Mathematical Geophysics

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Week - 02

Lecture - 06

Hello everyone Welcome to the SWAYAM NPTEL course on Mathematical Geophysics. In this lecture, we are going to start module number 2. This module is called Fundamental Theorems. This is the first lecture of module 2, titled Scalar Field and Gradient.

In this lecture, we are going to cover the following concepts: the scalar field and gradient. This is the overall focus of this lecture. The components of this lecture are as follows. The concept of a scalar field will be discussed first.

Then, the gradient of a scalar field, followed by directional derivatives of scalar fields. Then, we will be looking into some properties of the gradient and directional derivatives. Next, we will be considering gradients in different coordinate systems, as we have discussed in previous lectures. The applications in geophysics will be discussed side by side. So, let us begin.

The Concept of a Scalar Field, What is a scalar field? That we have discussed briefly in previous lectures. We have discussed what a scalar is. We have also discussed what a field is We have also discussed scalar fields in the context of temperature, pressure, etc., in geophysical applications. Now we will look at the details of scalar fields in particular. First, the physical concept. What is the physical concept behind a scalar field?

A scalar field is a concept that assigns a value, either positive or negative, in numerics or functional values, or both, for each point in space and time. They can also vary over space and time in a continuous manner. Mathematically, this can be represented as real-valued functions that depend on either spatial coordinates or both spatial and temporal coordinates. This can be represented as a function of x, y, z in Cartesian coordinates, which is only space-dependent, while a function of x, y, z, and t in Cartesian coordinates represents spatial-temporal functions or fields.

Next, we will look into the geometrical concept of a scalar field. This we will combine with the geometrical representation as we discussed in the previous lecture. Consider the figures shown adjacently. The scalar field can be represented in either one, two, or three dimensions.

For example, in one dimension, we can have a horizontal or axial profile of a scalar field.

We have taken the example of temperature as a scalar and the Cartesian coordinate system for the representation of a one-dimensional field. In this figure, the axial direction is shown by the arrow marked Z, and the temperature is only a function of Z, making it a one-dimensional field. The blue line shows a linear temperature profile, while the red line shows a non-linear temperature profile. Both are one-dimensional scalar fields. Similarly, we can extend this concept to two dimensions, taking temperature again as the scalar field as a function of x and z. This gives a two-dimensional domain over which we can represent the temperature scalar field in the form of contours.

These contours represent assigned values and are colored to represent a visually appealing structure. Like this, we can assign positive and negative values to temperature contours as the case may be. Extending this further, we look into the three-dimensional representation of a scalar field. In three-dimensional representation, we choose the isosurfaces as the geometrical representation of the scalar field. In this diagram, the scalar field is chosen as the axial helicity.

What is axial helicity? Axial helicity is defined as the scalar product of velocity and the curl of velocity. The scalar product operation and the curl operation were discussed in previous lectures. The H vector is the helicity vector, and its Z component is the axial helicity. We have chosen the Z component, H_Z, as the scalar field for plotting the three-dimensional isosurfaces in a spherical domain.

Have a look at this picture. This represents a three-dimensional figure with all the axes, X, Y, and Z, shown clearly. The positive and negative values of the axial helicity are shown in different colors. This is a three-dimensional geometrical representation of a scalar field. Such representation can be adapted for any scalar field and appropriately shown in either one, two, or three dimensions.

In geophysics, these scalar fields and their representations are of paramount importance. For example, in geothermal studies, the map of the temperature field within the Earth and its surface is represented using scalar fields. Using that, the heat flow, energy, and thermal gradients within the mantle or crust can be easily depicted and further analyzed for better understanding. Also, another geophysical application concerning hydrogeological studies considers the pressure field in a groundwater reservoir to assess water flows, groundwater contamination, and other studies.

Gradient of a Scalar Field Now let us look into the gradient operator acting on scalar fields. The gradient of a scalar field is used in various applications where the differences in the values of the scalar are important. For example, in the transfer of heat energy, the temperature is not as important as the gradient of temperature because the gradient of temperature is what drives the heat transfer.

Similarly, in fluid flows, it is the gradient of pressure that forces the fluid to flow. Thus, it is very important to understand the concept of the gradient of a scalar field. The physical concept of a scalar field gradient is described here. The physical concept is how a scalar field varies in space or time; its rate of change is considered the gradient of the scalar field. In most applications, the predominant use of the gradient occurs spatially.

The mathematical concept behind the gradient of a scalar field results in a vector field, which points in the direction of the steepest rate of increase or decrease of that scalar field. We can understand this with the help of the adjacent diagram. Consider this diagram shown here, depicting a paraboloid surface. Now, this paraboloid surface represents the values of a scalar field over a two-dimensional domain in x and y coordinates. The values of the scalar field assume a paraboloid structure.

Now, at any point, as shown here on the paraboloid surface, we can look into the gradients in various directions. For example, at this point, the gradients or the change in the values of the scalar fields in any of these directions can be calculated. But the gradient, as defined by the formula here, gives the direction in which the change is maximum. This is the gradient of the scalar field *F*. The gradient operator in Cartesian coordinates is defined as:

$$\nabla F = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}$$

This results in the gradient vector. Now, this is the mathematical form. Now, it is important to understand that for any complicated structure, apart from paraboloid surfaces, the gradient of a scalar field will give the direction of maximum change. Whether it is increasing or decreasing, the change occurring in the direction where it is maximum will be obtained if one calculates the gradient of a scalar field. Now, coming to applications in geophysics.

The most prominent applications of scalar fields and their gradients in geophysics occur with respect to the thermal gradient and pressure gradient. The thermal gradient is used to study the rate of heat flow in the lithosphere and mantle. Now, recall that Earth's interior is made up of various layers. In a broad sense, this can be considered as a schematic diagram of the interior of the Earth. The inner core is at the center with an overlying fluid outer core.

On top of it lies the mantle, and the surface and depth of the subsurface can be included in the lithosphere or crust. As we know, the temperature is very high in the center of the Earth, approximately 7000 Kelvin, and it decreases upwards right up to the surface of the Earth. So, this gradual change in temperature across the depth of our planet presents the thermal gradient. As a result of this thermal gradient, heat flows outward from the center of the Earth towards the surface. This is also known as secular cooling.

Also, near the surface, geothermal reservoirs present a practical area to understand and measure the geothermal gradients. We also know of volcanoes and tectonic regions where the activity involves thermal gradients, where molten lava is thrown out from volcanic eruptions and tectonic plates grind against each other, developing thermal gradients. So, we can see and understand various geophysical phenomena with respect to the thermal gradient and understand the heat transfer and energy transfer occurring in the planet's interior as well as near the surface. Similarly, the pressure gradient.

The pressure gradient is usually considered where fluid flows occur in geophysical applications. Prominently, groundwater flows and aquifer behavior are modeled using the pressure gradient. It is important to understand and measure what the pressure gradient behavior is for groundwater flows to determine the extent of groundwater and the flow to efficiently trap the groundwater. Another aspect where the pressure gradient is used in geophysical applications is in the oil and gas industry. In hydrocarbon reservoirs, oil and gas move through porous media and get collected in reservoirs.

So porous media are nothing but solid structures with perforations through which fluid can flow, and these flows are driven by pressure gradients. Thus, geophysical applications in the gradient of a scalar are abound. Next, we come to the discussion on the directional derivative of a scalar field. The mathematical concept behind the directional derivative is shown here.

Directional Derivative of a Scalar Field The directional derivative of a scalar field can be understood by looking at the adjacent figure. In this figure, F is a surface or any curve in a scalar field. And the gradient is shown as the gradient of F. Now, the gradient is calculated using the formula we discussed in the previous slide. This represents the direction of the scalar field.

The gradient of F can either increase in one direction or decrease in another direction. Therefore, the rate of change of the scalar field profoundly occurs in a direction-dependent quantity. Therefore, the rate of change of a scalar field profoundly the gradient of F can either increase or decrease in directions. The gradient of a field can increase either in one direction or decrease in another direction.

Therefore, the rate of change of a scalar field is a directionally dependent quantity. This makes it important to understand the gradients in particular directions. Thus, the directional derivative $D_u f$ of a scalar field f in the direction u is defined as follows. Here, u is the chosen direction. The blue arrow shows the direction in which the directional derivative of the scalar field is to be computed. Thus, $D_u f$ represents the gradient of the scalar field projected onto the particular chosen direction.

This can be simplified as:

$$D_u f = \nabla f \cdot \mathbf{u}$$

where ∇f is the gradient of f and **u** is the unit vector in the chosen direction. Thus, the directional derivative of a scalar field is a vector quantity. Thus, the directional derivative of a scalar field can be computed and obtained as a scalar function itself. In geophysics, various applications make use of the concept of directional derivatives.

For example, seismic wave propagation. In seismic wave propagation, the rate of change of the amplitude or travel time of seismic waves in particular directions is determined to calculate the energy that is being dissipated in that particular direction. The wave propagates in all possible directions. However, it may be imperative to locate the amount of energy that is propagating along a fixed direction. Thus, in that scenario, the directional derivative will come into the picture.

This is also used to compute the anisotropy of various materials. What is anisotropy? Anisotropy is the direction-dependent properties such as density and elastic modulus of any rock samples. For example, if this is a rock sample, its density may be different along different directions. This can happen because the rock may be composed of various combinations of minerals having different densities. It may be important to understand the density variation along particular directions. In other applications such as atmospheric and oceanic studies, directional derivatives also play a very important role. We all know by now that the temperature gradient and pressure gradient are of utmost importance in oceanic and atmospheric flows and atmospheric behavior. Thus, it may be important to understand the flow in a particular direction as it can help in navigation and understanding the behavior of weather and climate.

Also, the salinity variations in ocean currents can help us determine the interaction of river flows with oceanic flows and other aspects such as the migration of species and the harvesting of oceanic energy through tidal waves. In all these applications, the directional derivatives are important where we obtain the entire field of directional derivatives indicating the rate of change of that scalar under consideration. While the gradient gives the direction of maximum change, the directional derivative gives the entire scalar field. Now let us look into the various properties of gradient and directional derivatives. The gradient may be positive, negative, or equal to zero, indicating an increase, decrease, or no change in any direction of the scalar field.

Directional derivatives, on the other hand, if positive, indicate that f is increasing in that particular direction or vice versa. As directional derivatives' negative values indicate the decreasing of f in the particular direction. The equipotential surface is the surface where the directional derivatives of a scalar function obey these constraints, which indicate that all the points of the surface have the directional derivative equal to zero. This defines the equipotential surface. The equipotential surface may result either from the angle being 90 degrees, which makes the scalar product equal to zero, or the magnitude of the gradient itself going to zero, indicating a constant scalar field or equipotential surface.

Thus, the gradient of F is always perpendicular to the constant surface of a scalar field.

Now, we look at the gradient in different coordinate systems. In specific, we look at the spherical coordinate system where the applications of geophysics are of very much importance. This is the expression for calculating the gradient of a function in spherical coordinates, as we have discussed in previous lectures. Particularly, the geomagnetic field of the Earth is a very nice application where the gradient is used in the spherical coordinate system.

The geomagnetic field, being a conservative field, has the potential V. This scalar potential for the geomagnetic field can be expressed in terms of functions known as spherical harmonics. The radial components of the magnetic field can be obtained by the gradient operator. For example, B_r denotes the radial component of the geomagnetic field, which is obtained by taking the gradient of the scalar potential V. The radial magnetic field is nothing but the negative of the partial derivative of the scalar potential with respect to the radial coordinate. Thus, from the expression for the scalar potential, the expression for the radial magnetic field can be obtained as given here.

Notice the change in the exponent for r, which has changed due to the gradient operation. This indicates the use of gradient in the spherical coordinate system, which is very common for geomagnetic field calculations. Thus, from this lecture, we can obtain the following conclusions. Scalar fields and gradients play a very vital role in geophysics. They describe various physical quantities and their spatial variations.

The scalar fields and gradients also help to analyze and interpret geophysical data. That geophysical data can be used in the oil and gas industry for exploration or natural phenomena such as oceanic or atmospheric flows. The scalar fields, gradients, and directional gradients are essential concepts in geophysics. They provide a mathematical framework and a geometrical representation for describing and analyzing various geophysical phenomena and their spatiotemporal variations. One can refer to the following references for further understanding.

Thank you.