Mathematical Geophysics

Swarandeep Sahoo

Department of Applied Geophysics

Indian Institutes of Technology (Indian School of Mines), Dhanbad

Week - 08

Lecture – 37

Hello everyone, welcome to the SWAYAM NPTEL course on Mathematical Geophysics. We continue with module number 8, Data-Driven Analysis in Geophysics. This is lecture number 3, *Sampling Theory for Time Series*. In this lecture, we will cover the following concepts. First, we will look into the analog-to-digital converter.

Second, the sampling process. Third, the Nyquist theorem, which controls the accuracy of the sampling process. Fourth, we will look into the aliasing feature. Finally, we will look into various applications in geophysics. So, let us begin.

What is an analog-to-digital converter, and why do we require it? An analog-to-digital converter converts an analog signal to a digital signal. All natural phenomena occur in a continuous manner with time. The measurement devices we use in geophysical applications also record these continuous natural phenomena.

Now, the computer on which these data are being stored requires the data to be in digital form because an analog signal contains an infinite number of data points, which cannot be stored in the finite memory of a computer device. Hence, the digital form of the data is required to be stored on a computer. This conversion from the naturally occurring analog phenomena to the storage requirement of a digital signal is the function of an analog-to-digital converter. Now, this is a device that converts continuous analog signals into discrete signals. These signals can be processed by computers.

This is essentially a requirement for processing by computers in modern-day data analysis. Now, analog functions are continuous in time, while digital functions are sequences of numbers. The continuous-time function is represented by $x(t)$, while the discrete sequence of numbers is represented by x_n, where n is an integer. The numbers x_n such that $n = 1,2,3$, etc. are some values that are picked from the original analog function, which is continuous in time, that is, $x(t)$. Thus, $x(n)$ is nothing but x at a particular time instance $t(n)$. This is the first equation that governs the ADC conversion. Now, let us look at the adjacent diagram for better understanding.

For example, let us consider the recording of raw voltage. This is given as this diagram, where it appears that the voltage is continuous in time. However, these voltage signals are placed very close to one another and are discrete in time. They are measured using a measuring device, which stores the data onto a recording device that is discrete in time. Now, the advantages of the discrete or digital form of the data are that it is very useful for processing and analyzing the physical measurements from the sensors and instruments.

Note that in previous times, the sensors and instruments used to record the data on sheets of paper or some media, which is a recording continuous medium such as magnetic tapes, etc. Now, these types of recordings were not as efficient in post-processing or analyzing the data as today's modern digital devices are. Now, let us consider the working principle of ADC. These are the three steps in which the ADC operates. First is *sampling*.

That is, the analog signal is measured, or the analog signal values are picked up at regular intervals. That is the sampling process. Then we have *quantization*. The sampled signals are then mapped to discrete levels. This is shown here.

The middle figure shows the filtered voltage, which is essentially the averaged-out voltage of the raw voltage. We have the low-pass version of the raw voltage. This is obtained using low-pass filters, which we have studied in the previous lecture. The low-pass filter removes the high-frequency fluctuations present in the raw voltage and gives a smooth-looking function. Now, sampling this filtered voltage would give us discrete sample values.

These are the discrete sample values obtained from the filtered voltage at regular intervals. Now, these are mapped to the discrete levels by horizontal lines, as shown in the right-hand side figure. This is the stepped voltage, and the horizontal lines corresponding to each sampled signal (the dot) represent the discrete level of the signal at that instant of the time interval. This is the quantization process of ADC. The final process is *encoding*.

In encoding, the quantized levels are converted into binary form for the computer to process further. Binarization is the important step for converting data into a computer-readable form. Now, this is shown as a block diagram in this figure. We start with raw voltage, then we pass it through an anti-aliasing filter. An anti-aliasing filter removes high-frequency components, which is essentially a low-pass filter.

The reason why it is called an anti-aliasing filter will be discussed in a further lecture. Now, this filtered signal, which is in this form, is then moved on for sampling. After sampling, it is mapped to discrete levels, which gives the stepped voltage. After the stepped voltage, a discrete sequence is formed. Now, this discrete sequence is then quantized.

The quantizer converts the discrete sequence into quantitative values. And finally, we have the digital sequence. The digital sequence maps the raw voltage to a discrete set of values within a certain range that is decided by the quantization process. For example, let us consider that the range of the voltage is 5 volts. Now, this 5 volts can be divided into, let's say, 10 parts.

So, it will give us the quantization levels as 0, 0.5, 1 volt, etc. Now, if the same range of voltage of 5 volts is quantized into 100 parts, then the quantized levels are 0.05, 0.10, and so on. These quantization levels are decided by the quantizer. And hence, the digital sequence is obtained. The analog-to-digital converter is very important for converting analog signals into digital form for geophysical instruments and measurements.

The analog-to-digital converter is an important step and process in geophysical applications where the final form of the sequence or the discrete data is given in this form. We have the digital data, which can now be processed further by computers. Next, we discuss the first step, which is sampling in ADC, in more detail since this is the fundamental aspect of ADC. Sampling is the process of measuring and recording physical signals, which are continuous in time, in discrete form. Now, in converting analog signals to digital form, it has to be kept in mind that this process ensures the data obtained after sampling can be processed and analyzed very effectively.

Thus, some conditions have to be taken care of while sampling. To understand those conditions and the effective sampling process, we look into the mathematical procedures of sampling. Suppose $x(t)$ is a continuous-time signal. Consider this as input to the system which is sampling. The output is the discrete signal.

Now, the signal is to be sampled at intervals of time, which is T_s. T_s is the sampling interval or sampling period. Now, the time is discretized as n multiplied by T_s, where n is an integer. Now, let's say this is a continuous-time interval. Taking this discrete instance would convert it into a discrete sequence of time, that is, nT_s.

The individual interval is the sampling period. Obtaining the value of x from $x(t)$ at these intervals of time would give us x_n. Thus, $x(n)$ is a sequence of numbers, which is nothing but the picked-up functional values of the function $x(t)$ at nT_s. Substituting $t = nT_s$ for various values of n would give us a discrete sequence of numbers x_n, as there are a number of values for n. The sampling frequency ω_s is defined as the reciprocal of the sampling interval:

$$\omega_s = \frac{1}{T_s}.$$

This is the sampling process, which is more clearly depicted in the adjacent diagram.

The original function $x(t)$ is shown here. Now, after sampling, it becomes a discrete set of values taken from the original function. These are nothing but these values. Thus, this is the sampling process. Note that in regions where the original signal has steeper gradients, the concentration of the sampled signal over a domain is higher.

Thus, the sampled signal maintains the shape and structure of the original signal. We come to the Nyquist theorem, which is very important for the sampling process. The Nyquist theorem states that a continuous signal can be accurately reconstructed from the samples if and only if the sampling frequency is at least twice the highest frequency component of the signal. Now, we have to understand this from a Fourier spectrum point of view. Consider this example function $x(t)$. Let us consider this is a signal $x(t)$. It has low-frequency components and high-frequency components.

The high-frequency components occur when the signal is fluctuating rapidly. Now, if we take the frequency representation of this function, We may get a Fourier transform $\tilde{X}(\omega)$ like this. Note that this is just an example, and we have not evaluated the exact Fourier transform here. This is just for understanding purposes.

The $\tilde{X}(\omega)$ is the frequency spectrum of the $x(t)$ function. Now, it contains low-frequency regions and high-frequency regions. Now, there is a highest frequency, which is the cutoff frequency. One has to obtain this cutoff frequency for all the inputs for which sampling is to be conducted. The Nyquist theorem states that the sampling frequency, which is ω_s, should be at least twice the cutoff frequency ω_c. This mathematical representation is given by:

$$\omega_s \geq 2\omega_{\max},$$

where $\omega_{\max}$ is the maximum frequency in the signal. That can only be obtained after we have performed the Fourier spectrum analysis of the given signal. The Nyquist theorem states that only when this condition is observed and maintained while sampling can the signal be reconstructed from the Fourier spectrum. This backward procedure requires the Nyquist theorem to be followed. We define the Nyquist frequency f_n such that $\omega_n = 2\pi f_n$ and $\omega_n = \omega_s/2$.

This is the Nyquist frequency which is the minimum frequency at which the sampling should be performed for accurate reconstruction. Thus, signals having frequency components below the Nyquist frequency are accurately reconstructed. Now, signals with frequency components above ω_n will cause aliasing. This can also be understood in the following diagram. Suppose we choose ω_n as this frequency, then all the frequencies which are below ω_n will be reconstructed effectively.

And all the frequencies which are higher than ω_n would lead to aliasing while reconstruction. Thus, ω_n, which is the Nyquist frequency, is to be fixed such that it is at least $\omega_{\max}$. Such that all the components of the signal are reconstructed and aliasing is avoided. This example is a cleaner representation of what we have discussed just now. $\tilde{X}(\omega)$ is the Fourier transform of input signal $x(t)$. This is shown as a triangular function for simplicity.

The Fourier transform of the output signal $y(t)$ is given as this. Note that the output signal is nothing but the sample signal. The left hand side frequency representation is that of the input signal $x(t)$. The right hand side is the Fourier transform of the sampled signal which is discrete in nature. Thus discretization leads to periodic representation of the original spectrum. The original triangular spectrum repeats periodically when the signal is sampled in the frequency domain.

This is very important to understand aliasing, which we will discuss next. Now, have a look at this diagram here. This diagram represents the frequency spectrum of the sampled signal, which is the periodic version of the original non-sampled analog signal. Now, if we are to choose the Nyquist frequency, we can choose such that this is the Nyquist frequency. We have chosen this point.

Because the original signal consists of only these frequencies, which are less than ω_n. If we choose the Nyquist frequency as such, then we are taking into account all the frequencies present in the original signal, which is this. This brings us to the topic of aliasing. If the Nyquist frequency is chosen as anything less than this, then we will miss out on the original frequencies. The top and bottom cases are examples of oversampling and undersampling. Oversampling means that $\omega_s > 2\omega_{\text{max}}$.

While perfect sampling is $\omega_s = 2\omega_{\text{max}}$. Now, what happens in oversampling? If the sampling is done with a frequency that is greater than twice the maximum frequency present in the signal, then the repeating of the signal spectrum becomes such as This means there is a gap between the individual spectrums. If there is a gap, it is possible to choose ω_n as such.

We would take into account all the frequencies included in the original signal. All these frequencies are taken into account, as the frequencies less than ω_n would be considered for reconstruction. Now, consider the case of undersampling. Undersampling means ω_s, the sampling frequency, is less than twice the maximum frequency present in the original signal. It leads to the periodic representation of the original signal, such as this, which is the overlapping of the individual spectrum.

These are the overlapped regions. Now, we are in a difficult situation to choose the Nyquist frequency. We cannot choose the Nyquist frequency to recover the triangular structure of the original signal. For example, if we choose this as the Nyquist frequency, only this portion of the original spectrum is recovered. A small part is left out.

As a second example, if we choose This as the Nyquist frequency, then we include spurious frequencies, which are these parts of the adjacent triangular structure. This would also lead to aliasing. Thus, in the case of undersampling, it is very difficult or impossible to choose a Nyquist frequency that can accurately represent the original frequency spectrum of the signal. This is the phenomenon leading to aliasing.

This is a phenomenon in signal processing where the continuous signal is sampled at a rate that is too low. That is below the Nyquist criteria. This causes different signals to become indistinguishable when reconstructed. The reconstruction process is nothing but choosing the Nyquist frequency in a manner that recovers the original spectrum. The consequence of aliasing is distortion or false frequency components.

These additional frequencies can arise from the choice of the Nyquist frequency as such. They were not present in the original signal. Thus, aliasing occurs when $\omega_s < 2\omega_{\max}$. In frequency terms, high-frequency components fold back and appear as false or low-frequency signals, such as these spurious frequencies. This distorts the original signal and results in inaccurate reconstruction.

The original signal is not recoverable. Now, there are certain ways to reduce aliasing. Aliasing can be reduced if we increase the sampling rate or essentially enhance ω_s such that it is near to twice $\omega_{\max}$ or even greater than that. This is the way to reduce aliasing. Going from undersampling to perfect sampling or even oversampling, which is the safest way for accurate reconstruction. Now, if one is restricted from the instrument point of view in the maximum limit of the sampling of ω_s, then the alternative is to reduce $\omega_{\max}$.

Would make ω_s equal to or higher than twice $\omega_{\max}$. Hence, it would also lead to a reduction of aliasing. So these are the two ways to avoid aliasing and be able to reconstruct the original signal accurately. The analog-to-digital conversion and the sampling methods which we have studied just now are very important and fundamental to geophysical signal processing. In geophysics, data in the form of time series is obtained from various measuring instruments in the field and recorded in natural phenomena.

Now, analog-to-digital converters play a crucial role in this analysis. They enable the use of advanced signal processing analysis techniques and reduction methods for efficient interpretation of geophysical phenomena. Sampling is also fundamental in geophysical surveys for understanding the earth's subsurface structure in the form of discrete data. Accurate sampling ensures high-resolution imaging, analysis, and interpretation of subsurface geological structures. The Nyquist frequency is critical in geophysical data acquisition.

It has to be followed when the conversion from analog to digital is taking place. Otherwise, the reconstruction of the original signal would be inappropriate and inaccurate. In other words, there would be a certain degree of loss of information which is already available from the measurements. This would lead to inaccurate interpretation and loss of information in the understanding of various geophysical phenomena.

All these techniques are widely used in various fields of geophysics, such as seismic, electromagnetic, and gravity surveys. To conclude, we can say that ADCs, which are very useful tools, should be of high resolution for efficient and more accurate conversion from analog to digital versions of physical data. The higher the resolution, the higher the amount of information stored from the actual measurements. The sampling process converts geophysical signals into a digital format. Ensuring Nyquist criteria in sampling is of paramount importance to ensure effective and accurate reconstruction of the original signal.

In summary, efficient sampling balances data quality with manageable data volumes, reducing processing time and storage costs. A very high frequency of sampling would increase the data volume and storage costs. Thus, sampling has to be balanced, as a low sampling frequency would reduce the data quality. Hence, a manageable and optimized sampling technique should be followed, always satisfying the Nyquist criteria. These references can be referred to for more details on the techniques we have discussed in the present lecture.

Thank you.