Mathematical Geophysics

Swarandeep Sahoo

Department of Applied Geophysics

Indian Institutes of Technology (Indian School of Mines), Dhanbad

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Lecture – 36

Hello everyone. Welcome to the SWAYAM NPTEL course on mathematical geophysics. We start the final module, number eight. This is the data-driven analysis in geophysics. This is the first lecture of this module: Fourier analysis and transform.

In this lecture, we will cover the concepts related to Fourier analysis and Fourier transform. This lecture is composed of five components. First, we will look into the basic Fourier transform theory. Second, the Fourier integral theorem. Third, the similarity and shifting properties of Fourier transform.

Fourth, we will look into the derivatives and integration of functions and their Fourier transforms. Finally, we will look into the applications of Fourier transform in geophysical studies. So, let us begin. The Fourier transform theory is a very important aspect of geophysical data analysis. In geophysics, data is measured using various geophysical instruments over space and time.

Analyzing this data is of utmost importance to interpret the subsurface properties and gain a better understanding of various geophysical phenomena. Analyzing the measurements, which are in the form of time series data, is extremely important to perform, and further understanding of geophysical processes can only be obtained through proper analysis of these data. Now, Fourier analysis and techniques have proven to be very useful in analyzing data sets in all domains of science and technology. The Fourier series and Fourier transform comprise the two components of Fourier analysis. First, we will look into the Fourier series.

The Fourier series is a method to represent a function in terms of sines and cosines. We have seen such representations in the spherical harmonic studies in previous lectures. Now, we will focus on the Fourier series part. Let us consider $x(t)$ as a function of time. This function of time can be represented in terms of Fourier components as given here.

$x(t)$ is the summation over n equals zero to capital N of all the components, which are given by the Fourier components. $e^{2\pi i f_n t}$. This is the complex Fourier exponential term. Its amplitude is given by x_n or Fourier series components. x_n are constant coefficients, and f_n is the frequency.

Now, since x is any arbitrary time series function, the Fourier series is capable of representing various geophysical signals in this way.

One typical example is that of tides which occur in ocean. Tides represent displacement of water above the sea level which can be very appropriately represented in terms of Fourier series. Also, these Fourier series are known as harmonic development of a function. Next, we come to the Fourier transform. The Fourier transform is a method of representing the function in a continuous frequency domain.

Note that the frequency components of Fourier series f_n are discrete. Whereas the frequency component of the Fourier transform are continuous. What does this mean? It means that Fourier series utilizes discrete frequencies such as 1, 2, 3 or 0.1, 0.2, 0.3 as discrete values to represent a function. While Fourier transform utilizes a continuous change in the frequency which is over infinitesimal changes of frequency values to represent a given function.

Thus the same function $x(t)$ is represented as an integral instead of a summation $x(t)$ can be represented as an integral over the entire domain $-\infty$ to $+\infty$ $e^{2\pi i f t}$ where f is the frequency which is a continuous variable $\tilde{x}(f)$ is the coefficient function. Thus, the major difference between Fourier transform and Fourier series is that Fourier transform can represent any arbitrary function while Fourier series cannot represent an arbitrary function. Functions which are amenable to accurate Fourier series expansion are periodic functions over an infinite domain. Now Fourier transform can also represent non-periodic functions because of its continuous frequency range.

The expression $x(t) = \int_{-\infty}^{\infty} \tilde{x}(f) e^{2\pi i f t} df$ is known as the *inverse Fourier transform*. Now, this integral holds only and only if $\tilde{x}(f)$ can be written as an integral over the entire domain of time $x(t)$ multiplied by $e^{-2\pi i f t}$. This is known as the *forward Fourier transform*. Now, these are the two parts of the Fourier transform. These two equations define the function $\tilde{x}(f)$, which is the Fourier transform of the time function $x(t)$. This can be written as $\mathcal{F}\{x(t)\} = \tilde{x}(f)$. f is the cyclic frequency.

Equations (2) and (3) represent, combinedly, the *Fourier transform pair*. Now, let us have a look at the Fourier transform in the radial frequency domain. The radial frequency domain is given by the radial frequency ω, also known as the angular frequency, which is 2π multiplied by the cyclic frequency. The inverse and the forward Fourier transform are represented here.

Next, we look into the Fourier integral theorem. Now, the Fourier integral theorem is a set of four theorems. It defines how and when the Fourier transform exists and what its behavior is. The first theorem is the Dirichlet condition theorem. Now, before we move on to the statements of this theorem, let us have a look at the adjacent diagram for notational and geometrical understanding.

This diagram is a function of time represented over a domain $-\pi$ to $+\pi$. We can see various features of this function. This function is continuous over certain regions and discontinuous at finite number of places. For example, these are the places where this function is discontinuous and they are eight in number. This is the finite number of discontinuities of the function.

In between these discontinuities the function is continuous such as this region. So this function can be treated as partly continuous and partly discrete with finite number of discontinuities. Thus such a function can be represented by $x(t)$ which is periodic over a domain $-\pi$ to $+\pi$ which means Any 2π periodic domain which is beyond this such as π to 3π, 3π to 5π etc. will be same as this part of the function shown.

Thus this is a 2π periodic function. Having this 2π periodic function and having finite number of discontinuities with finite number of maxima and minima values such that all the points in the domain are single valued, we can look into the Dirichlet condition. Note that the number of maxima and minima are finite. These are the location of maxima and minima which are finite in number. One additional condition which is required is the modulus of $x(t)dt$ is finite over integral over the entire domain which means it is absolutely integrable.

If all these conditions are satisfied, then the Fourier series converges to the function x at all points in the domain where the function is continuous. At the jump or discontinuities, the Fourier series converges to the midpoint of the jump. For example, At this point where the function is discontinuous, it has two values for this location. For the location shown by the small dot, the crosses given in two in number represent the two values present at this discontinuity. Now, the midpoint value is shown by the larger circle dot.

This would be the value where the Fourier series would converge if all these conditions are satisfied. That is the Dirichlet condition theorem for Fourier integrals. Now, we have the next Fourier integral theorem. The conditions for the Fourier integral theorem are that it satisfies the Dirichlet condition over every finite interval and the function $x(t)$ is absolutely integrable. Then, the Fourier transform given by equations (2) and (3) in the previous slide holds and is accurate.

This means that for the forward and inverse Fourier transform to hold, the function $x(t)$ must satisfy the Dirichlet condition and be absolutely integrable over the domain. Next, we have the Fourier sine transform. The Fourier sine transform relates only to the sine part of the Fourier series. It is very useful for representing odd functions more effectively. Since sine is an odd function, the representation of an odd function requires only the sine part, while the cosine part represents even functions.

Thus, Fourier sine and cosine transforms are complementary. Now, $x_s(t)$ is the time-dependent function, whereas its Fourier transform is given by $x_s(\omega)$. Now, the forward and inverse transforms are given as these quantities, where the exponential term in the integral is replaced by a sine term. Similarly, for the cosine transform, the cosine Fourier pair is given by $x_c(t)$. The cosine transform pair has the integral term with $\cos(\omega t)$ instead of $e^{2\pi ift}$. Now, these are the Fourier sine and cosine transforms, which are complementary to each other and, combined, they represent the total Fourier transform.

Now, let us look at the similarity and shift properties of the Fourier transform. These are very useful for geophysical applications. The *similarity property*. The similarity property refers to the scaling of the function in the time and frequency domains. It describes how compressing or stretching a function in the time domain affects its Fourier transform.

Thus, we will be looking into functions in the time domain which, when compressed, change accordingly in the Fourier domain. Now, let's say we have the function $x(t)$. This is in the time domain. Now, this function is scaled by a factor α. Now, the function becomes $x(\alpha t)$. How does this look in graphical form? For example, let's say this is the function $x(t)$. This is the expanded form.

Now, $x(2t)$ is the expanded version of the function $x(t)$, while $x(t/2)$ is the compressed version of $x(t)$. Here, $\alpha = 2$ and $\alpha = 1/2$, respectively. This is scaling by a factor α. Now, we will be looking at what the Fourier transform of this scaled function will be. That is given by:

$$\mathcal{F}\{x(\alpha t)\} = \frac{1}{|\alpha|}\tilde{x}\left(\frac{\omega}{\alpha}\right).$$

For example, if $x(t) \xrightarrow{\mathcal{F}} \tilde{x}(\omega)$, then $x(2t) \xrightarrow{\mathcal{F}} \frac{1}{2}\tilde{x}\left(\frac{\omega}{2}\right)$.

We have seen that if $|\alpha| > 1$, it indicates *stretching* in the time domain and *compression* in the frequency domain because the factor by which ω is multiplied is less than 1. If $|\alpha| < 1$, it indicates *compression* in the time domain and *expansion* in the frequency domain. From this, it can be understood that the effect of similarity—that is, expansion or compression—is opposite in the time and frequency domains.

Next, we'll look at the shifting property. Now, what is shifting? Shifting is depicted by the change in the location of the function. Suppose this is a function of time; then, the shifting would indicate either this, which is shifting to the right, or this, which is shifting to the left. Shifting to the right is given by $x(t - t_0)$, while shifting to the left is given by $x(t + t_0)$.

Here, the shifting property of a Fourier transform describes how the Fourier transform of a function relates or changes when the function is shifted. So, we would be looking at the frequency representation of $x(t + t_0)$ and $x(t - t_0)$ in terms of the shifting property. The shifting property allows various manipulations of the function without changing the fundamental shape. Now, mathematically, if the function is shifted by t_0, it is represented by $x(t - t_0)$. If $x(t) \xrightarrow{\mathcal{F}} \tilde{x}(\omega)$, then the shifted signal $x(t - t_0)$ transforms to $\tilde{x}(\omega)e^{-i\omega t_0}$.

Thus, shifting a signal in the time domain induces a *phase shift* in the frequency domain. However, the *magnitude* of the frequency spectrum $|\tilde{x}(\omega)|$ stays the same. The phase changes by $-\omega t_0$. Thus, the similarity and shifting properties of the Fourier transform are very useful in geophysical datasets. The geophysical datasets are to be combined and processed using similarity and shifting properties for better representation and interpretation of subsurface phenomena and geophysical aspects.

Next, we look into the derivative and integration of the Fourier transform. The *derivative property* of the Fourier transform relates the differentiation in the time domain to its effect in the Fourier transform domain. Now, if a function is differentiated in the time domain, in the frequency domain, it is premultiplied by a factor $(i\omega)^n$, where n is the order of differentiation. Thus, differentiation in the time domain is equivalent to multiplication in the frequency domain. This property is very useful in analyzing the rate of change of signals and enhancing specific features of the frequency spectrum.

Next, we consider the *integration* of Fourier transforms. If $x(t)$ is a signal and $\tilde{x}(\omega)$ is its Fourier transform, then the integral of the time-domain function leads to the Fourier transform given by:

$$\mathcal{F}\left\{\int x(t)dt\right\} = \frac{1}{i\omega}\tilde{x}(\omega) + 2\pi C\delta(\omega).$$

Now, the first part is the division of $\tilde{x}(\omega)$ by $i\omega$. Just as differentiation leads to multiplication in the Fourier domain, integration leads to the opposite effect of division in the Fourier domain.

Thus, the original Fourier transform $\tilde{x}(\omega)$ is divided by $i\omega$. The integration constant, which can arise due to the integration of the function in the time domain, is then also transformed into $2\pi C\delta(\omega)$. $2\pi C\delta(\omega)$ is the Fourier transform of constant functions, and thus it appears as the Fourier transform where an integration of the function in the time domain is considered. Here, $\delta(\omega)$ is the Dirac delta function. The Dirac delta function is infinity at $\omega = 0$ and 0 when $\omega \neq 0$.

Now, we look into the various applications of Fourier series in geophysics. We have come across these applications in previous lectures in the context of spherical harmonics. However, we will reiterate them to emphasize the importance of Fourier series in geophysical systems. First, *seismic wave analysis*. Seismic wave analysis decomposes seismic signals into frequency components to identify body and surface waves.

In seismic wave analysis, in the Fourier spectrum, the high frequency signals indicate shallow features, whereas low frequency components indicate deeper structures in the Earth's interior. Thus, representing a seismic signal in frequency domain helps to segregate deeper and shallow structures. In geophysical exploration, we have the gravity and magnetic survey methods. In these methods, Fourier transform separates the short wavelength and long wavelength features. The short wavelength features represent shallow structures, whereas longer wavelength features represent deeper subsurface structures.

This helps in locating the depth of the anomalies which are mineral deposits or other resources which are sought after using these geophysical methods. Also Fourier transform can be used to project the data to higher or lower altitudes in gravity and magnetic surveys. This is in line with the upward and downward continuation phenomena, which we have discussed in the context of potential theory. In the electromagnetic survey method of geophysical exploration Fourier transform decomposes the electromagnetic signals or the variations of electromagnetic field to assess the frequency dependent penetration of the signals. In this surveys, High frequency map allows the identification of shallow features while low frequency detect deeper features. Thus we come to the conclusion of the Fourier transform in the context of geophysical and general context. The Fourier transform is very vital for decomposing signals. It decomposes signals into sinusoidal components which are categorized into various frequencies. Based on the frequency content various physical interpretation can be found out which is otherwise very difficult to obtain from the time dependent function representation.

The Fourier transform provides deep insight into both time-domain and frequency-domain characteristics of geophysical data. The Fourier transform enables the construction of filtering techniques, which we will discuss in further lectures. These filtering techniques help remove noise in seismic, gravity, and magnetic data. Removing noise enhances the interpretation and the quality of estimation and exploration of the subsurface. In the context of natural phenomena, the separation of short-wavelength and long-wavelength structures in gravity, seismic, and magnetic studies is enabled through the use of Fourier transform methods.

Overall, it can be said that Fourier transform techniques are of great importance in geophysical data analysis. Thus, data-driven analysis of geophysical data cannot be done without the use of the Fourier transform. One can refer to the following references for more details on Fourier transforms and Fourier series. Thank you.