Mathematical Geophysics
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Hello everyone. Welcome to the SWAYAM NPTEL course on Mathematical Geophysics. We continue with module number 2, Fundamental Theorems. This is the second lecture of this module. This lecture is titled Flux of a Field.

In this lecture, the broad aspect covered is the flux of a field. The components of this lecture are the concept of flux, flux of vector fields, flux of scalar fields, and the application of flux in geophysical studies. Let us begin. First, what is the concept of flux? As you can see here, physically, the concept of flux means what is the amount of a quantity that is passing through a particular area? Consider this picture for simplicity. This is the picture of a container. Now, if this container is aligned in a manner so as to collect a dropping fluid or liquid in this direction, then the area available is the projection of the opening area of the container to the direction of the incoming fluid. However, if the direction of the incoming fluid is perpendicular to the maximum area of the container, then more of the amount of fluid can be collected.

This entire area is available. So we can understand from this simple example that there is a relation between a vector and the area through which we are considering the flux of that vector. Thus, we will look into the physical concept of the flux. Physically, the flux can be quantified as the amount of field lines of a vector or a field that passes through a given surface at any point in time. This can be understood by looking at the adjacent diagram.

We have looked at this diagram earlier. This diagram represents the interaction of magnetic fields between two planetary bodies, such as Earth and Jupiter. These are the field lines. Now, consider a surface shown in the grey shaded area. We can see that a certain amount of field lines pass through this area.

So these are the field lines which are passing through this surface. Now, if we change the surface, which is oriented in a manner slightly tilted from the original surface. Now, you can see that although the area of the surface is approximately equal, the number of field lines passing through this surface is somewhat lesser. This occurs due to the misalignment of the surface with the direction of the field lines. Thus, it is important to have a quantification of the relation between flux and the surface area.

This will be the theme of this present lecture. In vector calculus, for example, flux is represented as a scalar quantity. This scalar quantity is the total field flow across the surface and accounts for both the strength of the field and the orientation relative to the surface. In other applications, flux is represented as a vector quantity. These applications, such as transfer phenomena, use the flux as a vector quantity passing through a particular surface.

The flux represents both the magnitude of the flow of the field and the direction of the flow as well. In one dimension, flux is measured along a straight line. Consider this as the number line which we had discussed earlier. A vector may be directed along either the positive direction or the negative direction. So, the rate of change of the field across that line is the flux.

We can see that the field vector changes across the line. So, the derivative of the field along that line will be the flux in one dimension. Extending this idea to two and three dimensions, flux is measured by the total flow across two-dimensional surface areas or closed curves or closed surfaces. With the help of these sketches, one can understand the two-dimensional flux or three-dimensional flux as shown in the diagram on the right. In this diagram, we can see that the flux diverges from a point within the three-dimensional surface, and the flux lines and the field lines passing through the entire surface is a measurement of the flux.

Thus, one can summarize the concept of flux as the relation between the field line direction and the surface through which it is passing. Now, let us have a look at the concept of vector field flux. A vector field flux measures the amount of vector field lines that pass through a certain surface area at a given point in time. This can be understood using the following description and the adjacent diagram.

In this diagram, the surface area is shown as a grey shaded region denoted by S. The flux of a vector field denoted by **F** passes through this surface. But the normal to the surface is denoted by a blue line and **N**. We can easily see that the angle between the direction of the normal vector and the field lines is not zero. Hence, the flux is a function of the angle between **F** and **N**, and it increases or decreases depending upon the alignment of the vector field with the direction normal to the surface.

Mathematically, the vector field **F** and the flux through the surface S can be obtained given in this expression. The flux ϕ is the integral of the field along the surface area. This integral is nothing but the projection of the field onto the surface area S. This is integrated for the entire surface and simplified as the product of the magnitudes of the field, the area, and the cosine of the angle between them.

$$\phi = \int_S \boldsymbol{F} \cdot \mathbf{n} dS = |\mathbf{F}| \cdot |\mathbf{n}| \cdot \cos\theta$$

So ϕ is the flux, **n** is the unit normal vector to the surface, and θ is the angle between the field and the normal vector. From this, we can understand that if θ goes to 0, in other words, the field and the normal vector become aligned, then the flux will be maximum as $\cos\theta$ will attain the value 1. Physically, this can be understood as the flux will be maximum if the field line is aligned with the normal vector or maximum area is available for the field lines to cross through.

In simple language, flux measures the interaction strength between the field lines and the surface normal direction. As we have discussed, if $\theta = 0$, the field line will be parallel to the normal vector, then ϕ or the flux will be maximized. On the other hand, if $\theta = 90^{\circ}$, the field lines are perpendicular to the normal of the surface area. Hence, ϕ becomes zero, which means no flow or field across S. This can be represented as in this diagram. The field lines are directed as shown.

Suppose we consider a surface which is parallel to the field and the normal vector is perpendicular to the field lines, it can be visually understood that the field lines will just pass through the surface without entering it. No field lines are captured by the surface. Hence, the flux of the vector field goes to zero. For closed surfaces such as cubes and spheres, flux is positive if the field lines emerge outward. And the flux is negative if the field lines converge inward.

This indicates a positive flux, while this indicates a negative flux. Now we will look at the scalar field flux. Scalar fields, being bereft of any direction, have to be treated separately from vector fields as far as flux is concerned. Consider this diagram where we have the surface $F =$

constant and the flux indicated by the vector gradient of F. The gradient of a scalar, being a vector quantity, can be denoted as field lines pointed in a particular direction.

The **N** vector denotes the perpendicular to the surface. θ denotes the angle between them. Once we have the gradient of the scalar function in place, we have a vector field from a scalar field. Thus, we can use the previously discussed concepts of flux of vector fields to define the flux of scalar fields. Before that, we'll just define the physical concept of a scalar field flux.

Scalar field flux indicates the transport of a field through a surface by considering the gradient of that scalar field. And this can be mathematically represented as this expression. Note that this expression is similar to that for the vector field except for one distinction. Here, the gradient of the scalar field is used instead of the vector field.

$$\phi = \int_S (\nabla f) \cdot \mathbf{n} \, dS$$

Here, ϕ denotes the flux of the field, **n** denotes the unit vector normal to the surface, ∇f is the gradient of the scalar field f, and θ is the angle between the gradient vector and the surface normal. The gradient vector is projected onto the surface and integrated over the entire surface area to obtain the scalar field flux.

The gradient of the scalar field indicates the direction where the steepest rate of change of the scalar field occurs. Hence, the flux of the scalar field measures the amount of field that is transported through the surface in the direction where the change is maximum. If the gradient of f and **n** are aligned, this indicates that the field flows in the outward direction. If the gradient of the field and **n** are aligned in the same direction, the flux will be positive. This indicates the field flows in the outward direction.

Similarly, the gradient of f and **n** aligned in opposite directions would result in a negative flux. The magnitude of the flow is determined by the gradient of f and the surface area, as indicated in this expression. So, we can conclude that the scalar field and vector fields have fluxes indicated by the relation between the field lines and the gradient of the scalar field, oriented along a direction that is inclined to the surface under consideration.

There are various applications of flux in geophysics because geophysics involves the transport of energy, material, etc., from place to place, such as oceanic circulations, magnetic fields, transport of material in the atmosphere, such as particle-laden flows, etc. We will look at a few of these applications in this lecture.

The electromagnetic studies conducted as part of geophysical exploration, associated with the magnetic field **B** as a vector field, measure the magnetic field flux. The magnetic field flux is given by the integration of the magnetic field over a surface area *S*. The magnetic flux helps in understanding the properties of subsurface rocks, like those below the surface. In particular, the electrical conductivity is measured to explore the possible occurrence of mineral deposits and water resources, etc.

In geophysical fluid dynamical studies, the associated vector field is the velocity field. And the mass flux is used to constrain the conservation of mass, given by the divergence of **U** equals zero in the case of incompressible fluids.

In gravitational field applications in geophysics, the associated vector field is the gravitational field **G**. The gravitational flux, denoted by ϕ, equals:

$$\phi = -\frac{GM}{r^2}$$

Or otherwise obtained through an integral of **g** over the entire surface, is used to estimate various gravity anomalies, which are also used to explore mass distribution of mountains, oceanic trenches, or possible occurrences of high-density minerals in the subsurface.

The scalar field temperature is used in geothermal studies. Its gradient is the heat flux. Here, the **q** vector stands for the heat flux, which is nothing but the negative of the conductivity multiplied by the gradient of temperature:

$$\mathbf{q} = -k\nabla T$$

This is used to measure the amount of heat flow per unit area per unit time across a particular surface. This has immense applications in geothermal studies, energy production rates from geothermal reservoirs, such as those found in hot springs, etc. Also, the thermal evolution of the Earth involves secular cooling, where heat is transferred from the center of the Earth to the surface. This cools down the planet, and the flux of the scalar field, which is the heat flux, is used to obtain various quantities, one of which is the age of the inner core.

In hydrogeological studies, where the associated scalar field is the pressure field, the flux or the gradient is the groundwater flux. Here, Q denotes the groundwater flux, which is nothing but the product of permeability K and the gradient of pressure:

$$Q = -K\nabla P$$

This is used to understand the rate of recharging and discharging of aquifer systems used in groundwater studies.

Flux is a very important concept in geophysics, as we have seen in a few examples among many, where the flux determines various geophysical conclusions in particular studies. In this lecture, we come to the following conclusions regarding the flux of fields. The flux of fields is a fundamental concept in geophysics. It describes the flow of energy or physical quantities through a given surface area. The flux is a crucial concept for understanding energy transfer, flow transfer, or field lines passing through a surface, and these are very important in geophysical applications. We have also looked at various types of fluxes, such as magnetic, electric, and gravity fluxes, as well as temperature gradient and pressure gradient fluxes, which are relevant to various geophysical applications.

In summary, we can understand that accurate calculations and interpretations of fluxes are essential for reliable geophysical modeling and the exploration of mineral and energy resources. So, the concept of flux will be used in further studies where the detailed aspects of geophysical applications will be discussed. One can refer to these reference books for a deeper understanding of the concept of flux and its various applications in geophysical studies. Thank you.