Mathematical Geophysics

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Lecture – 37

Hello everyone, welcome to the SWAYAM NPTEL course on mathematical geophysics. We continue with module number 8, data-driven analysis in geophysics. This is lecture number 2: *linear time-invariant systems and filtering*. In this lecture, the following concepts are covered. First, linear time-invariant systems.

Second, filtering. Third, correlation. Fourth, convolution. Then, deconvolution. Finally, we will look into geophysical applications.

Let us begin. Linear time-invariant systems are a fundamental concept in signal processing and are widely used in geophysical applications. The linear time-invariant system is shortly called an LTI system. Now, the concept of a system is as shown in the adjacent diagram. There is an input and an output for the system.

The input here is denoted by $x(t)$ while the output is denoted by $y(t)$. The system converts the input signal to output signal. Thus the system can be understood as a function of function. Now the linearity and time invariance are simplifications for general systems. A linear system is such that it obeys the principle of superposition as given by this expression: suppose we have two input signals x_1 and x_2, their individual outputs are y_1 and y_2, then if we add the two signals x_1 and x_2, the outputs are also added.

This is the linearity property of systems which are linear systems. Now the time invariance property of a system indicates that even if there is a shift of the input signal, the output also will be shifted by the same amount. $x(t+T)$ indicates a left shift of the input signal. Recall the concept of shifting which we had discussed in previous lecture. $x(t+T)$ will move the signal towards left. The consequence for the output also is the same. The output signal y is also shifted towards the left. This is the time invariance property. In the original signal, the signal is converted as shown here, where the left hand side is input, the right hand side is output. If the original signal is shifted towards left, Then the output also is shifted towards left. That is by the same amount. Now these two properties are simplifications for general systems and are known as linear time-invariant systems. The advantage of linear time-invariant systems is that it makes the phenomena under study predictable and mathematically more manageable than general non-linear time varying systems. Although it can be said here that natural phenomena are inherently non-linear.

However, the linear time-invariant systems capture the essential physics and concepts which are relevant for these non-linear systems also. For example, let us look at some naturally occurring geophysical examples of input and output and systems. The three columns give the various cases of input, system and output respectively. Now consider the first example where the system is seismometer.

This is a seismological instrument which takes input as the ground motion. During the process of an earthquake, the ground shakes due to the passage of seismic waves. This ground motion is converted to voltage which is output. Let us consider another example: the upper mantle.

The upper mantle is the layer just below the crust. The input to this system is ice loads. Ice loads are large masses of frozen water lying on a particular region. Now it enhances the gravitational effect on that local region. Now the upper mantle deforms under this load.

This leads to the output: post-glacial rebound. This can be understood using the following diagram. Let's say this is an ice load. Under this ice load, over a duration of time, the crust and the upper mantle deform. Now, once the ice melts, post-glacial rebound occurs.

Now this is a thin layer of ice on a rebounded mantle. Similarly, we can take other examples in the atmosphere and ocean where solar radiation is the input and weather and climate dynamics are the output. Now one can see that these are very complicated systems, inherently nonlinear and time-varying. However, the simplest of them is the seismometer, where the input and output are mostly related through a linear relation. As we look into other systems, they become inherently nonlinear and chaotic.

Thus, linear time-invariant systems are a way to understand the input and the consequences of various geophysical systems that govern natural processes. One of the applications of linear time-invariant systems is the concept of filtering. The concept of filtering essentially changes an input signal into an output signal. These signals are obtained from geophysical measurements. Now, understanding how to analyze them through the use of filters is of paramount importance in geophysical analysis.

These help us interpret subsurface features and the dynamics behind various natural phenomena. So, let us consider the basics and mathematical aspects of filtering, which will be applied to geophysical applications later. In general, filtering can be described as a process of modifying or extracting specific frequencies from the original signal. Now, there are various types of filters. These are ideal filters, namely low-pass, high-pass, band-pass, and notch filters.

Now, these filters operate in the frequency domain, which can be obtained after Fourier analysis of the given signal in the time domain. Thus, after performing the Fourier analysis, we would have the spectrum or essentially the frequency-based representation of the original measurements. Now, using these types of filters, we can choose particular frequencies among others for useful information extraction. Now, there are certain components of a filter, which are the passband—the range of frequencies that are allowed— Next is the cutoff frequency, which is the critical frequency.

The next is poles. Poles are the locations in the domain of frequency after transformation through Fourier analysis where the Fourier spectrum becomes a singularity or it becomes undefined. These are the regions known as poles. Now the importance of filters is the maximum when the signal-to-noise ratio is high. That means the signal is much more dominant than the noise.

Thus we can filter out the noise and obtain the pure signal for accurate analysis. Let us look into the details of the individual type of filters in the form of the system concept. The simplest one is the low pass filter. Now let us have a look at the frequency domain representation of this low pass filter. In the frequency domain, ω, the cut-off frequency being ω_c and the number of poles m, the functional form for the low-pass filter is given by this equation:

$$F_L(\omega) = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

Now let us analyze this functional form a bit. What does this say? It says that suppose we have a frequency representation of a signal such as, let's say these are the frequencies and amplitude of the individual frequencies. ω_1 and ω_2 is the band of frequencies which are present in the measured signal.

Now, let us look at the functional representation of the low-pass filter. Here, we have ω, which is the frequency in the denominator. This means that as ω increases, $F_L(\omega)$ decreases. That means the system is such that it suppresses higher frequencies. So, if we have the input signal as shown here, the output signal would be something like this, where the higher frequencies are damped out.

These are the higher frequencies which are damped out. While the lower frequencies are maintained. Now, if the value of n becomes larger and larger, the damping also would become more and more stringent. This means that only lower frequencies are allowed to exist in the spectrum, while higher frequencies are removed. This is why it is called a low-pass filter.

It allows only lower frequencies to pass, that means, to the output. The lower frequencies go through the system and appear at the output, while higher frequencies are damped out by the system, which is defined in this functional form. Similarly, for a high-pass filter, we have $1 - F_L(\omega)$. This indicates the opposite effect of a low-pass filter. Whereas a low-pass filter damps higher frequencies, a high-pass filter can be understood to damp lower frequencies, as it is $1 - F_L$, which is the lower frequency system.

Thus, in a high-pass filter, lower frequencies are damped out while higher frequencies move on to the output. The bandpass filter is a combination of low-pass and high-pass filters, where only a certain portion of the frequency is allowed to pass on to the output while other frequencies are damped out. These frequencies, which are allowed to pass, are centered around ω_B. So, the frequencies around ω_B are allowed to exist. So, this would be the output for the input shown here. In the case of bandpass filters, frequencies close to a selected frequency ω_B are allowed to move on to the output while other frequencies are damped out.

The functional representation of the linear invariant system is shown here. Next, we move on to the concept of correlation. Correlation is a statistical measure. It helps identify the similarity between two field quantities. It describes the strength, direction, and linear relationship between two variables, signals, or measurements.

There are two types of correlations. One is cross-correlation, and the second is autocorrelation. These are very much related mathematically. Cross-correlation is used to measure the similarity between two different signals. Now, these two different signals may be separated by a time lag, or they may belong to different physical quantities.

Now, this is widely used in time series analysis and various geophysical applications, such as seismic data analysis, remote sensing, and fault detection. And any other mathematical processes which govern geophysical phenomena. Mathematically, let us consider two signals x and y. These are the two input signals. The cross-correlation feature can be understood as a system which will convert these input signals $x(t)$ and $y(t)$ to an output which is $c(\tau)$.

Thus, the output $c(\tau)$ is defined as the integral from $-\infty$ to $+\infty$ of $x^*(t)y(t+\tau)dt$. In the integral shown here, $x^*(t)$ represents the complex conjugate of $x(t)$, while $y(t+\tau)$ is the time-shifted signal for y. Note that these signals may be complex numbers; hence, the complex conjugate is used. If the signals are real functions, then $x^*(t)$ is equal to $x(t)$. Now, the cross-correlation can be interpreted as follows. $c = +1$ indicates perfect positive correlation.

While $c = -1$ indicates perfect negative correlation. If $c = 0$, then there is no correlation. Thus, if $c = 0$, it means that the two signals are not at all similar to each other. While $c = \pm 1$ indicates that, shape-wise and functional structure-wise, the signals are equivalent. However, depending upon the sign, the signals' magnitude may be the same, but the direction may be different. So, this is the utility of the cross-correlation process. Now, we have the next autocorrelation statistical measure. The autocorrelation measures the correlation of a signal with itself at different time lags. Now, this is important to understand the inherent characteristics or geometrical structures of a given signal, such as repeating patterns or periodicity in a signal. It helps identify useful trends or cycles present in a signal. Now, suppose we represent an input signal $x(t)$, which is input to the autocorrelation system, can be defined such that the output is given by $a(\tau)$ as the integral from $-\infty$ to $+\infty$ of $x^*(t)x(t+\tau)dt$. Note that the major difference between cross-correlation and autocorrelation is that the second signal y is replaced by the first signal itself with time shifting. Now, let us look at the convolution process.

The convolution process is also a very geophysically relevant system, which analyzes various geophysical measurements and obtains useful interpretations. It's a mathematical process to combine two signals to produce a third signal. In the convolution process, there are two input signals and one output signal. The input signals are a given signal $x(t)$ and the impulse response of the system.

Now, the impulse response of the system is the characteristic signal of the system in response to a delta function input. If the input is a delta function or the Dirac delta function, which we have discussed in previous lectures, then the output is $h(t)$. Now, $h(t)$ is the impulse response, which is essentially a fingerprint or a characteristic of the system. With these two inputs, $x(t)$ and $h(t)$, the output signal is defined as $y(t)$, given by the integral from $-\infty$ to $+\infty$ of $x(\tau)h(t-\tau)d\tau$.

Note the use of the two input signals. $x(\tau)$ is the original signal, while $h(t-\tau)$ is the time-shifted version of the impulse response. The mathematical notation for convolution is $y(t) = x(t) * h(t)$. Now, let us try to understand the convolution process with the help of the adjacent diagram. Suppose we have the input signal as a step function $x(t)$, and a Gaussian filter given by this fluctuating signal $h(t)$.

The convolution process effectively converts the signal into $y(t)$. This occurs when, keeping the original signal $x(t)$ fixed, $h(t)$ is time-shifted continuously. As the signal moves toward the right and gets multiplied with $x(t)$, it produces $y(t)$. You can note that the output signal $y(t)$ has characteristics of both $x(t)$ and $h(t)$. The characteristics of $x(t)$ are similar to $y(t)$ in the region of very few changes or constant regions, while the characteristics of $h(t)$ and $y(t)$ are similar in highly fluctuating regions.

Thus, one can say that the convolution process inherently takes in the important characteristics of the input function, the impulse response of the system and produces a combined output signal $y(t)$. If we perform the Fourier transform of this convolution, we would obtain the frequency domain representation of the convolution process. Here $\tilde{y}(\omega)$, $\tilde{x}(\omega)$ and $\tilde{h}(\omega)$ are the Fourier transforms of $y(t)$, $x(t)$ and $h(t)$. Note that the convolution process as defined here becomes just a simple multiplication in the Fourier transformed domain ω. The properties of convolution are listed here. We have commutative, associative, distributive properties which are obeyed by the convolution system.

The commutative property involves interchanging the product components $x(t)$ and $h(t)$ still remaining the output same. The associative property indicates that the order of convolution is immaterial. The distributive property indicates that two signals which are responses of two systems $h(t)$ and $g(t)$ can be convolved with the input system and that would be same as the individual convolutions for individual systems. Finally, we have the scaling property of the convolution.

The scaling property means that if the signal is scaled, that means the magnitude is changed by a factor A, and then it is convolved with the impulse response, then the output of the system that is the convolved system is also enhanced or diminished by the same factor A. So these are the various properties of convolution system which is inherently an LTI system or linear time-invariant system. Next we have the deconvolution process. The deconvolution process is the inverse operation of convolution. Just like we have addition, subtraction, multiplication, division, integration and differentiation, which are opposite operations. Similarly, we have convolution and deconvolution.

These are the inverse systems. Now, convolution inherently reverses the effect of deconvolution and vice versa. It is useful for recovering the original signal or an approximation of the original signal from a convolved system or a convolved system's response. Now, let us first look at the diagram on the side for more understanding before we move on to the mathematical aspects. We had seen the convolution process.

Now, the fourth diagram represents the deconvolved signal. You can see that the deconvolved signal is much similar to the input signal, where we have the property of a step function and a constant phase. However, note that there are various rapid fluctuations, which are inherent in the deconvolution process. Thus, one can say that the deconvolution process is useful to recover the approximation of the original signal from the convolved signal. Mathematically, the convolution and deconvolution processes, being opposite to each other, can be represented well in the Fourier transform domain.

Since we know that multiplication and division are opposite operators. Similarly, since the convolution process is represented as a multiplication in the Fourier spectrum domain, the deconvolution is represented by the division process. The original signal $\tilde{x}(\omega)$ in the Fourier spectrum domain is obtained as $\tilde{y}(\omega)/\tilde{h}(\omega)$. This is the deconvolution process.

Implementing the inverse Fourier transform on $\tilde{x}(\omega)$ would lead to the original signal $x(t)$. Thus, the approximation of the original signal $x'(t)$ can be obtained even if only the convolved system is available. Now, these are the various systems that are linear and time-invariant, useful in analyzing geophysical signals. These techniques, such as filtering, convolution, deconvolution, and correlation, are widely used in various geophysical applications, such as those listed here. First of all, let us consider seismic data analysis.

In seismology, filtering techniques are often used to remove noisy data and enhance the signal's quality. These systems are modeled as LTI systems. The filtering process can also be used to remove noise from the Earth's gravity data and magnetic field data and can be modeled as outputs of LTI systems. Also, it can be understood that the Earth's structure acts as a filter itself. For example, let us consider the Earth's outer core, where the geomagnetic field is generated.

Its representation at the Earth's surface is such that the small-scale structures or high-frequency structures are filtered out. This is filtered out due to the inherent property of the mantle. Thus, the Earth's structure also acts as a filter for various geophysical phenomena, such as the geomagnetic field. Essentially, the Earth's structure may be considered a low-pass filter. Different types of filters, which we have considered here, are tailored for specific geophysical applications.

Low pass filter, high pass filter and band pass filters are used to isolate signals which are of interest from noise to record seismic data. The correlation technique is applied to correlate various geophysical logs. Now geophysical logs mean measured data of quantities such as the electrical resistivity, porosity, gamma ray logs in deep bore wells. Deep bore wells are used to characterize subsurface information. Geophysical logs from these boreholes are correlated with seismic or geological data to interpret the findings.

These findings can give us information about the lithology or the structure which is geophysically and geologically characteristic, the fluid content, the reservoir properties etc. Correlating seismic data with well log data are useful to create accurate geological models of the subsurface which are very useful in petroleum exploration. Finally, let us look at the application of deconvolution process. If the signal is convoluted with the Earth's response such as attenuation or dispersion in various layers of the Earth for signals such as the seismic waves or the geomagnetic field, then deconvolution techniques can be used to remove the effect of the Earth's response and recover the original source formed signals. Thus, the primary signal at its origin can be found out even if it is not directly accessible through this deconvolution technique.

The deconvolution process is applied to seismic data to eliminate the effects of surface waves. After the surface waves are eliminated, the primary reflection from subsurface layers are detected more accurately. and hence the subsurface structure is revealed. Thus, we come to the conclusion of the present lecture. First, the convolution process is a fundamental tool to model seismic, electromagnetic and gravity processes and the waves interaction in the Earth's subsurface.

The second method, which is the correlation method, is a tool in geophysics used to detect relationships between two field quantities or measured quantities or to find inherent geometrical structures in a single signal. Noise reduction, signal enhancement, and specific feature extraction are the two targets of filtering techniques. These filtering techniques enable geophysicists to better analyze seismic, magnetic, gravity, and time-series data for various geophysical applications in exploration as well as fundamental geophysics. One can refer to the following references for a more detailed analysis of various tools discussed in the present lecture. Thank you.