Mathematical Geophysics

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Lecture – 35

Hello everyone, welcome to the SWAYAM NPTEL course on mathematical geophysics. We continue with module number 7, Thermofluidic Processes in Geophysics. This is the fifth lecture, titled Outer Core Double Diffusive Convection. In this lecture, we will cover the basic aspects of double diffusive convection and how it applies to the outer core dynamics. The first part is about double diffusive convection.

Followed by the dynamo action. The governing equations and numerical simulations related to these processes will also be discussed. Finally, we'll look into convective heat transfer in the Earth's outer core due to double diffusive convection. So, let us begin. What is double diffusive convection?

Double diffusive convection is the process of fluid motion involving the diffusion of two species. The fluid instability that arises when these two components diffuse with different diffusivities is known as double diffusive convection. These two components can be heat and solute particles, which are present in the fluid. This phenomenon is very much applicable to the Earth's outer core, Earth's oceans, and stellar interiors. Let us consider the driving forces.

For double diffusive convection in the earth's outer core, the two driving forces are thermal buoyancy and compositional buoyancy. The thermal buoyancy occurs due to temperature differences while compositional buoyancy occurs due to the difference in the concentration of light elements or heavy elements. Now have a look at this diagram. In this diagram there are two components which are driving convection. One is temperature.

The temperature gradient due to warm and cold regions drives convection and conduction. We also have additional solute concentration that is for example salt. The bottom region is salty which means it has higher solute concentration while the above regions have low salt concentration. These are fresh water. Now due to the difference in the concentration convection due to compositional buoyancy can arise.

Thus we have two mechanisms which can drive convection in a double diffusive model. One is thermal, other is concentration. Now note that for water-salt combination, the solute salt is heavier than water. and salty water is heavier than pure water but in case of earth we have the solutes as oxygen or sulphur whereas liquid iron is a heavier solvent now here the solute is lighter The configuration where the solute is heavier is applicable for Earth's oceans, whereas the configuration where the solute is lighter is applicable to the Earth's outer core.

Both cases are double-diffusive convection. However, because of the density relation between solute and solvent, the dynamics can be much different. Next, we look into the diffusivity contrast. This is the difference between the diffusion coefficients of the two components in double-diffusive convection. The diffusion coefficient of heat is much higher than that of the solute, which means heat diffuses much faster than solute or light elements.

This can lead to a situation where the thermal gradient is stabilizing while the compositional gradient is destabilizing, or vice versa. This means in double-diffusive convection, there may arise scenarios depending upon the gradients of temperature and composition, such that one induces convection while another opposes it. Based on such different possibilities, we have two major types of double-diffusive convection. The types of double-diffusive convection are finger instability and layered convection. First, let us consider the finger instability.

The finger instability type of double-diffusive convection occurs when the fluid is stable to thermal convection but unstable to compositional convection. Such a configuration can occur when warm, salty water lies over cooler, fresh water. The next is layered convection. Layered convection is opposite to the finger instability configuration. In the layered convection model, the fluid is stable to compositional convection but unstable to thermal convection.

Such a system can arise when cold fresh water lies over warm salty water. The diagram shown here illustrates the layer convection model. If we reverse or turn this diagram upside down, it would lead to a finger instability system. In the finger instability system, the flow structures are in the form of salt fingers, while in layered convection, the flow structures are mostly diffusive. Now, having understood the double-diffusive convection model and its thermal counterpart in the previous lecture, we are now in a position to understand the dynamo action, which is an important consequence of thermal convection and fluid motion in the Earth's outer core.

The double-diffusive convection occurs in the Earth's outer core, and this leads to various phenomena, including the dynamo action. The thermal convection occurring in the Earth's core is driven by heat released by cooling and latent heat due to inner core solidification. The compositional convection occurs due to the release of light elements, which are expelled due to inner core growth. Thus, the inner core growth induces two sources: one, the latent heat for thermal convection, and the other, light elements for compositional convection. The secular cooling of the Earth over a long duration induces thermal convection.

The gravitational source of sinking heavier elements from the top of the core toward the bottom drives compositional convection. This interplay of two convective processes leads to very complex dynamics while also providing an efficient mechanism for geodynamics. Now, the convective motion drives conductive liquid iron, which generates electric currents. This is the conversion of mechanical energy to electromagnetic energy. The mechanical energy of convective motion comes from heat energy.

This is the chain of heat transfer, mechanical energy transfer to electromagnetic energy transfer inside the Earth's outer core. The electric currents thus formed couple with Earth's rotation and produce magnetic fields. The role of Earth's rotation is to induce appropriate fluid structures which can efficiently generate the magnetic field. The interaction between various fields such as temperature, velocity, electric current, and magnetic field and composition maintains and generates the geomagnetic field through a self-reinforcing loop.

We will look into the mathematical aspects of this loop very shortly. Before that, let us look at this adjacent diagram. This indicates a schematic diagram of the fluid dynamical processes and the thermodynamic processes occurring in the Earth's outer core. Here, the fluid motions are shown in yellow spiraling structures. Due to the spiraling fluid dynamical motions, the induced magnetic field is shown in black lines.

These are the interplay of electromagnetic and fluid dynamical processes. Now, let us look at the governing equations. These are the comprehensive set of governing equations which determine the dynamo action for Earth and the generation of the geomagnetic field from thermal energy. It assumes an incompressible fluid of density ρ under the Boussinesq approximation. First, we have the mass conservation equation, which is $\nabla \cdot \mathbf{U} = 0$.

Second is the momentum conservation equation. The momentum conservation equation is nothing but Newton's second law of motion for the fluid. The left-hand side of the momentum conservation equation is nothing but acceleration, which is per unit mass. The acceleration term is in three parts. First is the non-inertial acceleration, $\frac{\partial \mathbf{u}}{\partial t}$. The second term is the inertial acceleration, $\mathbf{u} \cdot \nabla \mathbf{u}$. The third is the pseudo-acceleration, which is the Coriolis acceleration given by $2\boldsymbol{\omega} \times \mathbf{u}$.

Here, $\boldsymbol{\omega}$ is the rotation rate. The right-hand side consists of the sum of all the forces acting on the fluid in the Earth's outer core. First is the pressure gradient. Second is the thermal buoyancy due to temperature gradients. Third is the chemical buoyancy, or compositional buoyancy, due to concentration gradients.

Fourth is the Lorentz force due to the interaction of the magnetic fields. Lastly, we have the viscous friction effect given by $\nu \nabla^2 \mathbf{u}$. The Lorentz force is derived from Maxwell's equations as $\frac{1}{\mu \rho}(\nabla \times \mathbf{B}) \times \mathbf{B}$. One can look into the derivation of Maxwell's equations for further understanding of the Lorentz force term. Next, we have the energy conservation equation for temperature. This is the equation for the temperature evolution over time.

The first term is the evolution of temperature. The second term is the heat advection given by $\mathbf{U} \cdot \nabla T$. The third is the thermal diffusion term. The fourth is the heat source term. We have the fourth equation, which is the concentration equation.

The concentration equation governs the evolution of the concentration over the entire region. C denotes the concentration field over all space and time. $\frac{\partial C}{\partial t}$ is the evolution of the concentration field over time. Similar to the energy conservation equation, the concentration equation is also affected by advection, diffusion, and sources.

The advection term is given by $\mathbf{U} \cdot \nabla C$. The diffusion term is given by $\kappa_C \nabla^2 C$, and the source term is given by S_C. Now we will look into the details of these terms because this is a new equation as far as the equations governing the fluid motion are concerned. The light element advection is the motion of fluid which carries the concentration from one region to another.

For example, if we have a highly concentrated region and a low-concentration region with fluid motion given by this direction, then the parcel of fluid containing the high-concentration solutes will be transported toward the region of low concentration and will enhance the concentration in these regions. Thus, the concentration in these regions will be enhanced solely due to the advection of light elements from a region where the concentration is higher to a region where the concentration is lower. This advection is driven by the fluid flow. Next, we have chemical diffusion, which is nothing but the diffusion of concentration.

The diffusion coefficient is κ_C, which is different from the thermal diffusion κ_T. The light element source is the term that models the release of light elements from the inner core boundary. Next, we look into the induction equation. Recall that we have discussed the derivation of the induction equation in previous lectures. The induction equation governs the evolution of the magnetic field over time.

The change in the magnetic field is governed by two processes. One is the $\nabla \times (\mathbf{u} \times \mathbf{B})$, which is the induction term, also known as the transportation term. This term helps generate the magnetic field due to fluid motion. The second term is the ohmic dissipation term, which is related to the diffusion effect.

The induction process enhances the magnetic field, while the diffusion process diminishes the magnetic field. The induction process involves the transportation of magnetic fields through fluid flow, while the diffusion process also leads to ohmic dissipation, which results in the destruction of the magnetic field. This is the comprehensive set of governing equations that drive thermal chemical convection or double diffusive convection in the Earth's outer core. Similar to thermal convection, double diffusive convection can only be solved using numerical methods for the complicated regimes of the Earth's interior. For that, we need to non-dimensionalize the governing equations, as done for the thermal equations.

This set of equations is the non-dimensional version of the governing equations that we have just discussed. To obtain this non-dimensional version of the governing equations, we have to scale each of the physical quantities using a proper scaling factor. The scaling factors are listed on the right-hand side. The length scale is the shell gap, which is the difference between the radius of the outer and inner boundaries. The temperature scale is the temperature difference across the outer core.

The compositional scale is the change in concentration from the inner core boundary to the core-mantle boundary. That is the concentration difference across the outer core. The time scale is $\frac{L^2}{\nu}$. Here, the time scale is the viscous diffusivity time scale. Choosing the viscous diffusion coefficient ν in place of κ for thermal convection is done to ensure an unbiased diffusion time scale for both temperature and concentration.

Thus, neither temperature diffusion nor concentration diffusion should be used in the time scale. It is best to use viscosity diffusion as the factor for the time scale. Thus, $\frac{L^2}{\nu}$ is the diffusion time scale. Thus, $\frac{L^2}{\nu}$ is the viscous diffusion time scale. Next, the velocity scale is defined as $\frac{\nu}{L}$, in line with the viscous diffusion.

Finally, the magnetic field is scaled by the factor $\sqrt{\rho\mu\omega\eta}$. η is the magnetic diffusivity. Such scaling of the dimensional equation gives rise to a few non-dimensional parameters, which are the control parameters for thermal-compositional convection. These non-dimensional control parameters are listed as follows.

First, we have the thermal Rayleigh number. This thermal Rayleigh number is the ratio of thermal buoyancy to viscous effects. It controls the magnitude and strength of thermal convection. Next is the compositional Rayleigh number, which is the ratio of compositional buoyancy forces to diffusion. This controls the strength of compositional buoyancy.

Next, we have the thermal Prandtl number. The thermal Prandtl number is the ratio of viscous diffusion to thermal diffusion. Similarly, we have the concentration Prandtl number, which is the ratio of viscous diffusion to the concentration diffusion coefficient. We also have the electrical Prandtl number, also known as the electromagnetic Prandtl number, which is the ratio of viscous diffusion to magnetic diffusion. Thus, we have three sets of Prandtl numbers, which are the ratio of viscous diffusion to the corresponding diffusivities, as appropriate.

We also have the Lewis number, given by Le. The Lewis number is the ratio of the concentration Prandtl number to the thermal Prandtl number. We also have the Ekman number, which is the ratio of viscous diffusion to rotational effects or the Coriolis force. The typical values of all these control dimensional parameters for the Earth are given below. The thermal relay number is 10^{22}, while the compositional relay number is 10^{8}.

The thermal relay number is 10^{22}, which is much higher than the compositional relay number, which is in slight excess of 10^{6}. The Lewis number is 10^{3}, which is much greater than 1. This indicates that the concentration Prandtl number is much greater than the thermal Prandtl number. With this, we can also derive This means that the thermal diffusivity is much greater than the concentration diffusivity.

This can be stated as follows. In other words, heat transfer occurs much faster through diffusion than through concentration. The control parameters which appear in the governing equations are as indicated here. These are the control parameters as they appear in the non-dimensional version of the equations. Now we look into some typical characteristics of thermal-compositional double-diffusive convective heat transfer.

To understand that, we take the ratio of the Rayleigh numbers given by δ. δ is defined as the ratio of the thermal to the sum of Rayleigh numbers. The heat transfer is depicted in terms of Nusselt's number, Nu. The Nusselt number is the ratio of convective heat transfer to conductive heat transfer. The Nusselt number depends on both the thermal and compositional Rayleigh numbers.

In the adjacent diagrams, we have the Nusselt number as a function of these Rayleigh numbers. On the y-axis, we have the compositional Nusselt number minus 1. On the x-axis, we have the compositional Rayleigh number multiplied by the Ekman number. For various parameter regimes. These parameter regimes are approximately $E = 10^{-5}$.

The Nusselt number minus one indicates a straight-line relation with the ratio of the Ekman number. This indicates that as the combination or the product of the Rayleigh number and the Ekman number increases, the Nusselt number also rises, leading to more efficient convective heat transfer. A similar effect occurs with the thermal Nusselt number, which also follows a nearly straight-line or linear dependence on the product of the Rayleigh and Ekman numbers. Thus, we can understand that the product of the Rayleigh and Ekman numbers is proportional to the Nusselt number. Thus, we can understand that the Nusselt number, or convective heat transfer, is linearly proportional to the product of the Rayleigh number and the Ekman number.

Next, let us consider a few geometrical representations of the temperature and compositional fields, as obtained from numerical solutions of the governing equations we have discussed earlier. On the left-hand side, we look at the parameter regime where the Lewis number equals 10 and δ equals 0.8. A Lewis number of 10 indicates that the compositional Prandtl number is 10 times the thermal Prandtl number. A δ value of 0.8 means that the contribution of the thermal Rayleigh number is 80% toward the total buoyancy. The temperature contours are shown here in a diagram.

The temperature contours and the compositional contours are shown here. Each of them depicts a periodic nature. The diagram depicts an equatorial section of the spherical shell. This means that for a spherical shell, A line in the north-south direction like this—the dotted line indicates the cut section represented here.

Looking from above, the temperature field gives a four-lobed structure similar to the compositional field, but the compositional field distribution structure is much different from the temperature field structure. This is because of the high diffusivity of temperature with respect to composition. The larger diffusion of temperature smears the temperature into a large-scale structure. While the low diffusion coefficient of composition retains plume-like structures, which are concentrated in localized regions. These concentrated regions do not diffuse and become large-scale, unlike the temperature.

This is evidence of the high diffusivity of temperature compared to composition. Similarly, we have the temperature and composition fields for an infinite Lewis number and $\delta = 0.8$. This means that the composition is nearly non-diffusive, leading to concentrated small-scale structures of composition. Finally, let us conclude this lecture. Double-diffusive convection plays a crucial role in Earth's outer core dynamics.

It influences the generation and sustainability of the geomagnetic field. This process arises due to thermal and concentration gradients, which are driven by various geophysical processes such as the solidification of the Earth's inner core. The combined result is thermocompositional convection, or double-diffusive convection. In today's Compositional convection is thought to be as important as thermal convection.

However, as the inner core becomes larger and larger, compositional convection is said to become more dominant than thermal convection due to the larger amount of light element release. This would help to maintain the long-term energy budget of the dynamo, enhancing the life of the magnetic field and protecting our planet from harmful solar radiation, thus prospering life on Earth. We can look into these further references for more details on double diffusive convection. Thank you.