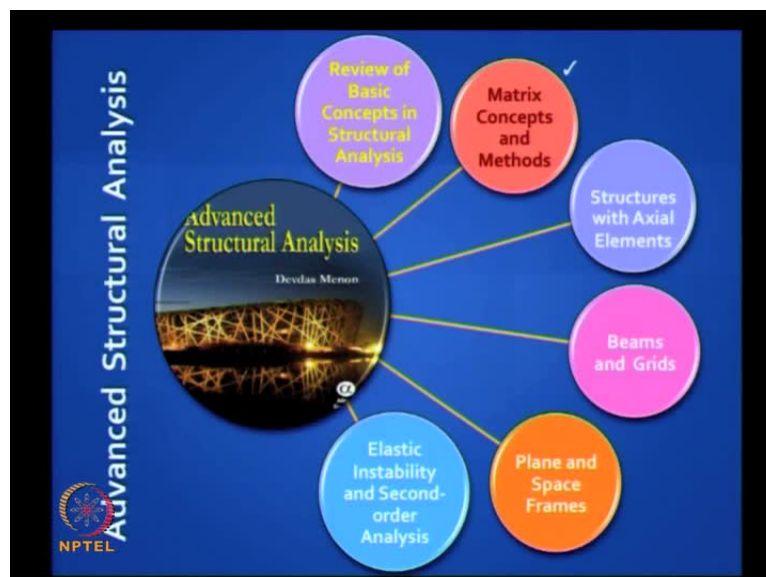
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Module 3: Basic Matrix Concepts

- Review of matrix algebra
- Introduction to matrix structural analysis (force and displacement transformations; stiffness and flexibility matrices; basic formulations; equivalent joint loads). ✓



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Equivalent Joint Loads


When intermediate loads act in between the joints of a structure, they can be converted to equivalent joint loads, to facilitate formulating the load vector in matrix analysis.

Direct Actions: F_{ef}^i


Indirect Loading: $\Delta F_{ef}^i = k_e^i D_{e, initial}^i$

$$F_f^i = T^{iT} (F_{ef}^i + \Delta F_{ef}^i)$$

Fixed end force vector:



$$F_f = \begin{bmatrix} F_{fA} \\ F_{fR} \end{bmatrix} = \begin{bmatrix} T_{DA}^T \\ T_{DR}^T \end{bmatrix} (F_{ef}^i + \Delta F_{ef}^i)$$



Eqvt joint load vector: $F_e = -F_{fA} = -T_{DA}^T F_{fR}$

Good morning this is lecture twenty one i hope to cover basics matrix concepts with this lecture so this will be the fifth lecture on this topic this is covered in the second and third chapters of the book on advance structural analysis there was one topic which we did not covered in the last session that is equivalent joints loads you are already familiar with the need for this you need this whenever you have intermediate loads we have used it when we did slope deflection method and moment distribution method

So when intermediate loads act in between the joints of a structure they can be converted to equivalent joint loads to facilitate formulating the load vector in matrix analysis that is because we want to limit the degrees of freedom.

Ah so you have two types of problems one where you have direct actions which are distributed and you can find the fixed end forces you have done it for beams but, it is possible you might get them in axial elements for example, a column which self weight that is a distributed load we will see how to handle that.

So this notation f_i what does it represent in the stiffness method in the i th element the fixed end forces that small subscript f stands for fix end forces along the a local axis so it is a local axis system right you can get fixed end forces from another problem indirect loading take a

truss when you let us say you have a lack of fit problem in the truss or let us say you have a temperature change problem that is indirect loading.

Ah how will you handle that how will you convert into fixed end forces.

(())

How will you find that end.

Elongation (()).

Where is now let us say it is statically determinate then why are you taking that as a fixed end force elongation will not give you the fixed end force it will give you elongation say Δ is a free elongation you get it in a bar from that where is the tension there is no tension in the bar yeah.

Then there is an elongation (())

Please understand the physics i have a statically determinate truss let us say i am doing by the stiffness method i heat all the bars let them all get heated where do i get tension.

No fixed force no tension.

No there is you are right but, you are not able to to give the physics behind the problem what you said is right you might get tension or you might get compression that is a different issue but, you will get an axial force when in which structure will you get an axial force.

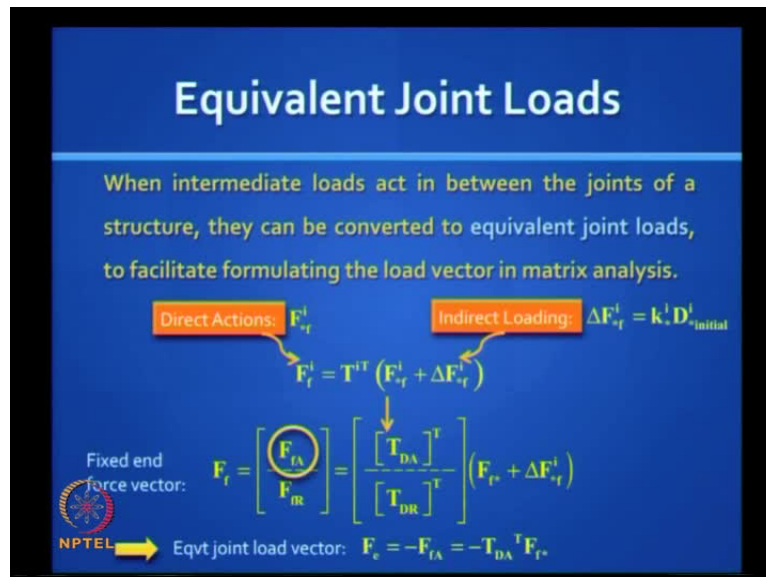
Sir if it is determinate we can move the structure at a.

We are doing by the stiffness method that is what you learnt till now right in the stiffness method how would you handle environmental changes in bars in a truss for example, it is a little tricky we will study it in detail in the next module but, i will give you a clue all structures are kinematically indeterminate even if they are statically determinate.

So we are always working as a base the primary structure so in the primary structure no movements are allowed at the joint so you have to arrest all the joints now you try heating those

bars you will get compression in those members or you cool those bars you get tension in those members so in the primary structure you get them and they get passed on to the joints that is how you handle it in stiffness method it is very interesting.

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So do you agree i can write down a set of initial displacements either contraction or elongation in a truss and if i just multiplied with the elements stiffness of that i get the fixed end forces we will see this in detail later so you got the local fixed end force vectors for all the elements either cause by direct actions that is intermediate loads or by indirect loading.

What do you do next in this conventional stiffness method which we studied you you have to convert these to the global access system how do you do that

Transformation matrix

What is the transformation matrix you will use.

(())

Will you use t i will you pre multiply by t i not t i then what t i.

T f

Don't mix up problems we are doing conventional stiffness method T^T comes in flexibility method so it is easy to get confused you remember the T enables you a transformation from the global axis system to the local axis system both for forces and displacements.

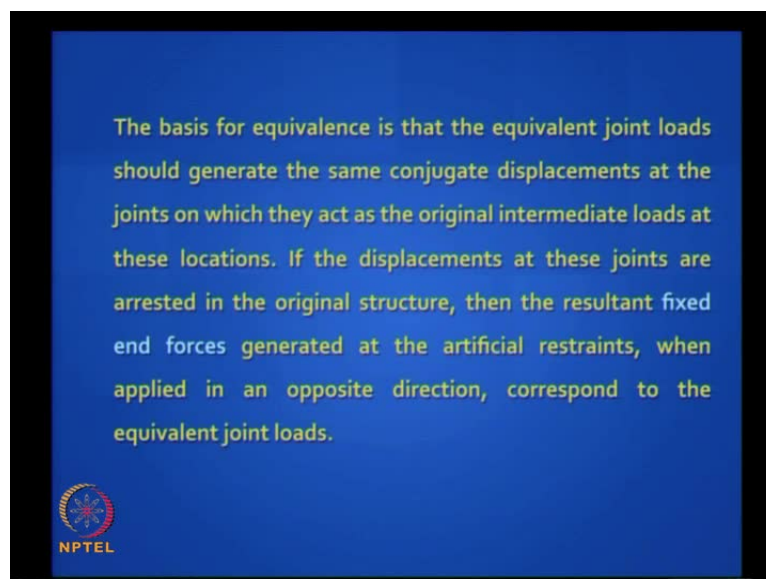
Now we are trying to switch from the local to the global so what is the transformation you have need to do.

T transpose

Where you are T transpose that is all you have to do so add up all your elements fixed end forces apply the transformation in the reverse direction so there you would go back to the fixed end forces and then then you can generate the fixed end force vector for the structure by assembling it in some appropriate way and you will realize you will get some fixed end forces coming to the active degrees of freedom they are going to be a loads and some will pylon to the restrained degrees of freedom they will add on to your reactions

So you pick up your fix equivalent joint loads from F^F a is at the active degrees of freedom and then you have to apply it in the opposite direction you remember in the classical stiffness method and that is your equivalent joint loads i'm just brushing up what we have done earlier but, we are doing it in a matrix format.

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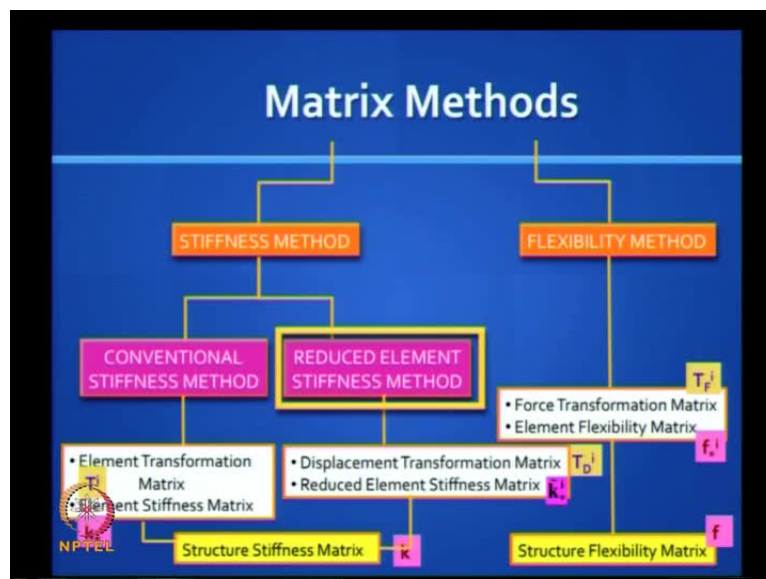


So the basis for equivalence is that the equivalent joint loads should generate the same conjugate displacements at the joints on which they act as the original intermediate loads at these locations remember we have shown you this earlier you have a original structure and you have first a restrained structure where you arrest all the degrees of freedom and then you have a structure in which you apply nodal loads

The joint displacements here are zero because you arrested everything the joint displacement the original structure are non zero so obviously if you do super position the equivalent joint load should act in such a way that you end up getting the same unknown displacements in the original structure at the joints not in between and that is the basic principle of equivalent joint loads.

If the displacements at these joints are arrested in the original structure then the resultant fixed end forces generated at the artificial restraints when applied in an opposite direction correspond to the equivalent joint loads you have to add up these equivalent joint loads with whatever additional joint loads that you may get ok

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Now we move on we had a quick look at the conventional stiffness method there is no way you fully understand it unless you apply to problems but, the idea here is to just lay road map now we

are going to look at the reduced element stiffness method we will see the flexibility method after that they are similar so it is easy to see them.

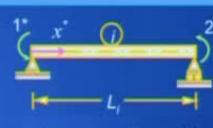
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Reduced Element Stiffness Matrix (local coordinate system)



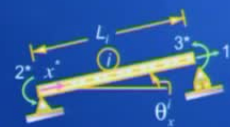
(a) Truss element
($\bar{q} = 1$)

$$F_1' = \left[\frac{E_1 A_1}{L_1} \right] D_1'$$



(b) Beam element
($\bar{q} = 2$)

$$\begin{Bmatrix} F_1' \\ F_2' \end{Bmatrix} = \frac{E_1 I_1}{L_1} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} D_1' \\ D_2' \end{Bmatrix}$$



(c) Plane frame element
($\bar{q} = 3$)

$$\begin{Bmatrix} F_1' \\ F_2' \\ F_3' \end{Bmatrix} = \frac{E_1}{L_1} \begin{bmatrix} A_1 & 0 & 0 \\ 0 & 4I_1 & 2I_1 \\ 0 & 2I_1 & 4I_1 \end{bmatrix} \begin{Bmatrix} D_1' \\ D_2' \\ D_3' \end{Bmatrix}$$

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If you recall we know why we can take advantage of these reduced element stiffness method because the rank of the element the conventional stiffness matrix at the element level is less than full so you can reduce it to the minimum to full rank and we've already discussed three types of elements we will see other types later but, this is elementary so we looked at the truss element you have only one degree of freedom reduced.

We looked at the beam element two degrees of freedom we have chosen just for convenience as simply supported arrangement you can very well choose a cantilever but, let us not get confused we will take one at a time and the plane frame element is a combination of the truss element and the beam elements so is the three degrees of freedom and three displacements correspond to one translation and two end rotations in terms of forces one actual force and two end movements

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Generation of components of the T_D^I matrix for a plane truss element

Kinematic approach to finding components of T_D^I (applying $D_j = 1$)

(Global coordinates can have arbitrary numbers)

$$\{D^I\}_{1 \times 1} = [T_D^I]_{1 \times 4} \{D\}_{4 \times 1}$$

$$e_j = \begin{bmatrix} ? & ? & ? & ? \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}$$

$$D^I = T_D^I D \Rightarrow T_D^I = \begin{bmatrix} -\cos \theta'_x & -\sin \theta'_x & \cos \theta'_x & \sin \theta'_x \end{bmatrix}$$

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$$D^I = T_D^I D = \begin{bmatrix} T_{DA}^I & T_{DB}^I \end{bmatrix} \begin{bmatrix} D_A \\ D_B \end{bmatrix}$$

$$T_{DA}^I = \begin{bmatrix} (D_1=1) & (D_2=1) & (D_3=1) & (D_4=1) & (D_1=1) & (D_2=1) & (D_3=1) & (D_4=1) \\ -\cos \theta_{x2} & -\sin \theta_{x2} & 0 & \cos \theta_{x2} & \sin \theta_{x2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

NPTEL

Now you may ask what about the shear forces and all that all that statically determinate once you got the end movements you got the shear forces right so we also looked at this recall we talked about the displacement transformation matrix and we derive this from first principles using what we call a kinematic approach i'll show you you can also get the same matrix the t d i matrix from

a static approach using this principle called the contra gradient principle which we are now going to look at.

But before that let us just to an application of the $t d i$ matrix let us take this frame which we have been playing along all this while you got this three element frame the primary structures are arrested we want to if you push d one equal to one what and if you have to generate the $t t$ two d a matrix which are put it with a circle here.

How will you get this matrix what will be this size of this matrix i'm just taking a second element which i have shown here second element has three degrees of freedom one two and three when i apply a unit displacement to the structure here arresting all the other displacements this will effect elements one and two just see the effect in element two what will how will you convert it to what what effect will it cause.

It will cause a compression in that member agreed and remember the you have only one degree of freedom in that element in the reduce format and that displacement is what we said it is an elongation right so we want to get that quantity what will it be it will be it is a component of one you know would not it be $\cos \theta$ will it be plus $\cos \theta$ or minus $\cos \theta$ minus $\cos \theta$ right and will you get so we have got this value $d i$ one star is going to be minus $\cos \theta$ what about $d i$ two star for element two will you get any movements will you get any rotations no because you are arrested that here you would not get anything there will you get any rotation here no ok

You'll understand when i show you the matrix but, at least tell me the size of the matrix what is a size of t two d a matrix you remember there are how many degrees of freedom here in this problem there were eight active degrees of freedom and four restrained degrees of freedom right

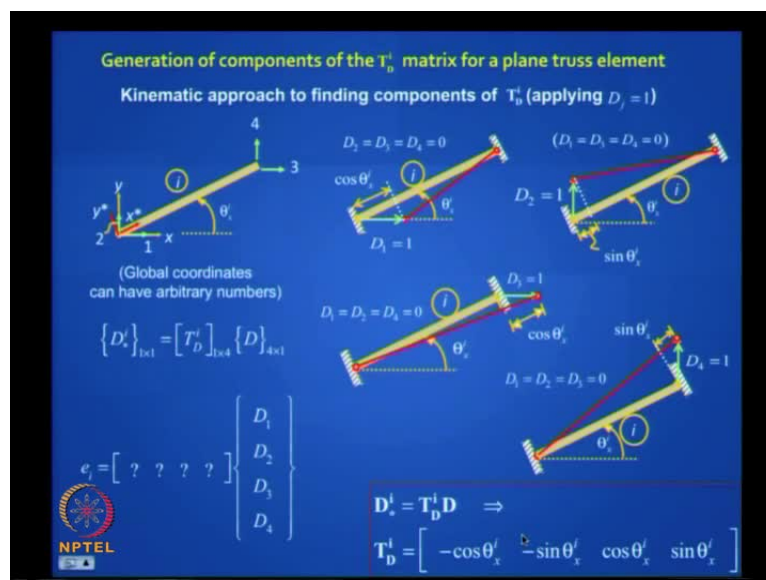
So this is the matrix will look like this am just pulling out the t two d a matrix here with the circle for this element if i have to generate the matrix the first column this is a three by eight matrix first column is what i write down when i apply d one equal to one to this structure do you agree when i apply d one equal to one to the structure i get just a contraction in that element and so corresponding to this first degree of freedom i get minus $\cos \theta$ of that element two i do not get a rotation at the left end i do not get a rotation at the right end.

So do you understand the physical meaning of first row second row third row the first row refers to what i get here when i keep applying a unit displacement to the structure the second row refers to the rotation here the third row refers to the rotation here likewise i can move ahead and go to apply d two equal to one which means i now lift up this joint and keep everything else restrained when i lift up this joint what do you think i will get i will get minus sin theta.

Do you understand or you do not you understand do not worry we will come back to this later.

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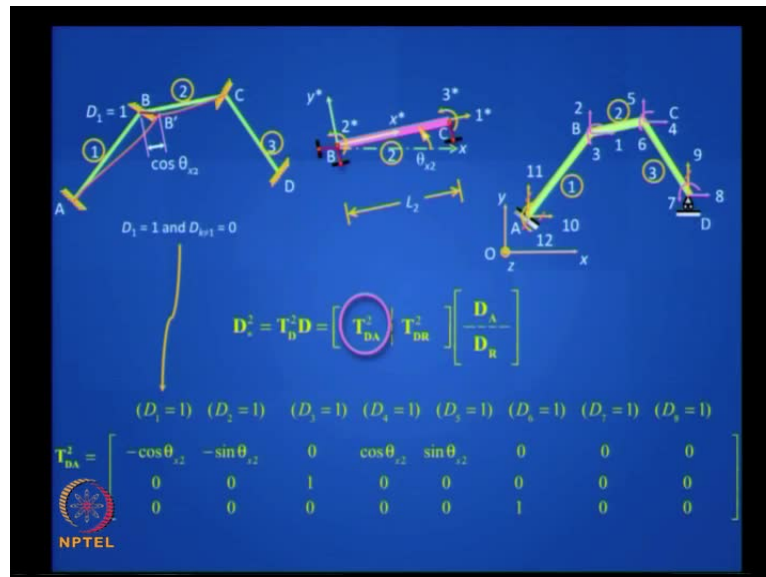
Because you will get a contraction the length of the material well that question you should have asks for why did you get a minus sin here see when i push this horizontally here i got a contraction so this was a minus sin when i lifted it up then also the length reduce i got a minus sin but, when i pull the right end here right i get a plus sign because i get you can see the red line is elongated and when i lift it up at the right end i get elongation right.

So what i have written here the displacement transformation matrix gives me the element elongations cause by a unit movement unit translation at the left end and the right end keeping all the other degrees of freedom clear so what i get minus whenever i push the left end i get plus when i push the right is that clear know i do not seem to see yes or no

yes.

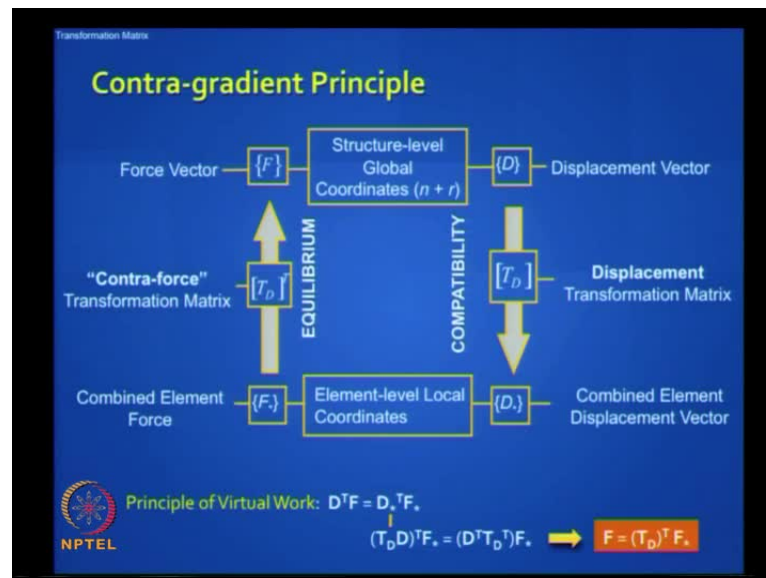
Give me a resounding yes if you understood but, do not worry with time you will understand.

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So broadly you get the picture hang of this matrix now this when you do reduced element stiffness method and you would do flexibility method you cannot program it as blindly as you could the conventional stiffness method because you are trying to take a shortcut you have to apply your brain a little bit especially in generating the fixed end force vector which we will see later so you can do this.

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Ah now let us look at the contra gradient principle i have shown this earlier but, here at the lower level i put the element level forces the combined element force vector f^* which actually refers in a case of a truss to all the axial forces in the truss the combined element displacement vector d^* which are the bar elongations in the truss the structure level displacement vector which are the joint displacement in the in the truss and the structure level force vector f ok.

F i can include both the active degrees of freedom and the restrain degrees of freedom or i can limit to the active degrees of freedom now the transformation we did was if i know the joint displacements when i get the individual member deformations in this case the bar elongation by applying the t_d matrix as a transformation i just demonstrated that and this actually gives a [com/compatibility] compatibility relationship between the structure joint displacements then the member bar elongations right

Likewise there is a set of equilibrium relationships which relate f^* with f and do you think this relationship is linked with the t_d matrix can we prove that relationship how do you prove that relation[ship]-.

This is a very important concept how do you relate the element level forces to the structure level forces in the way we related the element level displacements to the structure level displacements what principle will you use ah.

Principle of super positions like.

But whenever you are doing transformations of matrices you are using principle of super position whether you like it or not that is how the it is a linear combination of vectors so that is implicit what is the answer you have gave.

(())

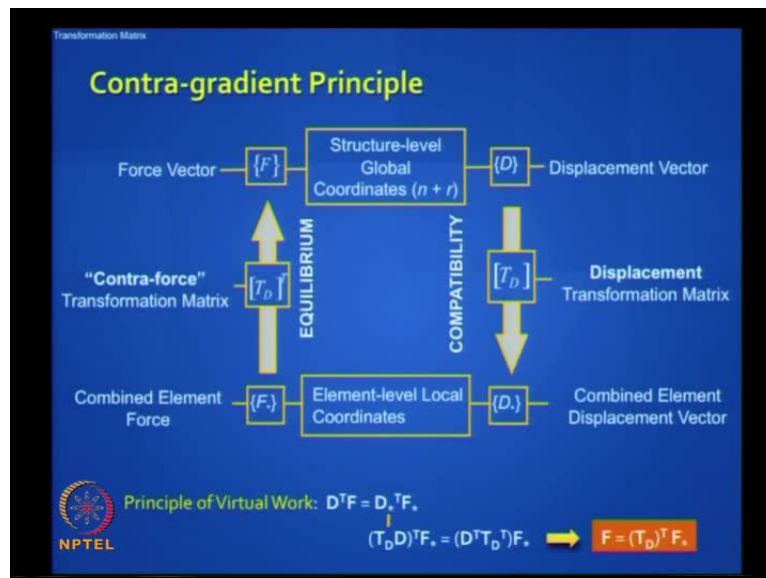
But you have to talk matrix language you are just saying what i wrote here i am asking you how do i build a relationship between this and this where you are just come out.

Using this so you use the principle of virtual work your real work is that principle which ties together the force field with the displacement field so do you realize the principle virtual work says the total external virtual work product is equal to the total internal virtual work product.

Now here all the forces are real all the displacements are real but, it is the product easily virtual because i am not applying any factor half am taking just taking the forces so do you agree to this equation that i have written here that if i take the structure level forces and find the total work done it simply given by either $d^T f$ or $f^T d$ similarly, if i take the internal virtual work i can write either $d^T f^*$ or $f^{*T} d$.

Now we will use the other one in the flexibility method now how to proceed after this because you will end up with a beautiful equation what can we now substitute in this equation.

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Instead of d with ah

Instead of.

D star.

No that is wrong what you need you have to bring in t d into the picture man.

(()).

Equal to t d.

Ah

T d into d star.

You replace d star with.

T d into d.

T d into d that' s that is how you bring into picture and you have to take the transpose of it.

Now $a^T b$ the whole transpose is $b^T a^T$ right so do not you get this nice relationship then what what is this establish

$(())$

Then you keep your eyes open and compare the left hand side with the right hand side and then you arrive at a simple equation

F is equal to.

F is equal to $t d (())$

So what does this mean this means that is your relationship that is a powerful relationship that means that means once you would establish the $t d$ matrix you are actually tying you you are also linking simultaneously a relationship between the force vector at the element level to the joint force in terms of a truss what we are saying if you know a bar forces in the truss then the joint loads that could have cause a forces are related in this way but, it is not a very useful relationship in the sense of normal problem is you have unknown bar forces when you have known loads ok

The real problem is that but, this is useful in building up the force transformation matrix now because you are doing it in the reverse direction now this was the normal direction for $t d$ you are you are moving in that contra direction the slope is moving in the reverse direction hence the word contra gradient gradient is the slope

you know normally things flow downhill but, here it is going to flow uphill when you are dealing with the force field and yeah.

$(())^T \times$ star into $t d$ into f star

Sorry.

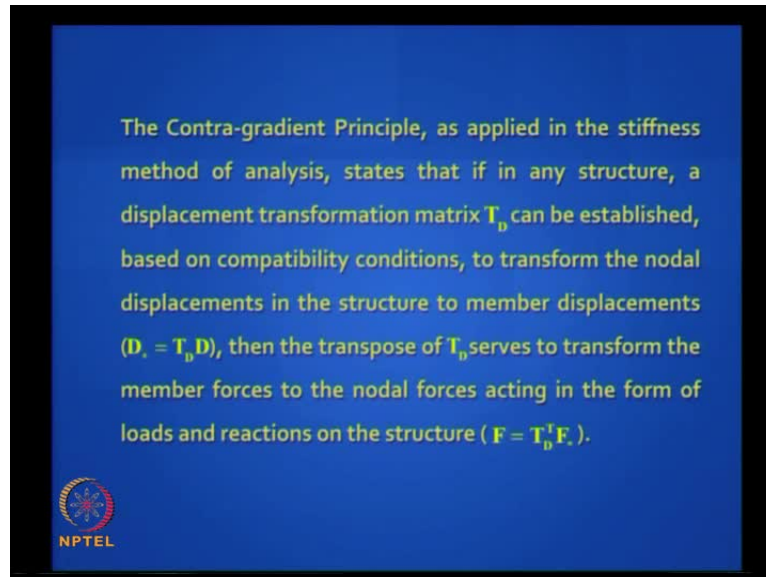
Similarly you can write f star equal to $t d$ into.

You cannot do all that.

Where $t d$ is not $(())$

You cannot do all that see in the stiffness method t d unique from first principle we derived it what you are saying is something we will do in the flexibility method there we will bring the t f matrix.

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Right now do not jump ahead because you make mistakes do it slowly is this proof crystal clear it is a simple proof beautiful proof let us apply this proof but, before that a statement the contra gradient principle as applied in the stiffness method of analysis states that if in any structure displacement transformation matrix t d can be established based on compatibility conditions to transform the nodal displacements in the structure to member displacements d star equal to t d d then the transpose of t d serves to transform the member forces to the nodal forces acting in the form of loads and reactions on the structure f is equal to td transpose f star ok

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Generation of components of the T_D^i matrix for a plane truss element
 Static approach to finding components of T_D^i (by finding $[T_D^i]^T$)

(Global coordinates can have arbitrary numbers)

$$\{F^i\}_{4 \times 1} = [T_D^i]^T \{F_*^i\}_{1 \times 1}$$

$$\begin{Bmatrix} F_1^i \\ F_2^i \\ F_3^i \\ F_4^i \end{Bmatrix} = \begin{Bmatrix} ? \\ ? \\ ? \\ ? \end{Bmatrix} \{N_i\}$$

$$\begin{Bmatrix} F_1^i \\ F_2^i \\ F_3^i \\ F_4^i \end{Bmatrix} = \begin{Bmatrix} -\cos \theta_x^i \\ -\sin \theta_x^i \\ \cos \theta_x^i \\ \sin \theta_x^i \end{Bmatrix} \{F_*^i\}$$

$$\Rightarrow T_D^i = \begin{bmatrix} -\cos \theta_x^i & -\sin \theta_x^i & \cos \theta_x^i & \sin \theta_x^i \end{bmatrix}$$

NPTEL

Remember this problem we wanted to find the T_D^i matrix well if you use static and find the T_D^i transpose matrix then you take the transpose of the transpose you will get the T_D^i matrix right now if i look at this relationship F_*^i in this you have four a four components here right F_i one F_i two F_i three F_i four this force internal force here is actually N_i you know that is what i called F that is what that is what the element force there is right.

Actually this is written the wrong way yeah the F_i is four by one right then what you get here what you get here these are the four vector joint that i need that means if i take this structure and imagine i have a bar and have a unit actual force in that bar what are that load components that i get at the two ends of the bar which will which will satisfy equilibrium that is a static approach can you tell me what these four components what will be the what should this force be equal to.

(())

Plus or minus.

Minus.

There is axial tension.

Minus plus plus minus.

Then how will you satisfy $\sum F_x = 0$.

Minus.

What about here this one will be plus or minus.

Plus sir.

And what will this be.

Plus.

Plus what.

Sin.

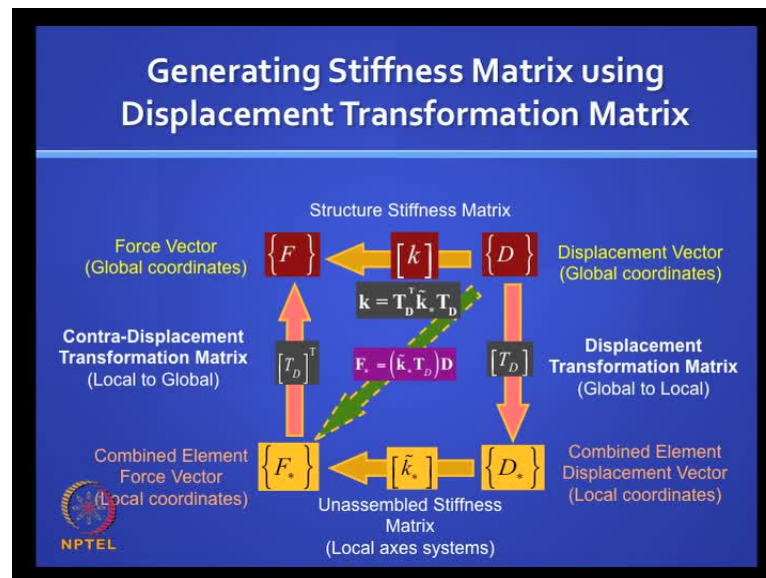
Sin theta what will this be.

Minus.

There i hope it does not take too long.

So you see you got this that is your $t d$ transpose matrix you take the transpose of that you get $t d$ you see very clever way of getting the $t d$ matrix in the reverse direction this is statics previously we did kinematic you have a choice normally students are weak in kinematics they love doing $\sum F_x = 0$ $\sum F_y = 0$ you can get it this way and take the transpose you get the $t d$ matrix but, if you are good in geometry you can do that i want you to be good in both and you can do either way you are is this clear

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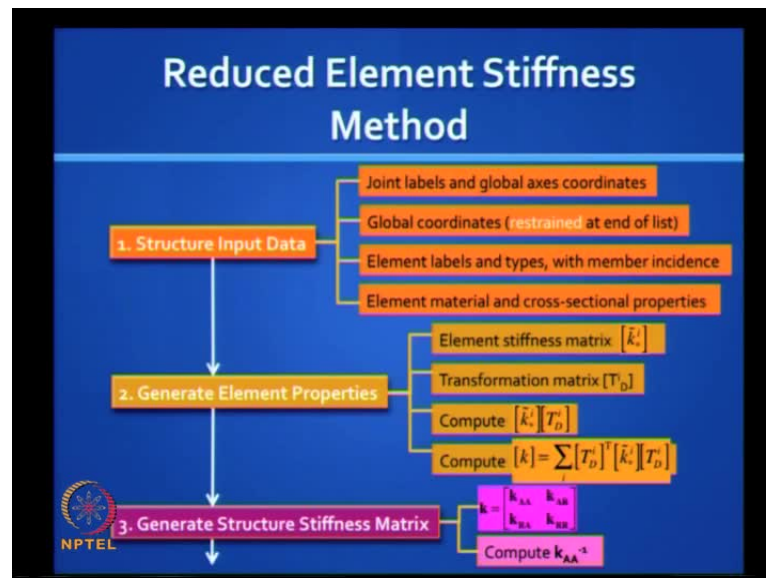
We will see examples of this in trusses later now you got you got the stiffness matrix at the element level you got the $t d$ matrix which relates the element level displacements to the structure level displacements our job is to find the structure stiffness matrix.

Remember we were not very comfortable with the conventional stiffness method where you took put the linking coordinates and you have to put it in the slot and so on here you will have no problems because you got this relationship this is $t d$ now you got the same play ground everything is familiar to you you got a new beautiful relationship from the contra gradient principle which is what which is this.

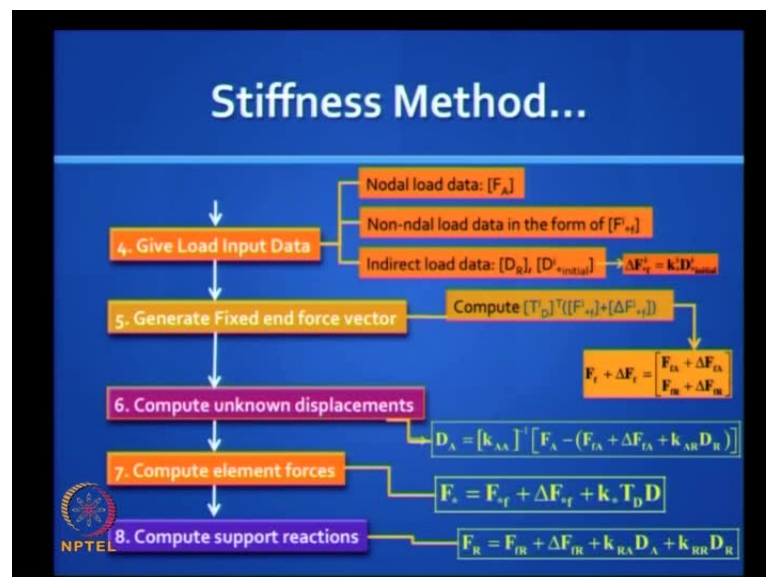
Does this make sense to you the structure level on top the element level below $t d$ flowing downward for displacements $t d$ transpose flowing upwards for forces then take the diagonal the diagonal just says that if you tell me what the displacements are in a truss joint displacement immediately i will tell you what the bar forces are by doing this transpose this matrix.

Remember i put a k tilde to to remain myself that i am dealing with the reduced stiffness matrix so agree so this is the unknown bar forces are nothing but, this times this times that is it clear then if i want to take this shortcut directly from d to f what do you think k will be k will be $t d$ transpose k tilde star $t d$

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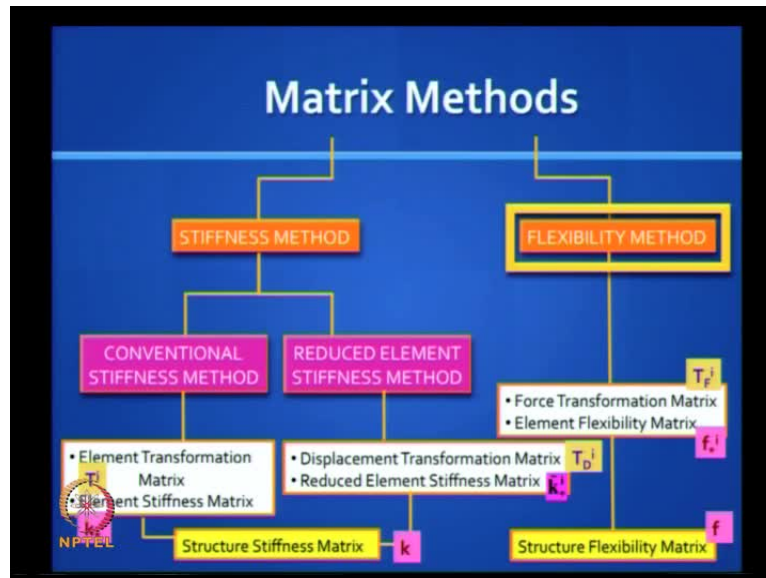
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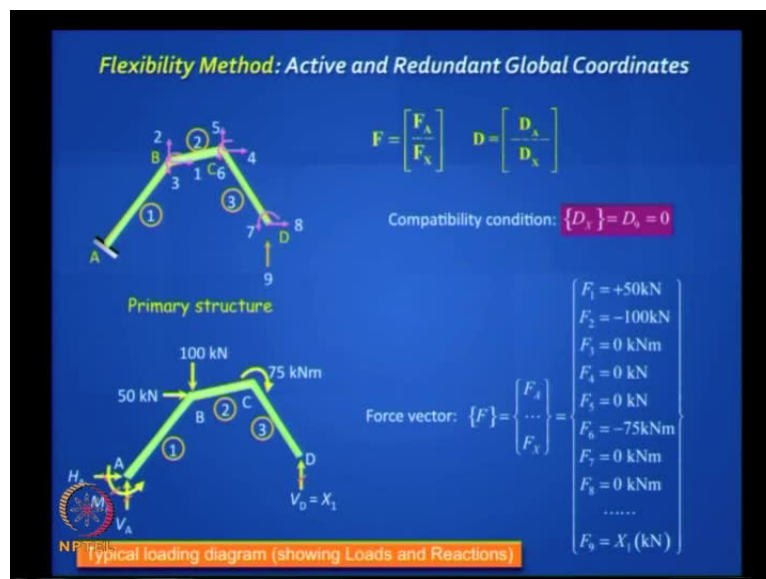
So you will find repeatedly this kind of pattern in merging you have to premultiply the stiffness matrix whether it is a regular stiffness matrix or the conventional stiffness matrix by the transpose of some transformation matrix in this case T_D^T and also post multiply with T_D so that is that is what you do when you do this you will get the full stiffness matrix without any effect so that is to sum of the reduced element stiffness method we will go through the details later it

similar to the conventional stiffness method the steps are similar but, you will really understand this only when we do some problems.

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So we are now set to move on to last topic which is the flexibility method the first problem you are notice the main [dis/difference] difference between flexibility and stiffness methods this you

have even at the global coordinates level you have to be careful- why because here you have a choice of redundancy now this particular structure is it statically indeterminate or determinate.

Indeterminate sir.

It is determinate.

Indeterminate.

Indeterminate what is the degree of indeterminacy.

One.

One which would you like to choose as a redundant

The roller support (())

The support is not redundant is a force you will so corresponding to that redundant so we want the primary structure to be cantilever say that that means you want to get rid of that roller support right.

So in terms of statics you have to replace it with a force right and that is your single redundant here so as far as your external loads are concerned there is an one to one similarity between flexibility method and stiffness method you have eight degrees of freedom active degrees of freedom and put the same numbering one two three four five six seven eight i am not showing the reactions yet actually flexibility method is more suited for manual analysis.

So you can always get the reaction once you got the free body so for the present let us not worry about reactions but, i will show you can find reaction if really want to also will skip that so what do you do remove it this is your primary structure and to complete the picture introduce introduce your redundant as the next coordinate so you put the f_x the redundant forces here there is only one supposing at three so you'll put nine ten eleven as your degrees of freedom in addition to the active degrees of freedom we will call them redundant degrees of freedom ok

So for example, if you had this problem which i showed you earlier then v_d is your redundant x one and this is how you would write the force vector right the first part the f_a part we have seen earlier you are just putting f_9 now f_9 is your additional unknown redundant ok

You need to pay attention right so f vector is f_a f_x d vector is d_a d_x so this is the major departure from the stiffness method because the stiffness method we have d_a d_r f_a f_r in flexibility method for statically indeterminate structure we have f_a f_x and d_a d_x as a force vector displacement vector does it make sense to you right ok

In these vectors what is known and what is unknown well f_a is known these loads are usually given to you f_x is unknown this is the redundant you need to find d_a is not known and d_x is known because d_x gives you your compatibility conditions right so what is your compatibility condition well in this particular problem d_x is having only one element d_9 and it is zero but, suppose you add a support movement settlement then it will not be zero you'll put the correct value we are familiar with this we did method of consistent deformation so it falls into place so far so good ok.

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Dealing with support reactions and displacements in flexibility method

In the flexibility method of matrix analysis, we can replace supports by appropriate rigid links, and thereby find support reactions as the internal forces in these links, if required (not usually done in manual analysis).

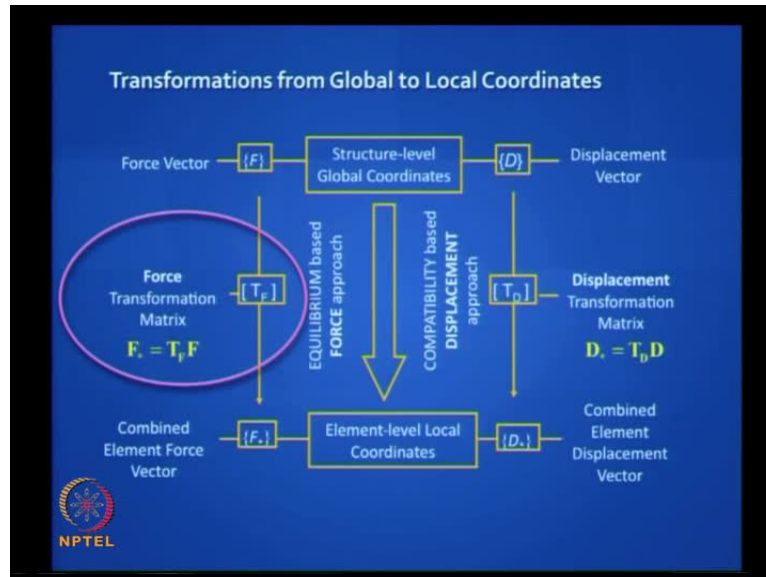
Similarly, we can assign support displacements, if required, by way of initial deformations in the rigid links.

NPTEL

Next point if you want to find support reactions you can do it we will see it later in the flexibility method of matrix analysis we can replace supports by appropriate rigid links so the force in a

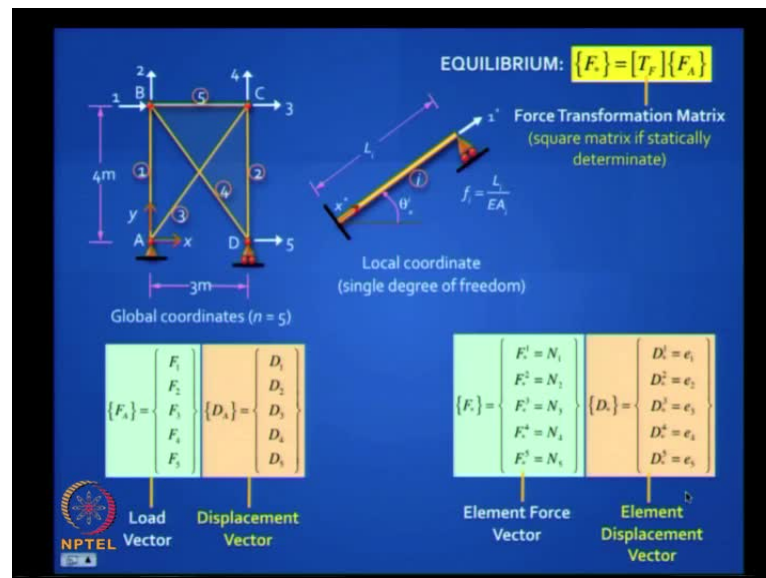
rigid link becomes your you convert the external reaction to an internal force in an imaginary rigid link it is a little trick you can do but, it adds to a size of your matrix you might not find it convenient to do in in practice ok

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Similarly we can assign support displacement if required by way of initial deformations in the rigid links we will see that later now we are going to look at what one of you asked know can we do this transformations so this is our t f transformation remember we did t d transformation we looked a t d transformation in the stiffness method both the conventional stiffness method and the reduce stiffness method now you do not worry about t d you look only at t f and we have based obviously t f must be based on based on equilibrium alone.

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So you can uniquely define $t f$ only for a statically determinate structure because you have a unique statically admissible solution for a statically indeterminate structure you cannot uniquely define $t f$ please note that demonstrate with an example very familiar example this is plane truss you identify the active degrees of freedom there only five of them right one two three four five ah

You have one element remember you done the reduced element stiffness method the flexibility method follows the same as same system as the reduce element stiffness method so what is what is your unknown displacement in that element it is a bar elongation what is the unknown force in the element it is a bar force tension assume positive elongation assume positive ok

So that is a degree of freedom we want to write down the $t f$ matrix we want to write down the $t f$ matrix which is f^* is equal to $t f$ into f a how do you write the $t f$ matrix

(())

What do you need to do to write the to generate the $t f$ matrix.

Apply external

Apply

Unit load.

Unit load how many of them one at a time how many of them you answer the question that i have raised.

(())

What's the size of t f matrix.

Five by five.

Five by five by five so far so good how do i get the first column in the t f matrix.

(())

Apply unit load where.

(())

At one so it is called apply f one equal to one then how do i get the second column.

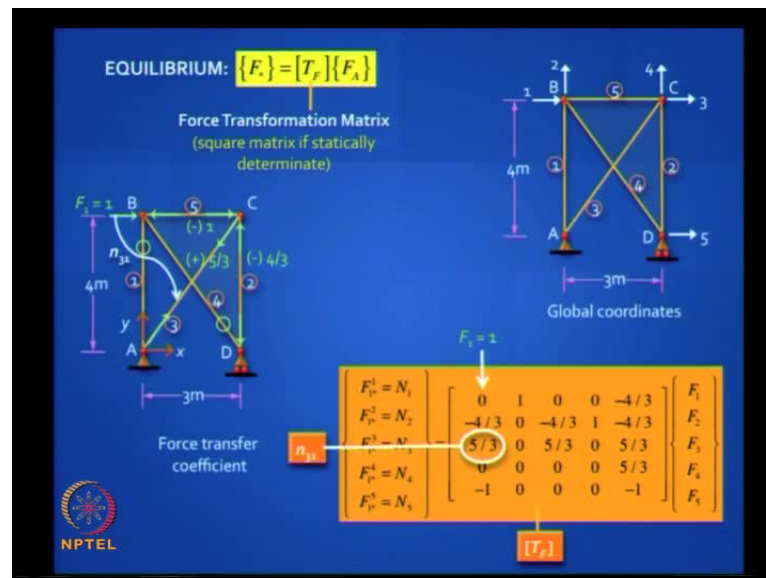
F two equal to one.

F two equal to one so i apply so i have to apply five times and one at a time is it clear

Let's do one of them so this is my first transformation matrix five by five it turns out to be a square matrix it is a unique matrix it is interesting so f a has five components d a has five components they refer to the joint movements right you know what f one is you know d one is f two d two etcetera this is external what about internal f star has also five because there are five bars right

Now i written a f one star f two star f three star but, you can write if you want f one one star because really your degree of freedom is one star so whether i write just star here or one star here in the case of a truss it does not matter you can choose either i'll show you alternatively how you can do it.

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So this is nothing but, the bar force and the internal d one star is nothing but, the bar elongation do you understand what this means everything make sense now we need to generate the t f matrix so let us do that so let us take that matrix apply f one equal to one can we quickly analyze this other any zero bar forces here by inspection tell me which bar force ah.

One four.

One and four it is quite fast well look at the support d roller support from there you can make out one of those two bar forces zero which is zero four is zero right then you go up to the joint b which is zero one is zero then what is the force in bar five one is it tension or compression.

Compression.

It's compression so minus five then you go to joint c what is the force in bar three.

Three four three by four.

The component the horizontal component of the axial force in bar three must be equal to one so how much would plus or minus.

Plus plus plus

Plus very good then the vertical component of this must be the force end bar two how much is that.

Four by $(())$ minus four by three.

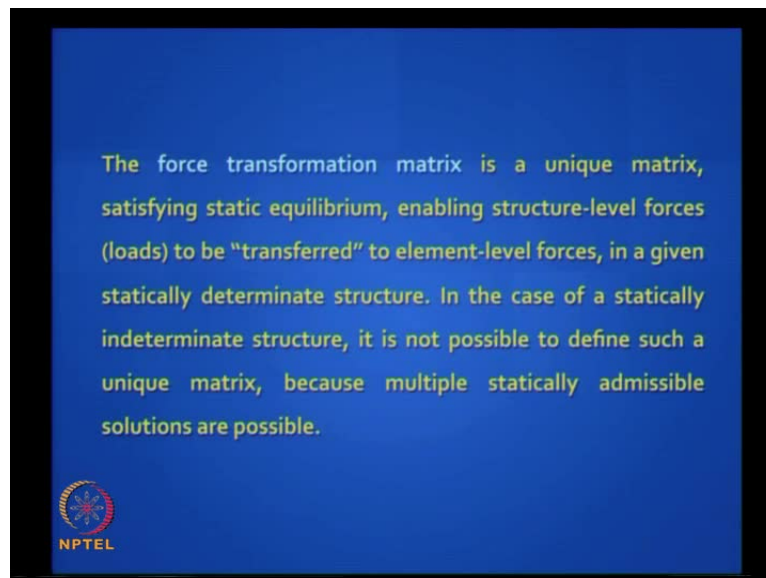
That's all like this you do five times you can get that matrix and some of them are really easy that is what it looks like so do not gets scare t f matrix is very comfortable because these we were raised on statically determinate structures that is of orders so it is very easy to generate the especially the second one f two equal to one has only one element ok.

So you understand what t f matrix is good now let us take any element in that that element we use to call n three one remember small n for unit loads n three one refers to the force in bar number three cause by f one equal to one remember we have use this notation earlier it is called the first transfer coefficient why it why it is called transfer coefficient.

$(())$ what is that $(())$

Yeah it is like all these five bars form on organization a partnership and whatever money comes or goes out to that you know it is an immediate transaction online bank account some money come in some fraction of it goes so that is exactly what is happening you apply a load f one equal to one on the structure can you see what i have done you apply f one equal to one immediately you got a transfer n three one to bar number three in this case five by three but, do not take that analogy too far because you are actually transferring more money than that came in right

(Refer Slide Time: 37:47)



So this is the meaning of force transfer coefficient right the force transformation matrix is a unique matrix satisfying static equilibrium enabling structure level forces or loads to be transferred in inverted commas to element level forces in a given statically determinate structure in the case of a statically indeterminate structure remember it is not possible to define such a unique matrix because multiple statical admissible solutions are possible.

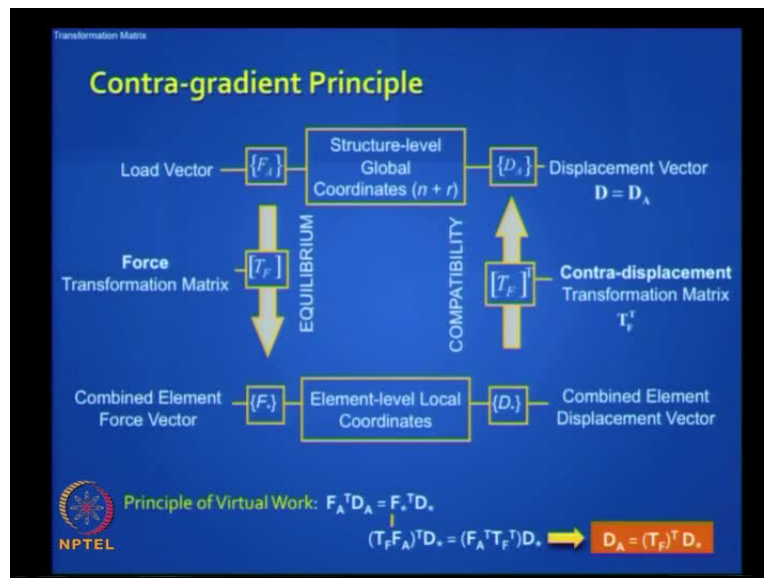
So how do you handle an indeterminate structure primary structure.

(())

Primary structure primary structure is statically determinate and apply.

(()).

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Apply super position right that is what we did in consistent deformation so that is what we are going to do so here again you have a contra gradient principle an actually have a principle which is the one that proves why you do not need to satisfy explicitly compatibility in statically determinate structure you remember all structures have to satisfy three requirements what are the three requirements equilibrium compatibility and force displacement relationship which include the restrain relationship constitute relationship.

Statically indeterminate structures equilibrium alone cannot help you get the unknown forces we have force to look into the other two statically determinate you are getting away we just managing looking at equilibrium the question arises a good student should ask this question does it mean some of automatically compatibility satisfied yes prove it.

So the contra gradient principle actually proves it how do you prove it we we did a similar proof earlier we did principle of virtual work can we do the same principle then we will get the same old result no there we proved $t d$ transpose here we have to prove $t f$ transpose.

So how do you use the same principle you flip what.

(()) f star n a

Ah you there we did d transpose f now we do f transpose d now because we are keeping reactions out of this game anyway reactions do not do any work if the supports you are move we have f a transpose d a agreed is equal to f star transpose d star agreed total external virtual work product is equal to total internal virtual work product what do you need to substitute now.

F star equal to t f .

Where now you are good this is simple logic what do you end up proving by comparing.

(())

D a is stable.


(())

Fantastic so without your knowledge when you are satisfying equilibrium in a statically determinate structure you are also simultaneously satisfying compatibility provided a structures statically determine ok.

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Contra-gradient Principle

The Contra-gradient Principle, as applied in the flexibility method of analysis, states that if in a statically determinate structure, a force transformation matrix T_f can be established, based on equilibrium conditions, to transform the loads on the structure to member forces ($F_m = T_f F_A$), then the transpose of T_f serves to transform the member deformations to the active structure displacements ($D_A = T_f^T D_m$).

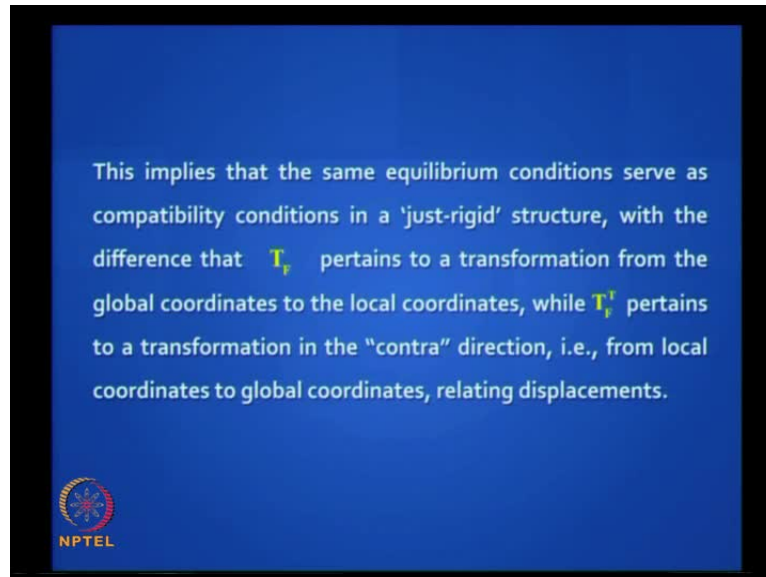


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The contra gradient principle as applied in the flexibility method of analysis states that if in a statically determinate structure a force transformation matrix t f can be established based on

equilibrium conditions to transform the loads on the structure to the member forces f^* is equal to $t^T f^*$ into f^a when the transpose of $t^T f^*$ serves to transform the member deformations to the active structure displacements d^a is equal to $t^T f^*$ transpose t^* .

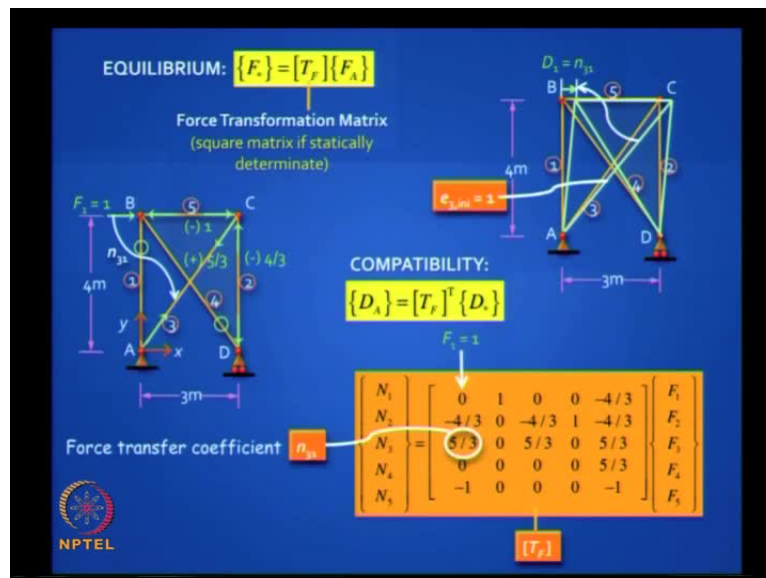
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This implies that the same equilibrium conditions serve as compatibility conditions in a just rigid structure with a difference that $t^T f$ pertains to a transformation from the global coordinates to the local coordinate while $t^T f$ transform transpose pertains to a transformation in the opposite direction in the contra direction in the contra gradient direction that is from local to global when you are relating displacement that is it

So you see you will never forget the contra gradient principle it has two two parts to it one part when you find $t^T d$ transpose that is in stiffness method the other part is when you find $t^T f$ transpose you will you will very comfortable with flexibility method unfortunately our focus is not the flexibility method because the computers do not to do that method for reasons we will see shortly.

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Let's see the implication of this t f transpose we have done this right now what are the implications let us take a bar in the truss the third bar for the fun of it let us heat that bar how does this t f transpose matrix help us find find what.

(())

Ah

(())

Displacement at c the end this if i say e three initial that is a initial displacement it could be a lack of heat it could be heating that bar by say one mm what do i end up finding let us take n three one five by three we did a bank transfer recently right for the forces what is five by three going to be displacement it must be a displacement where at b or c.

(()) at c

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Lack of fit:
 Bars '1' and '2' are too long by 1.5 mm; bars '3' and '4' are too long by 3 mm; bar '5' is too short by 1 mm. Find the various joint displacements.

$$\{D_{\text{initial}}\} = \begin{Bmatrix} +1.5 \\ +1.5 \\ +3 \\ +3 \\ -1 \end{Bmatrix} \text{ mm}$$

EQUILIBRIUM: $\{F_r\} = [T_r] \{F_s\}$

$$\begin{Bmatrix} F_{1r}^1 \\ F_{1r}^2 \\ F_{3r}^3 \\ F_{4r}^4 \\ F_{5r}^5 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & -4/3 \\ -4/3 & 0 & -4/3 & 1 & -4/3 \\ 5/3 & 0 & 5/3 & 0 & 5/3 \\ 0 & 0 & 0 & 0 & 5/3 \\ -1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix}$$

COMPATIBILITY: $\{D_s\} = [T_r]^T \{D_r\}$

$$\begin{Bmatrix} D_{1r}^1 \\ D_{1r}^2 \\ D_{3r}^3 \\ D_{4r}^4 \\ D_{5r}^5 \end{Bmatrix} = \begin{bmatrix} 0 & -4/3 & 5/3 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -4/3 & 5/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -4/3 & -4/3 & 5/3 & 5/3 & -1 \end{bmatrix} \begin{Bmatrix} +1.5 \\ +1.5 \\ +3 \\ +3 \\ -1 \end{Bmatrix} = \begin{Bmatrix} +4 \\ +1.5 \\ +3 \\ +1.5 \\ +7 \end{Bmatrix} \text{ mm}$$

NPTEL


Well you what you need to do is to c well all the bars will move while heated you know the compatibility will satisfy remember this five by three is the force you got in bar three when you applied f one equal to one so when you change the length of the bar three you will get the displacement at one where is one where is f one d this much so it is fantastic where you derived from statics of matrix you can get that matrix to get joint displacements and let us give a so d a is t t f transpose d star let us give another demonstration of it same truss now i have a lack of it i arbitrarily have different you know it it came to the site the bar is came with different lengths with that is how bad manufacturing is.

So bars one and two are found be two long by one point five mm bar is three and four two long by three mm bar five too short by one mm find the various joint displacements well most people will have two necessarily either do this by geometry or actually do the experiment and then measure how much it moved.

What about intelligent if you are brilliant you can invoke this beautiful contra gradient principle and get it in a jiffy how do you get it in a jiffy well first you have to convert these into the language of matrices how will you write that you have an initial displacement element level matrix which vector which you can write like that i n i short for initial do you agree i put plus when i have a larger length put minus when i have a shorter length does it make sense right.

Then i invoke this powerful principle these are my this is my t f matrix and i take the transpose of that matrix i have got all the answers in one shot if i am using matlab correct you you have to agree matrix methods very powerful if you know how to use it when to use it and when not to use it when not to abuse it when not to misuse it and in practice the misuse and abuse is wide spread ok

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For a 'just-rigid' structure

$$\left(\mathbf{T}_F^T\right)^{-1} = \mathbf{T}_{DA}$$

$$\left(\mathbf{T}_{DA}\right)^{-1} = \mathbf{T}_F^T$$

The contra-displacement matrix \mathbf{T}_F^T (which is based on a flexibility formulation) for a statically determinate (just-rigid) structure and the active displacement matrix \mathbf{T}_{DA} (which is based on a reduced stiffness formulation) for the same structure, are both square matrices describing compatibility relationships between the local and global coordinates, one matrix being the inverse of the other.

So if you compare now the two transformations you get another interesting result for a statically determinate structure t f transpose inverses t d a and t d a inverse is t f transpose so you can get a potential question in the exam where i give you one of them and ask you to figure out the other you have to bring the same.

The contra displacement matrix that is a name i have given but, they it is not a very common name you can say t f transpose matrix which is based on a flexibility formulation for a statically determinate just rigid structure and the active displacement matrix t d a which is based on a reduce elements stiffness formulation for the same structure of both square matrices describing compatibility relationship between the local and global coordinates one being the inverse of the other


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FLEXIBILITY MATRIX

The flexibility coefficient f_{ij} may be defined as the displacement D_i generated at the coordinate i on account of a unit load at the coordinate j ($F_j = 1$), with no other load acting on the structure ($F_{i \neq j} = 0$).

Element flexibility matrix: $\{D^e\} = [f^e] \{F^e\}$ \Rightarrow $\{D\} = [f] \{F\}$
(Unassembled)


Structure flexibility matrix: $\{D\} = [f] \{F\}$



Flexibility matrix definition you know just like the stiffness matrix each element is called the flexibility coefficient f_{ij} is defined as a displacement d_i generated at the coordinate i on account of a unit load at the coordinate j that is f_{ij} is equal to one with no other loads acting on the structure in stiffness method we said the same thing as it with all other displacement arrested then only the definition is complete you can write it this way you can put all the elements together this way ok

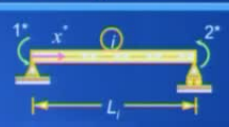
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Element Flexibility Matrix $\mathbf{f}_e^i = \mathbf{k}_e^i$ (local coordinate system)




(a) Truss element
($\tilde{q} = 1$)

$$\mathbf{D}_e^i = \left[\frac{L_e}{E_e A_e} \right] \mathbf{F}_e^i$$



(b) Beam element
($\tilde{q} = 2$)

$$\begin{Bmatrix} D_{e^i}^1 \\ D_{e^i}^2 \end{Bmatrix} = \frac{L_e}{6E_e I_e} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} F_{e^i}^1 \\ F_{e^i}^2 \end{Bmatrix}$$



(c) Plane frame element
($\tilde{q} = 3$)

$$\begin{Bmatrix} D_{e^i}^1 \\ D_{e^i}^2 \\ D_{e^i}^3 \end{Bmatrix} = \begin{bmatrix} \frac{L_e}{E_e A_e} & 0 & 0 \\ 0 & \frac{L_e}{3E_e I_e} & -\frac{L_e}{6E_e I_e} \\ 0 & -\frac{L_e}{6E_e I_e} & \frac{L_e}{3E_e I_e} \end{bmatrix} \begin{Bmatrix} F_{e^i}^1 \\ F_{e^i}^2 \\ F_{e^i}^3 \end{Bmatrix}$$

NPTEL

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Structure Flexibility Matrix for a Statically Determinate Structure

$$\{D_s\} = [f_s] \{F_s\}$$

$$\begin{Bmatrix} \{D_s^1\} \\ \{D_s^2\} \\ \vdots \\ \{D_s^m\} \end{Bmatrix}_{m \times 1} = \begin{bmatrix} [f_s^1] & & \\ & [f_s^2] & \\ & & \ddots \\ & & & [f_s^m] \end{bmatrix}_{m \times m} \begin{Bmatrix} \{F_s^1\} \\ \{F_s^2\} \\ \vdots \\ \{F_s^m\} \end{Bmatrix}_{m \times 1}$$

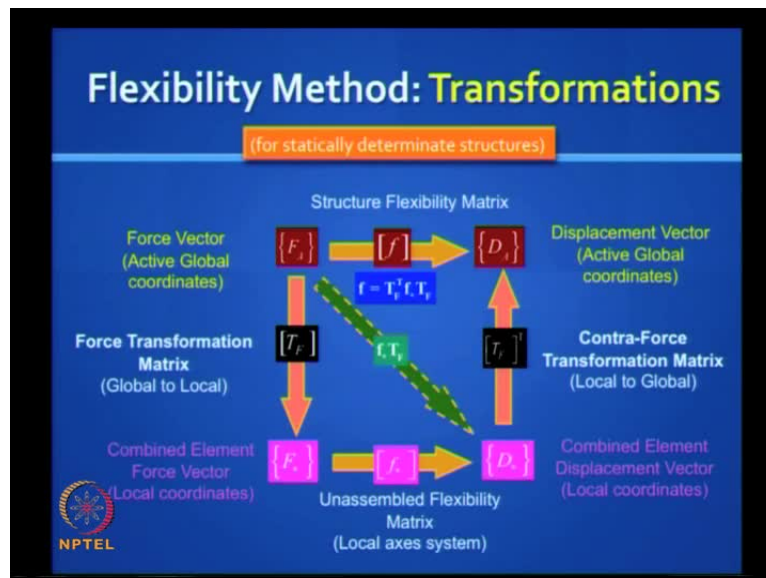
$$\mathbf{D}_A = \mathbf{T}_F^T \mathbf{D}_s = \mathbf{T}_F^T (\mathbf{f}_s \mathbf{F}_s) = \mathbf{T}_F^T \mathbf{f}_s (\mathbf{T}_F \mathbf{F}_A) = (\mathbf{T}_F^T \mathbf{f}_s \mathbf{T}_F) \mathbf{F}_A$$

$$\{D_s\} = [f_s] \{F_s\}$$

$$\Rightarrow [f] = [T_F]^T [f_s] [T_F]$$

NPTEL

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It's call the unassembled with the diagonal form and you need to get the structures flexibility meth[od]- now it is simple we played this game so many times it is not difficult you know the element flexibility stiffness is is just the inverse of what we did for the reduce element stiffness matrix and so this we know if you expanded it will look it will have a diagonal form like this remember we did it recently and then you can either write equations if you have found of writing equations or you can play this game

I i suggest you write this game it is very easy right first at the element level f star is now you see the direction is moving to the right because flexibility always oppose to stiffness there are two different paths so if you know the forces element level forces you multiply by the flexibility matrix you get the element level displacement right in in a bar you multiply the bar force in a truss multiply the bar force with its flexible axial flexibility you get the bar elongation right.

At the structure level you need to find f relating f a with d a how do you get f from f star small f i put small f because capital f goes to force well let us do it the right royal will play the game by its rules so first this this transformation it flows this way and on the left side t f transpose and contra gradient says this flows this way you take the diagonal it will flow this way as you take the shortcut it goes that way that is your proof t f transpose f star t f is your structure flexibility matrix.

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Statically Indeterminate Structures

EQUILIBRIUM: $\{F_s\} = [T_F]\{F_A\}$

Cannot be uniquely defined using equilibrium alone!

However, for a chosen primary structure (statically determinate), the force transformation matrix is unique. Thus, the T_F matrix in a statically indeterminate structure depends on the choice of redundants F_X .

$$\{F_s\} = [T_{FA}]\{F_A\} + [T_{FX}]\{F_X\} \Rightarrow F_s = \begin{bmatrix} T_{FA} & T_{FX} \end{bmatrix} \begin{bmatrix} F_A \\ F_X \end{bmatrix}$$

$$F_s^T = T_F^T F = \begin{bmatrix} T_{FA}^T & T_{FX}^T \end{bmatrix} \begin{bmatrix} F_A \\ F_X \end{bmatrix}$$


If you got these matrices with you you do not multiply those matrices put kill the dam computer the job to do it press the button get your f matrix so you got statically indeterminate structures here the problem is d f cannot be uniquely defined so what you need to do primary structure for a chosen primary structure d f is unique what does it mean each student in this class can arrive at the different t f matrix so it is difficult for the examiner to check the results so he has to wait for the final results to come out.

But whoever does it has to follow the same path now i said t f a and t f r i you know partition it t f a refer as to that part which is dealing with the loads on the primary structure and t f r sorry its t f x is that which deals with the unknown redundance and a you can do that the individual element level or you can take the combine level right.

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$$\begin{bmatrix} D_A \\ D_X \end{bmatrix} = \begin{bmatrix} D_{A,initial} \\ D_{X,initial} \end{bmatrix} + \begin{bmatrix} f_{AA} & f_{AX} \\ f_{XA} & f_{XX} \end{bmatrix} \begin{bmatrix} F_A \\ F_X \end{bmatrix}$$

$$\Rightarrow \{D_A\} = \{D_{A,initial}\} + [f_{AA}]\{F_A\} + [f_{AX}]\{F_X\}$$


$$\{D_X\} = \{D_{X,initial}\} + [f_{XA}]\{F_A\} + [f_{XX}]\{F_X\}$$

$$\Rightarrow \{F_X\} = [f_{XX}]^{-1} (\{D_X\} - \{D_{X,initial}\} - [f_{XA}]\{F_A\})$$

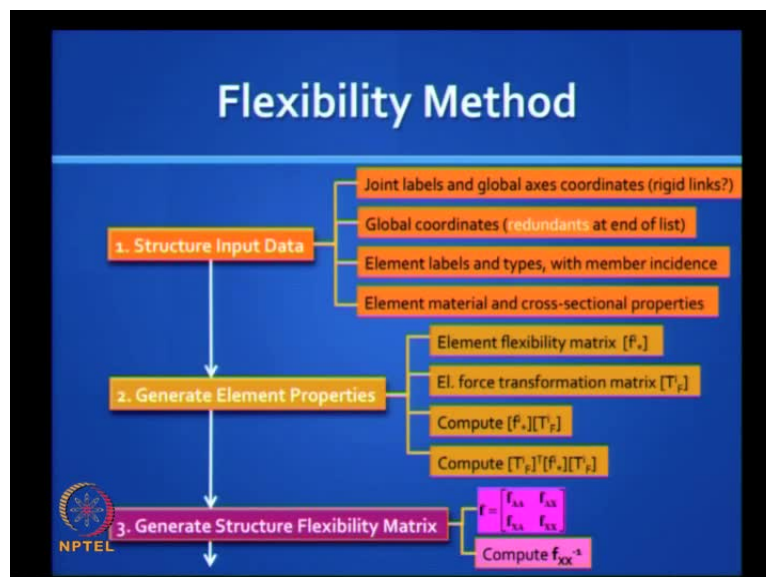
$$D_X = 0 \Rightarrow \{F_X\} = -[f_{XX}]^{-1} [f_{XA}]\{F_A\}$$

Structure Flexibility Matrix for a Statically Indeterminate Structure

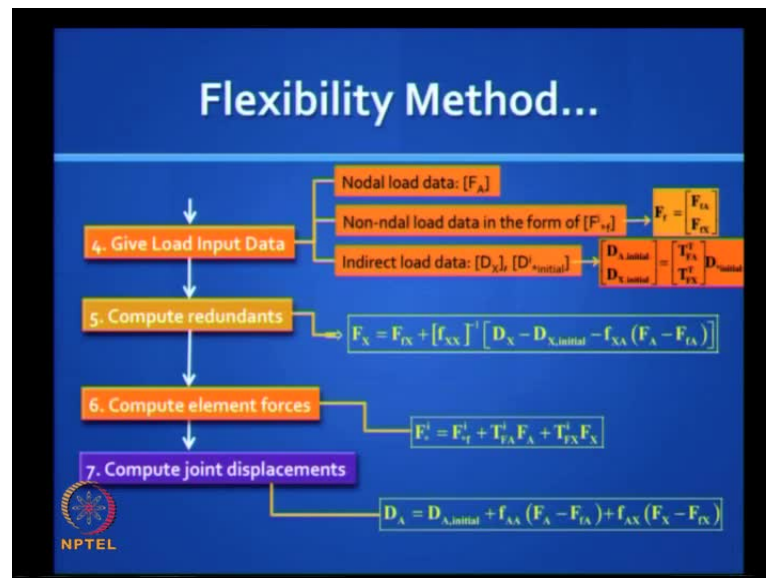
$$\{D\} = [\bar{f}]\{F\}$$

$$[\bar{f}] = [f_{AA}] - [f_{AX}][f_{XX}]^{-1}[f_{XA}]$$


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Stiffness Method vs Flexibility Method

1. Basic Unknowns?	Displacements	Forces
2. Type of Indeterminacy?	Kinematic	Static
3. Unique primary structure?	Unique !	Not unique !
4. Force-displacement relation format?	Stiffness	Flexibility
5. Matrix always well-conditioned?	Yes !	No !
6. Governing equations?	Equilibrium	Compatibility
7. Intermediate loads?	Equivalent joint loads	Adopt from stiffness method!

NPTEL

So you can write down these equations they are compatibility equations we will study them at in great depth i just want to rush through it because this is just a road map and you can write down the procedure just as we wrote the procedure earlier for the flexibility method i want to conclude with this last slide summing up all that we've learnt till now.

Broadly stiffness in flexibility method differ in some some terms what are the basic unknowns in stiffness method.

(())

Stiffness methods they are forces.

Of course yes (())

in stiffness method the unknown are

Displacements.

In flexibility method the unknowns are this is child's play ok

Next one what is the type of indeterminacy in stiffness method.

Kinematic

Kinematic verses.

Static static.

Brilliant what is a unique do you have a unique primary structure in the stiffness method.

No.

Do you have a unique primary structure that means will all the students in a class.

Yes.

Yes what about the flexibility method.

No.

Well depends if all of you if all fools think alike then it will be unique but, otherwise potentially it will be different right.

What about the force displacement relation format what is the format you have in stiffness method.

(())

Yeah but, if you want to write it in word what is a format.

(())

It's a stiffness format whenever flexibility format come on is a matrix that you get the stiffness of flexibility always well condition yes in a stiffness method.

(())

Well sometimes yes though but, you cannot say always right.

Governing equations.

Ah compatibility.

Compatibility in which one.

(()) flexibility method compatibility.

Equilibrium in stiffness compatibility in flexibility lastly how do you deal with a intermediate loads.

(())

In stiffness method equivalents are in flexibility method.

(())

No no on your computer you are number of degrees of this is the real this is the achilles heel of flexibility method this is what destroys the beauty of the method you have to borrow the same dam concept from the it is a tragedy because there is no other way you can convert to that means you have a you have a statically determinate portal frame with the uniformly distributed load

load on top we have to ascertain that that beam is fix fixed make it statically indeterminate and borrow from stiffness method how they calculate the equivalent then put it here that is a problem so flexibility method is fantastic for trusses and not particularly great when you deal with intermediate loads thank you