Mathematical Geophysics

Swarandeep Sahoo

Department of Applied Geophysics

Indian Institutes of Technology (Indian School of Mines), Dhanbad

Week - 01

Lecture – 04

Hello everyone. Welcome back to the Swayam NPTEL course on Mathematical Geophysics, Module 1, Basic Concepts, Lecture 4, Theory of Fields. In this lecture, the following concepts are being covered. The concept of a field, such as the gravitational and magnetic fields, which we have already discussed as examples from the geophysical context. We would be looking into the fundamental aspects of the theory of fields, its classification, and the proper mathematical definition of such fields.

Next, we will look into the properties of conservative and non-conservative fields, which are the major broad divisions of the fields applicable to geophysical aspects. Then, we will summarize by comparing and contrasting these conservative and non-conservative fields. Let us begin. First, we will look into the concept of a field.

What is a field? The field is a mathematical concept that is partly abstract but can be understood with practical examples. These practical examples hail from our day-to-day life as well as scientific applications. So, a field is an abstract concept with distinct interpretations depending upon the context. These contexts can range from physics, chemistry, and geophysical applications. In particular, the concept of a field represents a quantity or a property of the system at each point in space, which is varying continuously. Simply put, a field can be described as a single-valued function in mathematical terms.

Now, in mathematical concepts, the field can be understood as an algebraic structure containing a set of elements and followed by properties such as addition and multiplication, which are defined for the particular field. For example, the real number field and the complex number field. So these are examples from pure mathematical concepts. Now, coming to the general physical context, the field can be described as a physical quantity that is well-defined at every point in a region of space. The field can also vary in time.

Now, based on the type of data, the field can be of three types: scalar, vector, and tensor. The concept of scalars and vectors in relation to fields has already been discussed at the start of this lecture series, which can be recalled from previous lectures. The tensor field is a generalization of the extension of scalars and vectors. We will be looking into it very briefly.

Examples of these types of fields are temperature, which is a scalar field; velocity, which is a vector field; and stress, which is a tensor field. The stress tensor applies to various geophysical contexts and engineering aspects, such as the strength of materials and deformation. From a geophysical perspective, a field represents various properties of the Earth's interior, exterior, atmosphere, and surface, which are essential for interpreting geophysical processes. For example, the geophysical context in which the concept of a field is used can be listed as follows.

First, the thermal field, which is a scalar field as discussed previously, represents the temperature distribution in the Earth's interior as well as in the Earth's atmosphere. In the atmosphere, the temperature varies as a function of height, and this variation is the thermal field.

Next, the gravitational field encompasses the entire planet and spreads from the center of the Earth to infinity. It is considered as a vector field. It represents the gravitational force exerted by the planet Earth and also applies equally to various other planetary bodies with mass.

The third example is the stress field, which is a tensorial field used for studying earthquakes and tectonic plate motions in the context of plate deformations and stresses occurring when plates merge or diverge. The stress field also develops when an earthquake occurs and is dissipated throughout the volume of the Earth.

We will look into the detailed classification and definition of the fields next. So, first of all, the basis of classification is to be determined. So, based on the interaction mechanism of a field with energy, work, or path dependencies, the fields can be classified.

So, the basis is the energy or work done and path independence or dependence. So, the field is broadly classified into two classes: the conservative and non-conservative fields. The conservative fields have the property that the work done by a force F to displace an object in this particular type of field is independent of the path taken. For example, the gravitational field, the electric field, and the pressure field. This can be understood from the diagram shown here.

If a mass moves from point A to point B using two different paths as shown here, for a conservative field, the work done by the force under which the mass is moving is the same. It does not change because of the change in the path undertaken.

Next, we will look at the classification and definition of fields. The fields can be classified based on the interaction mechanism of the field, on the basis of the energy or work done, and path dependencies. This leads to the broad classification of conservative and non-conservative fields.

The conservative fields, as shown here, are defined as the fields where the work done by a force F to displace an object inside the field becomes independent of the path taken. The work done only depends on the initial and the terminal points. Let us consider two points, A and B. If these two points are in a conservative field, the work done to move an object from point A to point B along any path, two of which are shown here, remains the same. The work done only depends upon the location of points A and B.

For non-conservative fields, this does not hold true. In non-conservative fields, the work done by the force to displace an object from point A to point B is a function of the path undertaken.

So look at The diagram where the points P and Q are separated by a distance with two paths which can be used to move an object from point P to point Q. In the case where P and Q lie in a non-conservative field, the work done would depend on the path undertaken. Now, why do we have these two divisions?

These divisions are necessary to differentiate the simplicity of physical problems. This division enables us to understand physical problems from fundamental perspectives and with additional complexities due to natural phenomena. The naturally occurring constraints often lead to non-conservation, while fundamentally conservative fields can describe the aspects of physical problems to a large extent. Thus, conservative fields simplify physical problems by enabling energy conservation and scalar potential analysis.

On the other hand, non-conservative fields often require additional considerations, constraints, and calculations of dissipative forces or time-dependent forces, which do not conserve energy or involve the transfer of energy. Hence, the path dependencies. Now, in geophysics, the fields which are commonly occurring are such as the gravitational field and magnetic field, known as geodesy. It is used to determine the geoid to study the mass distribution and internal structure of the Earth. We will discuss more about such quantities, such as the geoid and various properties of the mass distribution, in further lectures. In the context of the magnetic field, the geophysical applications are to explore minerals and hydrocarbons, the naturally occurring geomagnetic features, and the study of paleomagnetism involving rock magnetism and minerals which are magnetic in nature.

The seismic velocity fields are also vector fields, which are widely used and studied in geophysical applications. These velocity fields are obtained using the propagation of seismic waves, and seismic wave propagation inherently encounters dissipation. The seismic velocity fields are used to image the Earth's interior structure, assess earthquake hazards, and explore other aspects of the deep regions of the planet. Similarly, the thermal field is used to explore geothermal reservoirs and the thermal evolution of the planet's core and mantle. The thermal field also encounters dissipation.

It converts to various other forms of energy and hence becomes a non-conservative field in the context of geophysics. Now, let us have a look at the properties of conservative fields in detail. As we have seen earlier, conservative fields offer path independence. This can be represented mathematically as the work done by a force F becoming zero, shown by this formula, which gives the integral of the work done over a closed loop as equal to zero. A conservative field is irrotational.

$$\oint F \cdot dr = 0$$

The irrotationality of a conservative field can be indicated as the curl of the force going to zero.

$$\nabla \times F \neq 0$$

The curl is a vector operation, which we have discussed in previous lectures. The curl of a force going to zero essentially means that the field has a characteristic which is different from rotational fields. The next property of a conservative field is that there exists a scalar potential function F, such that the force can be expressed as the gradient of that function.

This is the scalar potential function V. The gradient operation of the scalar potential function results in the vector quantity, which is the conservative force field. In geophysics, conservative fields are The gravitational field, which can be represented as g equals minus the gradient of phi.

$$g = - \nabla\Phi$$

Here, phi denotes the gravity potential, and its gradient gives the gravitational field. The next is the pressure field in a fluid system.

The pressure gradient, as shown here, equals the weight of the body above the point under consideration. This is the hydrostatic pressure gradient.

$$\nabla P = \rho g$$

This is applicable for geophysical studies, such as geothermal exploration. Another conservative field in geophysics is the thermal field. The thermal field can also be expressed in terms of a gradient, such as the heat flux, which equals minus K multiplied by the gradient of temperature.

$$Q = - k \nabla T$$

Here, K is the conductivity of the material, and Q is the heat flux. The thermal field is used to understand the evolution of temperature and heat transfer or energy transfer inside the Earth's core and mantle throughout the planet's history from its beginning. Next and final is the fluid velocity field. The fluid velocity field in two dimensions can also be represented as the - $\nabla\Phi$. Here, the potential function phi is called the stream function.

So the velocity field can be obtained as the gradient of the stream function. This sort of analysis in geophysical applications is abundant in groundwater exploration using Darcy's law. Now moving on to the properties of non-conservative fields along the same lines, we will discuss the various aspects. First, the path dependence. This is in contrast to conservative fields.

So here the path dependence can be mathematically described as this shows that the line integral of a force along a closed loop is not equal to zero;

$$\oint F \cdot dr \neq 0$$

essentially, the work done is non-zero. The field is also rotational. A non-conservative field is rotational, indicated by the curl being non-zero.

$$\nabla \times F \neq 0$$

Unlike conservative fields, the scalar potential function does not exist.

This makes the analysis of non-conservative fields much more difficult compared to conservative fields. In a geophysical context, the non-conservative fields are the time-varying magnetic field, which is given by the curl of the electromotive force E equals minus the time derivative of the magnetic field B.

$$\nabla \times F = -B$$

This time-varying magnetic field undergoes magnetic dissipation and is converted to heat energy or other forms of energy. In geophysics, this magnetic field variation in time is used in various applications such as geomagnetic secular variation and electromagnetic methods for mineral exploration.

The second type of non-conservative fields in geophysics is the viscous forces. This is denoted by F equals minus eta Laplacian of the velocity field vector.

$$F=-\eta\nabla^2 V$$

Now, the viscous force is dependent on η, which is the viscosity of the material. The viscous forces are applicable to various geophysical fluid flows, such as flows in porous media and in groundwater exploration.

In particular, the flows in porous media encounter small spaces where the viscous forces become of paramount importance and are significant. Now, let us look at the differences between conservative and non-conservative fields from both mathematical and geophysical contexts. From a mathematical perspective, the conservative fields and non-conservative fields are differentiated in terms of the three quantities shown here. First, the existence of curl.

Conservative fields do not have a resultant curl, whereas the non-conservative fields possess a finite curl of the field, indicating its rotational character. Next, the conservative fields have the existence of a scalar potential, while such potential functions do not exist for non-conservative fields. Third, the energy is conserved for conservative fields, while for non-conservative fields, the energy is transferred in different forms. Apart from the mathematical aspects, it is important to understand the geophysical context and how, in that context, conservative and non-conservative fields differ. We are going to have a look at these five geophysical applications in the context of differentiating conservative and non-conservative fields.

The gravitational field is used to study the steady and static fields of the Earth, while non-conservative fields in the gravitational context are used for gravitational interactions with the dissipation of energy, for example, tidal energy in oceans. In the geomagnetic field context, conservative fields are used in the crystal magnetization of the static magnetic field in permanently magnetized rocks. The non-conservative fields occur in the non-static main magnetic field, which results from the geodynamo action. In seismic and seismological aspects of geophysical applications, the conservative fields are appropriately used to describe the elastic rebound theory in earthquake seismology, while non-conservative fields are used in inelastic media, attenuation, and scattering of seismic waves. In the context of various thermal processes that occur in planetary systems, conservative fields abound, such as heat conduction, geothermal gradient, and magma chamber cooling, etc.

While the non-conservative fields occur in heat convection and dissipation systems where the thermal energy of the Earth gets dissipated. And results in various other forms of energy. Finally, in geophysical fluid dynamics, conservative fields occur in non-dissipative laminar flows that drive plate tectonics and interactions among the plates. The non-conservation occurs when dissipation is involved, resulting in turbulent flows. The non-conservative fields occur in dissipative turbulent flows, which have a high contribution of viscous dissipation and transfer to heat energy. Such flows occur in the Earth's outer core, atmosphere, and oceanic circulations. Also, in mantle convection, where the typical viscous values are very high.

Resulting in high dissipative forces and non-conservation of energy. From this lecture, we obtained the following conclusions. First, in terms of data interpretation. Understanding the differences between conservative and non-conservative fields is very important for the accurate interpretation of various geophysical data. Secondly, modeling and inversion.

Conservative and non-conservative fields require different modeling strategies. The inversion approaches are also different for conservative and non-conservative fields. This impacts the obtained results and the understanding of the relevant geophysical phenomena significantly. Finally, in geophysical exploration and characterization, the distinction between conservative and non-conservative fields helps geoscientists design more effective strategies for exploration that are cost-effective, safe, and accurately characterize the subsurface. These are the references that can be used to understand the properties of fields in a more detailed manner.

Thank you.