Mathematical Geophysics

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Week - 02

Lecture - 09

Hello everyone. Welcome back to the SWAYAM NPTEL course on Mathematical Geophysics. This is Module 2: Fundamental Theorems, and we are going to begin Lecture Number 4 in this module. This lecture is titled Curl and Stokes' Theorem. In this lecture, the following concepts are covered.

The overall idea of Curl and Stokes' Theorem is divided into various components. First, the basic concept of a curl. Then we look at different formulations of curl in various coordinate systems. Next is Stokes' theorem and then the interpretation of Stokes' theorem. Finally, we look at various applications of the curl operator and Stokes' theorem in geophysical studies.

So let us begin. What is a curl? The physical concept behind the curl is the requirement to measure the rotational tendency of a field. The rotational tendency of the field can be understood with the help of the adjacent diagrams. In this diagram, you can see that the field lines circle around a given point P.

The direction of the field lines is indicated by the red arrows. Note that the direction of the field is nothing but the tangent to the field lines. Due to this rotational character of the field, the field lines appear as circular lines. The field lines may also be elliptic or any other closed loop for the curl to exist. Thus, the measure of how much a field can circulate around a given location gives the curl.

It is also important to consider the direction of the rotation. In these two figures, the direction of rotation is opposite to one another. We can see that the first picture indicates an anti-clockwise direction of the rotation, while the second figure in blue shows a clockwise rotational tendency. The curl of a field **F** is positive for an anti-clockwise rotational tendency, while it is negative for a clockwise rotational tendency. Next, the mathematical concept of a curl.

Curl is a vector quantity which is obtained by the operation of the gradient vector as a cross product upon a velocity field. Next, the mathematical concept. Mathematically, the curl is a vector quantity. It can be obtained by operating the cross product between the gradient operator and the vector field. Thus, in vector calculus, the curl of a vector field **F**, having the components F_x, F_y, F_z in Cartesian coordinates, is represented as:

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}.$$

For the evaluation of the curl, let's say we consider the direction of the unit vector $\hat{\mathbf{i}}$. Neglecting the corresponding row and column, we could take the determinant of the remaining elements. So, the **i**-th component of the curl would result as shown above.

Now, we will look at the various interpretations of the curl from a physical point of view. First, what happens if the curl of a field is zero? We have seen this in the previous lecture as an irrotational field. The field becomes irrotational, which indicates that the tendency of rotation is absent. For example, if the field lines, as shown in the sketch, are perfectly parallel to one another, then $\nabla \times \mathbf{F} = 0$.

If $\nabla \times \mathbf{F} \neq 0$, it indicates a rotational field. Now, if the curl of a field is positive, it indicates that the field has an anticlockwise rotation or a counterclockwise rotation. Then, on the other hand, if the curl of the field is negative, the field has a clockwise rotation. So, we can understand that the curl represents the rotational tendency or curvature of the field lines.

Thus, we can understand that the curl physically represents the curvature effect of the field lines. If the field lines are more curved, then the curl is higher. And if the field lines are parallel to one another, then the curl vanishes. Next, we look at the expression for curl in different coordinate systems as relevant for geophysical applications. In the spherical coordinate system (r, θ, ϕ), co-ordinate, the curl is given as this expression.

Note that the coefficients are not constant. In the cylindrical coordinate system (ρ, ϕ, z), the curl is given as this expression. Just like the divergence operator, cylindrical coordinates also have non-constant coefficients for some of the terms, while other coefficients are constant.

Next, we look at various vector identities, which are important to understand and interpret geophysical fields.

The first vector identity indicates that the curl of the linear combination of two fields F and G is equal to the linear combination of the curl of the vector fields.

$$\nabla \times (a\mathbf{F} + b\mathbf{G}) = a(\nabla \times \mathbf{F}) + b(\nabla \times \mathbf{G}).$$

Next, we have the curl of a scalar multiple of the field equal to the gradient of the field crossed with the field itself plus the scalar function f multiplied by the curl of the vector field.

$$\nabla \times (f\mathbf{F}) = (\nabla f) \times \mathbf{F} + f(\nabla \times \mathbf{F}).$$

This is, in some sense, similar to the commonly used differential operator, where the derivative of the multiplication of two functions is obtained by keeping one of the functions constant and taking the derivative of the other. Since here a scalar function f and a vector field capital F are involved, we have the following two terms on the right-hand side. On the first term of the right-hand side, the field is kept constant while the scalar function undergoes a gradient operation, while the second term has the function f, a scalar field, held constant while the curl operator is operated on the vector field F.

The next vector identity is the curl of F cross G.

Which is the curl of the cross product of two vector fields. It equals four terms. These four terms can be categorized into two classifications. First involving divergence quantities, and the second involving gradient quantities. This indicates that if a cross product of two vector fields is considered for an assessment of the curl operation or its curvature effects, then the concept of the divergence of individual fields and the gradient of individual fields comes into consideration. If the gradients and divergence exist in the field, then the curl will have a finite value.

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}).$$

Next, we have the fourth identity, which is the curl of the curl of F, which is essentially a double curl operation. This is similar to the second-order derivative of a function. The double curl operator can be broken down into two components. The first component involves the gradient of the divergence of F. We can understand that all these terms involve second-order derivatives, as one derivative occurs for the gradient while the other occurs for the divergence operation. The second term is nothing but the Laplacian operator on the field F.

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}.$$

The fifth identity indicates that the curl of any gradient of a scalar field equals zero. Next, the divergence of the curl of any vector field equals zero. These two identities can be interpreted as follows. When the curl of a scalar field goes to zero, it means that the scalar field is behaving as parallel lines, which indicates that the gradient of F is irrotational. Thus, we can assume or conclude that the gradient of scalar fields are irrotational quantities.

$$\nabla \times (\nabla f) = 0.$$

Similarly, the curl of F has a divergence equal to zero, which means that the curl of any vector field does not diverge or converge.

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

With the basic ideas in place regarding the curl operator, we have the essential tools to understand the Stokes theorem. Stokes theorem is essentially a bridge between the rotational behavior of a field and the circulation of the field along the boundary. Consider this closed loop. **S** denotes the surface area which is enclosed by the loop **L**.

The Stokes theorem relates the behavior of a field at the boundary and the interior of this loop. Consider this surface and the loop bounding it to be in a field **F**. Then the Stokes theorem is an understanding of the behavior of the field inside the loop while just looking at the properties of the field on the bounding line. Mathematically, it relates surface integrals to line integrals:

$$\oint_L \mathbf{F} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Recall that the Gauss theorem is also a similar quantity which relates volume integrals to surface integrals.

Thus, Gauss theorem and Stokes theorem can be understood in similar lines. Have a look at the adjacent diagram. This diagram is a clearer version of the sketch just drawn. We have the field **F** vector and the area **S** and the bounding line **L**. Consider a small region which has the normal vector $\hat{\mathbf{n}}$ having an area dS. This region has the boundary elemental lines as dx and dy.

Now, Stokes' theorem relates the circulation with the flux of the curl. What is circulation? Circulation is the line integral of the field along the bounding line. So, the circulation of the field **F** per unit area can be given as the ratio of this circulation to the area over which the circulation is calculated along the boundary line. The curl of **F** in the direction of the perpendicular vector $\hat{\mathbf{n}}$ can be obtained as follows: Thus, in the limit as the area goes to zero, that is, $dS \rightarrow 0$, the area element is approximately an element of the tangent plane.

The surface at the point in dS. This leads to the fact that $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ equals the limiting value of the line integral. This can be understood by considering the convergence of the entire area to a single point. If this area reduces to a single point, then the line integral and the flux become equal because the field lines are nothing but the tangent to the bounding lines. Thus, Stokes' theorem can be obtained by integrating on both sides, which results in the flux of the curl.

Thus, Stokes' theorem can be obtained as follows. Note that Stokes' theorem involves first the circulation and second the flux of the curl. So, it is an interesting theorem that relates the flux of the curl of a field to the line integral of the field. Next, we look at some interpretations of Stokes' theorem. If $\nabla \times \mathbf{F} = 0$ at each point in a region, then the line integral would also go to zero.

This happens because, by Stokes' theorem, the line integral is equal to the flux of the curl. Since the curl is zero, the flux also reduces to zero. So, we can obtain without further calculation that the circulation of the field becomes zero. This is valid for every simple closed loop inside the region or enclosing the region. This has the implication that **F** is a conservative field.

For conservative fields, the circulation is independent of the path of integration from **A** to **B**, which are two points in that region. For example, if $\nabla \times \mathbf{F} = 0$ over this entire surface **S**, then any line integral within the surface would be equal. This would indicate that **F** is a conservative field. In line with previous discussions, we have understood that if **F** is a conservative field, then it can have a scalar potential. The scalar potential is a quantity V such that:

$$\mathbf{F} = -\nabla V.$$

Now, having gone through the various aspects of curl and Stokes' theorem, we look into the various applications of the curl and Stokes' theorem in geophysics. First of all, the electromagnetic method, which is used in geophysics for exploration and related energy resource estimations. Faraday's law is of utmost importance. Faraday's law indicates that:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

where **E** is the electric field and **B** is the magnetic field. This means that if the magnetic field varies with time, the curvature of the electric field increases or decreases as the case may be.

So this is used in natural electromagnetic phenomena like the Earth's magnetic field and magnetotelluric investigations. In the second application, which is related to geophysical fluid dynamics and oceanography, vorticity is a very important quantity. The vorticity field is nothing but:

$$\boldsymbol{\omega} = \nabla \times \mathbf{v},$$

where **v** is the velocity vector. This means that if the velocity vector is such that the particles are moving in such a manner that the velocity field lines have curvature, then the vorticity field becomes non-zero. So we can understand that the circulatory motions of various fluid particles would give rise to vorticity fields.

Now, vorticity fields are used to measure local rotational tendencies, which are used in modeling ocean currents and eddies. They are also used to analyze atmospheric circulations such as cyclones, anticyclones, and jet streams. Thus, the curl operator denotes a source or sink of circulatory motions in the form of a vorticity field. Having gone through the various aspects of the curl and Stokes' theorem and their relation to geophysical applications, it can be understood that the curl and Stokes' theorem are of utmost importance in various geophysical applications such as oceanic and atmospheric circulation patterns, electromagnetic field dynamics, and geomagnetic surveys.

We have also looked at seismic wave propagation and rotational components in seismology as applications of various fields and vector operations in previous lectures. It is also important to understand that the curl and Stokes' theorem can be applied to such geophysical applications. Plate tectonic stresses and mantle convection analysis also make use of vector fields, and their curves help in interpreting various geophysical phenomena. Thus, the curl and Stokes' theorem would be very useful tools for geophysical applications. One can look at the following references for a better understanding of the curl and Stokes' theorem and their applications in geophysical studies.

Thank you.