Mathematical Geophysics

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Lecture – 39

Hello everyone, welcome to the SWAYAM NPTEL course on mathematical geophysics. We continue with module number 8: data-driven analysis in geophysics. This is lecture number 4: *transforms for geophysical analysis*. In this lecture, we cover the general methods for geophysical data analysis, which are related to various transforms. First, we will look at the Laplace transform.

Next, the Z-transform. Then, we will continue with the discrete Fourier transform, the properties of the discrete Fourier transform, and fast Fourier transforms. Finally, we will look into various geophysical applications. So, let us begin. What is the Laplace transform?

The Laplace transform is the generalized version of the Fourier transform for continuous signals. Later, we will look at transforms for discrete signals. First, let us see the transform that is the Laplace transform. The Laplace transform is an integral transform, and it is used for analyzing linear time-invariant systems or LTI systems. Now, this transform is also a linear transformation.

It obeys all the rules of linear superposition and linear functions. Basically, the Laplace transform converts a signal which is continuous in time into its Fourier domain, which is the complex Fourier domain in s. Here, s is a complex number, and this complex function of s, $X(s)$, is the transformed function through the Laplace transform. Now, $x(t)$ is also a complex function. This means that the functional values of both these functions are complex numbers.

Thus, the Laplace transform converts a signal into its complex Fourier domain, while the Fourier transform converts a signal into its real frequency domain. The Laplace transform solves differential equations and can also be used in understanding and interpreting the signal in the complex domain. This is the notation for the Laplace transform. $X(s) = \mathcal{L}\{x(t)\}$. $\mathcal{L}$ is a symbol representing the Laplace transform. The mathematical definition of the Laplace transform is given by:

$$X(s) = \int_0^\infty x\,(t) e^{-st} dt.$$

You can compare this with the Fourier transform, where instead of s, we would have $i\omega$. Here, s is a complex number which is $a + ib$. Here, a denotes the growth rate or fall rate of the complex exponential such that we can write:

$$e^{-st} = e^{-(a+ib)t} = e^{-at} \cdot e^{-ibt}.$$

In this complex exponential, the first part e^{-at} denotes growth or fall of the function as the sign of a is negative or positive. Now, the second part e^{-ibt} represents the oscillatory nature of the function. In Fourier transform, the growth and fall of the function part is not present, while only the oscillatory part is included. Thus Laplace transform includes the growth and fall part of the function and hence is a generalized version of the Fourier transform.

In other words, while Fourier transform is only capable of analyzing periodic functions, Laplace transform is capable of analyzing periodic rise and fall functions. For example, the functions of this category which are periodic, oscillatory and also decaying with time. Such type of functions are more suitably analyzed using Laplace transform, while Fourier transform would be more appropriate if the amplitude of this oscillatory function would remain constant. Thus we look at the more properties of Laplace transform as delineated below. We have discussed that Laplace transform is a linear function which means the Laplace transform taken for the sum of two functions x and y is equal to the sum of the Laplace transforms.

That is Next we have the time shifting property. Which means the Laplace transform of any shifted signal is equal to exponential multiplied by its transform. By multiplying e^{-aT}, the phase of the transform is shifted. This causes the shift in phase of the Laplace transform.

Next, we have the frequency-shifting property. Now, suppose we multiply e^{at} with $x(t)$. This changes the frequency of the signal. This is the Laplace transform, which is shifted in the complex domain. Thus, if we multiply an exponential function with the original signal, the Laplace transform will be frequency-shifted.

The next property is about derivative and integration. The integration—the Laplace transform of derivatives—is given by multiplication in the Laplace domain. Thus, derivative or differentiation in the time domain is equivalent to multiplication in the Laplace domain. Similarly, integration in the time domain is equal to division in the Laplace transform domain. Finally, we come to the convolution property, where the Laplace transform of the convolution of two functions is equal to the product of the two functions in the Laplace transform space.

This is also in line with the property of the Fourier transform, where the Fourier transform of two convoluted functions equals their multiplication in the Fourier domain. Thus, the Laplace transform is a transform used for generalized signals for analyzing in the continuous-time domain. Now, we look into the next transform, which is meant for discrete signals. It is called the Z-transform. The Z-transform is used in signal processing and systems theory to analyze discrete-time signals.

In geophysical measurements, we have data which is obtained from field measurements or other natural phenomena monitoring that uses discrete time signals. Thus, the Z-transform is very useful for analyzing such signals. Systems that operate on discrete data need to be analyzed using Z-transforms rather than Laplace or Fourier transforms. In fact, the essentiality of the Z-transform is that it is the discrete counterpart of the Laplace transform. Thus, whatever is the action of the Laplace transform on continuous signals, similar is the action of the Z-transform on discrete signals.

Thus, the Z-transform converts a discrete time signal given by X of N into a complex function X of Z, which is in the Z-complex plane. Now, Z is in the Z-transformed domain, while N is in the real physical domain. N is an integer. The mathematical form of the Z-transform is given by this equation number 2, which states X of Z, which is the Z-transformed function, equals the summation of X of N into Z to the power minus N. Now, let us look at the right-hand side of the Z-transformed equation in more detail. We have two parts of the function within the summation sign.

One is the signal itself, which is $x[n]$. The other, z^{-n}, is the basis function. This basis function is similar to the functions used in Fourier transform or Laplace transform, such as $e^{-i\omega t}$ or e^{-st}. These are the basis functions which are multiplied by the signal and integrated or summed over. To understand the Z-transform in line with the Laplace transform, we can look into the following analogy. The function $x(t)$ in Laplace transform is replaced by $x[n]$ in Z-transform.

While the basis function e^{-st} in the Laplace transform is replaced by z^{-n} in the z-transform. The integral sign is replaced by the summation sign due to the discrete characteristics of the data. Hence, we have the z-transform, which is an analogous transform to the Laplace transform. Note that z is a complex number and can be represented in Euler notation as $re^{i\theta}$, which is magnitude r multiplied by $e^{i\theta}$ (phase angle θ). Now, let us consider the region of convergence.

The idea behind the region of convergence is that the z-transform may or may not exist at all domains within the transformed space. Thus, the region in the transformed space where the z-transform is finite is known as the region of convergence. This means that the sum, which is expressed on the right-hand side of equation 2, converges, and thus the region of convergence is defined as such. The region of convergence is crucial for determining the stability and behavior of discrete-time systems. Now, let us look at the next discrete Fourier transform, which is the discrete version of the Fourier transform.

We have looked into the continuous version of the Fourier transform in the previous lecture. This is the transform used to analyze discrete signals in the frequency domain. Now, the discrete Fourier transform is the most commonly used tool in signal processing, including geophysical applications. Since geophysical instruments sample analog signals, the data is in discrete form, and the frequency content of this discrete dataset is more appropriately obtained using the discrete Fourier transform than the continuous Fourier transform. The fields of application are seismic and electromagnetic applications in geophysics.

The mathematical representation of the discrete Fourier transform is given in equation number 3. We have $x[n]$ as a discrete signal. Now, this discrete signal is made up of N samples, where k is the counter and the total number of samples is N. Thus, k is the transformed variable, which is written on the left-hand side. n is the counter for the discrete signal $x[n]$ in the time domain. Thus, n and k are integers, and they are counters for time and frequency domain functions, respectively.

Thus, we have the basis function given by $e^{-i\frac{2\pi}{N}nk}$. This is a discrete basis function. Note that we had continuous basis functions previously for Laplace transform, Fourier transform, as well as Z transform. But in this discrete Fourier transform, the basis function itself is a discrete function. Thus, the resulting transform is $X[k]$. The discrete Fourier transform converts the time-domain sequence into the Fourier spectral-domain sequence, as given here, where $X[k]$ represents the frequency component of the signal.

Note that for $k = 0,1,2,3$, etc. These are the frequencies, which are discrete in nature. Let us understand this with the help of this example and diagram. The left-hand side represents a discrete sequence, which is a set of points on the x-axis and the values given on the y-axis. n is the counter, and $x[n]$ is the discrete function.

Now we have for $n = 11$, 11 data points which are discrete in nature. Now converting it to the discrete Fourier transform domain, we have k on the x-axis and $|\ X[k]\ |$, that is the magnitude, on the y-axis. The discrete Fourier transform gives us the discrete frequency representation of the original signal. These are the discrete points. It has been joined by a thin line to show the trend.

However, note that there is no continuity in the signal. This is also a sequence of discrete points. For example, the minimum value of the magnitude $X[k]$ occurs for a discrete frequency equal to 5. This is the discrete Fourier transform. One can also define the inverse discrete Fourier transform to recover the original discrete signal from the transformed signal. This is given by:

$$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X\,[k]e^{i\frac{2\pi}{N}nk}.$$

One can also understand the discrete Fourier transform as an expansion or representation of the discrete signal $x[n]$ as a sum of various components. These components are nothing but the discrete Fourier transform, which acts as the amplitude for the basis functions. This is the representation of a discrete function in terms of a set of discrete functions. Just like in Fourier series, we discussed that any given continuous function can be represented by a linear combination or sum of various continuous functions which are sinusoidal in nature.

This is essentially the discrete counterpart of the Fourier series. Thus, the inverse discrete Fourier transform is the discrete version of the Fourier series expansion. Now, let us consider the various properties of the discrete Fourier transform and the fast Fourier transform algorithm. We denote $\mathcal{D}$ as the DFT operator. The DFT operator obeys linearity.

This means that the discrete Fourier transform of the sum of two discrete signals is equivalent to the sum of the discrete Fourier transforms of the individual signals. This is in line with the linearity property of the Laplace and Fourier transforms. Next is the periodicity. The discrete Fourier transform is N-periodic. This means that the discrete Fourier transform repeats after N points, where N is the number of points in the original dataset.

Finally, we have the convolution property. The convolution property states that the discrete Fourier transform of two convolved discrete signals x and y is equivalent to the multiplication of X and Y in the transform domain. This convolution in the Fourier-transformed space is nothing but the scalar product of the two vectors, X and Y, which are discrete sequences of the same size. Next, we look into the fast Fourier transform. The fast Fourier transform is an algorithm that is nothing but an optimized version of the numerical implementation of the discrete Fourier transform.

If we try to implement the discrete Fourier transform as per its definition, it would take a lot of time and computational power. The Fourier transform in its discrete version is optimized using the algorithm known as the fast Fourier transform to reduce the number of calculations from order $O(N^2)$ to order $O(N\log N)$. Order N^2 is the number of operations one has to perform for direct definition-based computation of the DFT. Whereas implementing the fast Fourier transform algorithm, the number of computations is reduced to order $N\log N$, which is much faster compared to order N^2. To get a feel for it, let us consider $N = 10{,}000$.

If we have $N = 10{,}000$, N^2 becomes 10^8. While $N\log N = 10{,}000 \times \log(10{,}000)$, which is $10^4 \times 4$, resulting in $40{,}000$. Note that the log of N is equal to 4. $40{,}000$ is much less than 10^8 computations. Now, the FFT algorithm takes advantage of the symmetry and periodicity of the DFT definition and makes the number of operations much lower.

In a nutshell, the FFT algorithm is shown here. Let us consider two signals: $X[0]$ and $X[1]$. We have $X[0] = x[0] + x[1]$. This is the discrete Fourier transform. And the discrete Fourier transform at $k = 1$ equals $x[0] - x[1]$.

You can verify this by implementing the calculation as shown here. Here, the total number of points, N, equals 2, and we have taken $n = 0$ and 1, and $k = 0$ and 1. This is given in the diagram. There are only two components within the summation sign here. Now, this can be represented as $X[n] = x[n] + (-1)^n x[N]$.

This can be represented in the general form as shown here or diagrammatically. Shown here is a two-point DFT where $x[0]$ and $x[1]$ are mapped to $X[0]$ and $X[1]$. These are the coefficients obtained from this expression. Now, the symmetry property of the mapping and the periodicity of the mapping are useful for defining the fast Fourier transform algorithm.

The details of the fast Fourier transform algorithm can be found in the references provided at the end of this lecture. These are the various versions of the algorithms for the fast Fourier transform, each advantageous depending on the application. Now, we move on to the geophysical applications of the various transform methods discussed in this lecture. The Laplace transform is used in seismic wave attenuation studies and diffusion studies. This is because attenuation and diffusion involve decay or growth in time, which is appropriately modeled using Laplace transforms.

The loss of energy as propagation through heterogeneous media occurs for seismic waves and is modeled using Laplace transforms. The Laplace transform simplifies the calculation of such wave attenuation and helps in the inversion of seismic data to obtain subsurface properties. The Z-transform is useful in designing digital filters for applications involving geophysical signals. Such geophysical applications range from seismic to acoustic waves. These waves are transformed into the Z-domain, making it easier to apply low-pass, high-pass, or other filters in the discrete domain.

This helps enhance the signal quality and remove noise. The discrete Fourier transform is highly useful in electromagnetic geophysical methods such as GPR. GPR stands for ground-penetrating radar. Time-domain electromagnetic surveys also use the DFT, or discrete Fourier transform, to analyze the subsurface frequency response. The DFT helps transform time-domain electromagnetic signals into their frequency components.

It helps identify subsurface structural anomalies, which are geologically very important. This can involve highly conductive regions or highly resistive regions. This can include regions or geological structures with high or low anomalous conductivity. The fast Fourier transform, being an optimized algorithm of DFT, is also useful in such applications, which range from seismic waves to electromagnetic signals and seismic noise. Nevertheless, FFT is also used in earthquakes, volcanic activity, and environmental change applications in geophysics.

Thus, overall, we can say that these methods, which are mostly designed for analyzing discrete signals, are of paramount importance in various geophysical applications. Now, in conclusion, we can say that geophysical datasets are primarily sampled data, which are in the form of discrete datasets. This makes the use of discrete transforms more effective and appropriate. The discrete transforms also represent the data in discrete frequency form. This is more suitable for post-processing and further analysis using computers.

One can look at the following references for more details on discrete transforms and geophysical analysis. Thank you.