Mathematical Geophysics
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Hello everyone. Welcome to the Swayam NPTEL course on Mathematical Geophysics. We will continue with the third lecture of module two, the fundamental theorems. This lecture is titled Divergence and Gauss Theorem. In this lecture, the following concepts are covered. The divergence and Gauss theorem. The components of this lecture are concepts of divergence, the divergence in different coordinate systems, the Gauss theorem, and application of Gauss theorem in geophysical studies. Let us begin. What is divergence?

As the name suggests, divergence indicates a spreading out or convergence of field lines. It is a measure of the degree of spreading or converging of field lines at a particular point in space. It provides an idea about how a source or sink of a field behaves at a particular locality in space. It can be understood with the help of adjacent diagrams. Consider these two diagrams.

The left diagram shows a divergent field. This is the point in consideration, and the divergent field acts in such a manner that the field lines spread out from this point. Such a divergent field can also be categorized as a source. Second, we look at the convergence of a field. The converging field acts in such a manner that the field lines converge to a specific point.

It can also be termed as a field sink. Mathematically, the divergence is a scalar quantity. It can be obtained as a result of the divergence operation on a vector field. From vector calculus, we have already looked at the concept of gradients. Now, the gradient operator operates as a scalar product with a field in a manner shown below, giving the divergence.

Thus, the divergence operation is nothing but the scalar product of the gradient operator with the field **F**. The divergence operator results in a scalar quantity. This shows the divergence operator is the sum of three individual components, such as the partial derivative of the field with respect to x, with respect to y, and with respect to z of the respective components. These are the components of the field in Cartesian coordinates x, y, z. And the divergence of **F** is the divergence operator. Consider this special case where the point *P* under consideration observes or experiences zero divergence.

The zero divergence is indicated as the number of field lines passing through point *P* is equal to the number of field lines emanating from point *P*. Thus, this is neither a source nor a sink. It can be considered as a neutral or stagnant point. The various physical interpretations of divergence follow. Before that, we look at some of the mathematical interpretations of divergence.

What is meant by the divergence of **F** greater than 0? Divergence of **F** being positive means the field emanates in the outward direction and indicates that the field is associated with a source. If the divergence of a field is negative, it means that the field converges inward and indicates the presence of a sink. Consequently, if the divergence of **F** is zero, then the field neither creates nor destroys. This indicates the absence of a source or sink.

Thus, the divergence can be understood as a measure of the presence of emanating field lines or converging field lines, and the geometrical representation of various fields in terms of divergence can be very useful for understanding and interpreting geophysical data. Now we will have a look at the details of divergence in different coordinate systems. First, let us have a look at the spherical coordinate system, where the vector field is given as the three components in radial, azimuthal, and polar angles. In the spherical coordinate system, the divergence is given by this formula. The peculiarity of the divergence of a field in spherical coordinates is the non-constant coefficients of the partial derivatives.

Next, the cylindrical coordinate system. In the cylindrical coordinate system, a vector field is represented by the three components in radial, azimuthal, and axial directions. The divergence of a field in cylindrical coordinates can be represented as the following expression. In cylindrical coordinates, the divergence operator has non-constant coefficients for the radial coordinate as well as the azimuthal coordinate. While the axial coordinate has a constant coefficient.

Next, we look at some vector identities concerning the divergence. Consider two vector fields denoted by $\vec{\boldsymbol{F}}$ and $\vec{\boldsymbol{G}}$. A scalar field F and two scalar values A and B. Now, the first vector identity is given as follows. This vector identity reads as the divergence of $A\vec{\boldsymbol{F}}+B\vec{\boldsymbol{G}}$. equals A times the divergence of $\vec{\boldsymbol{F}}$ plus B times the divergence of $\vec{\boldsymbol{G}}$.

$$\nabla\cdot(\mathrm{A}\,\vec{\boldsymbol{F}}+\mathrm{B}\vec{\boldsymbol{G}}) = \mathrm{A}\nabla\cdot\vec{\boldsymbol{F}}+\mathrm{B}\nabla\cdot\vec{\boldsymbol{G}}$$

The left-hand side of this vector identity indicates the divergence of the sum of the vector quantities $\vec{\boldsymbol{F}}$ and $\vec{\boldsymbol{G}}$ combined through a linear combination. While the right-hand side denotes the sum of the divergence of individual vector fields weighted by the coefficients A and B. The identity signifies that the divergence operation and the sum operation are interchangeable. The next vector identity concerning divergence is given as: The divergence of the curl of a vector field is always equal to zero.

This identity can be better understood with the help of a simple sketch. Consider the field lines occurring as closed loops, with the field direction indicated by the arrows. We can see that these field lines are concentric circular curves. These field lines have a finite curl since the field lines are closed loops. But the divergence of these field lines is zero because the number of field lines passing through any point is equal to the number of field lines leaving the point.

As we can see, a point P has one field line entering and one field line exiting it. Thus, the divergence of the curl of any field is always equal to zero.

$$\nabla\cdot(\nabla\times\vec{\boldsymbol{F}})=0$$

The next identity is given as: The divergence of F times vector field $\vec{\boldsymbol{F}}$ equals the gradient of F dotted with $\vec{\boldsymbol{F}}$ plus F multiplied by the divergence of $\vec{\boldsymbol{F}}$.

$$\nabla\cdot(\mathrm{F}\,\vec{\boldsymbol{F}})=(\nabla\mathrm{F})\cdot\vec{\boldsymbol{F}}+\mathrm{F}(\nabla\cdot\vec{\boldsymbol{F}}$$

The left-hand side of this identity indicates the divergence of a vector which is weighted by the scalar field F. On the right-hand side, we have two terms.

The first term indicates the projection of the gradient of F onto the field $\vec{\boldsymbol{F}}$. In addition to that, the second term indicates F weighing the divergence of the field $\vec{\boldsymbol{F}}$. The final identity which we are discussing in this lecture is given by the divergence of $\vec{\boldsymbol{F}} \times \vec{\boldsymbol{G}}$ equals the curl of $\vec{\boldsymbol{F}}$ projected onto the $\vec{\boldsymbol{G}}$ vector minus $\vec{\boldsymbol{F}}$ dotted with the curl of $\vec{\boldsymbol{G}}$.

$$\nabla\cdot(\vec{\boldsymbol{F}}\times\vec{\boldsymbol{G}}) = (\nabla\times \vec{\boldsymbol{F}})\cdot \vec{\boldsymbol{G}}-\vec{\boldsymbol{F}}\cdot(\nabla\times\vec{\boldsymbol{G}})$$

The left-hand side indicates the divergence of a cross product of two vectors. The right-hand side has two terms.

The first term is the projection of the curl of the first vector field onto the second vector minus the scalar product of the vector field $\vec{\boldsymbol{F}}$ with the curl of the second vector field. These identities are very useful in understanding in furthering our understanding and analysis of geophysical fields. Next, we will look into Gauss's theorem. Gauss's theorem is the relation between the flux of a field through a closed surface and the divergence of a field within the volume enclosed by that surface.

Consider this as a closed surface. Now, this surface has a surface area and an enclosed volume V. Gauss's theorem is a relation between the flux of a field passing through the closed surface and the divergence of the same field within the volume of this contoured surface. This is more clearly depicted in the adjacent diagram. Here, V is a large volume, and dV are the corresponding volume elements. $\vec{\boldsymbol{G}}$

The outflow of the field F from each such elemental volume dV_i is given by the divergence of $\vec{\boldsymbol{F}}$ dotted with dV_i. Thus, the total outflow can be measured or obtained as the sum of all such fluxes. As dV tends to the 0 limit, the total outflow is given by the integral of the divergence of $\vec{\boldsymbol{F}}$ over the whole volume V. The shaded area in this volume indicates the region occupied by that volume. The field lines are indicated in red color, while the normal vector to the surface S is indicated by the blue line and vector $\mathbf{n} \cdot dS$.

Now, the flux of the field F through this surface dS, which is an elemental surface area in the direction of this unit vector indicated by the blue line, is given by $\vec{\boldsymbol{F}} \cdot \mathbf{n}dS$. This we have seen in the previous lecture. In the limit as dS tends to 0, the total flux can be obtained by the integral of this elemental flux. As the two quantities are the same from physical considerations, the Gauss theorem can be stated as:

$$\int_V (\nabla \cdot \vec{\boldsymbol{F}}) dV = \oint_S \vec{\boldsymbol{F}} \cdot \mathbf{n} dS$$

Thus, the Gauss theorem can be stated as the left-hand side is the integral of the divergence of $\vec{\boldsymbol{F}}$, which is multiplied with the elemental volume and integrated over the entire volume. The right-hand side indicates the flux of $\vec{\boldsymbol{F}}$ passing through the surface and integrated over the entire surface gives the equal magnitude. Thus, the Gauss divergence theorem indicates the content of the field line within a three-dimensional body. The left-hand side depends on the field lines contained in the body, while the right-hand side is the measure of the same quantity just from the consideration of the surface area. Gauss's theorem is very useful when one has the measurements on the surface but doesn't have the measurements in the interior of the body.

In that case, having the knowledge of the flux over the entire surface, one can infer about the properties and characteristics of the field that exist within the entire volume which is covered by that surface. Applications of Gauss's theorem abound in geophysics. For example, in atmospheric and oceanic circulations, it is well known that the divergence of wind velocity can be used to study atmospheric processes. Due to the pressure gradient, the wind flow field diverges from high-pressure to low-pressure areas. It can be understood by looking at this figure.

H indicates a high-pressure area, while *L* indicates a low-pressure zone. It can be seen that the field lines of flow diverge from the high-pressure zone. While the flow field lines converge into the low-pressure zone. Thus, the divergence of the velocity field is positive for the high pressure zone, while the divergence of the velocity field is negative for the low-pressure zone. The interaction between the high and low-pressure zones creates an interacting vector field between these two local zones.

This results in a complicated wind pattern and air circulation patterns. In between this high and low-pressure zone, there exists a pressure which is midway, or the gradient of the pressure at this point may be equal to the minimum or maximum. In between these two zones exists a point P where the field lines behave in such a manner that the divergence goes to zero. If we consider the flux of a surface over this entire high and low-pressure zone, then we might conclude the features of this combination of the high and low-pressure zones. However, if we take a surface which encloses only the low-pressure zone, for example, we can conclude that the divergence of the velocity will be negative as the field lines will be entering this surface.

From the surface measurements only, we can understand from the converging field lines that a low-pressure zone may be existing inside this closed surface without making any measurements in the interior of this surface. This is the usefulness of Gauss's theorem. This is the usefulness of Gauss's theorem. Next, we have the example from hydrogeology. In hydrogeology, the divergence of the vector field **U** is used to investigate the recharge and discharge rates in aquifers.

If the divergence of **U** is positive, then the aquifer is recharging. On the other hand, if the divergence of **U** is negative, then it is discharging. Thus, from the ideas and details of the discussion in this lecture, we come to the following conclusions. First, to understand the Earth's structure and dynamics from local point-based to global regional scales, divergence and Gauss's theorem are of paramount importance. Gauss's theorem and divergence concepts can be applied to conservation laws very effectively.

For example, conservation of mass, energy, and momentum can be expressed in terms of divergence. Divergence and Gauss's theorem can also be used to model complex interactions in Earth's atmosphere, oceans, as well as crust and deep interior, such as the core of the Earth.

So, we have discussed various concepts of flux, gradient, and divergence, and Gauss's theorem, which are fundamental theorems in these lectures, and we will be looking into various geophysical applications in further lectures. Thank you.