Mathematical Geophysics

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Lecture – 31

Hello everyone, welcome to the SWAYAM NPTEL course on mathematical geophysics. We will be starting module number 7, which is thermofluidic processes in geophysics. This is the first lecture of this module: Laplace's and Poisson's equations. In this lecture, we are going to cover the mathematical aspects, such as Laplace's equation and Poisson's equation, related to thermofluidic processes in geophysics. The thermofluidic processes that depend on these equations will be considered in further lectures.

In this lecture, the four components are as follows: Laplace's equation and its solution, the extension to Poisson's equation, and various applications in geophysics. So, let us begin. First, we consider Laplace's equation. We have seen the Laplacian operator in previous lectures. This is the Laplacian operator.

We have the Laplace equation based on this operator acting upon a field and equaling 0 on the right-hand side. The Laplacian of any scalar function equaling 0 is the Laplace equation. This is a second-order partial differential equation. We have seen the occurrence of the Laplace equation in potential theory, such as gravity and electromagnetism. It describes the conditions for the potential—whether gravitational or magnetic—such that there are no net sources or sinks of the field.

Thus, the Laplace equation can be understood in a physical sense as representing the steady state or equilibrium condition where there are no net sources or sinks of the field. In other words, Laplace's equation governs systems where a conserved quantity, such as velocity potential or temperature, balances out over a region. This means that the Laplacian equals zero. Its solutions are known as harmonic functions. Now, we will look into the various forms of the Laplacian equation in different coordinate systems.

We had looked at a glimpse of such equations in very few previous lectures. In the very first few lectures. In Cartesian coordinates, the Laplacian equation is represented as:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

Now, these are constant coefficient partial differential equations. In spherical coordinate systems, which are more prevalent and appropriate for geophysical systems, the Laplace equation is given as:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}=0.$$

Now, these three terms represent the individual Laplacian components of the Laplace equation in radial, azimuthal, and meridional angular directions. Note the presence of variable coefficients in this partial differential equation. These variable coefficients also occur in the cylindrical coordinate system representation of the Laplace equation. Thus, in spherical and cylindrical coordinate systems, the Laplace equation is a variable coefficient PDE. Now, we will attempt the solution of the Laplace equation.

We have chosen the spherical coordinate system because the solution of Laplace's equation in spherical coordinate systems is very important for geophysical applications. The solution of Laplace's equation in a spherical coordinate system leads to the Legendre polynomial system and the spherical harmonics. Thus, it is very important to understand the solution of Laplace's equation in a spherical coordinate system for geophysical purposes. We have the Laplace equation in spherical coordinates, into which we introduce the variable separation form of F. F being a function of r, θ, and ϕ can be separated out as the product of capital R, capital T, and capital P functions.

These are individual functions of r only, θ only, and ϕ only coordinates. Now, substituting this variable separation form of the function F into Laplace's equation as given by equation 1, we obtain equation 3, where the appropriate substitutions have been made. Note the coefficients of these three terms as they appear in the Laplace equation components. The first term, which is the radial component, has T and P as the functions in the coefficient, which means R is absent.

Similarly, the θ component has the function capital T absent in its coefficient. Function P is absent in the coefficient for the ϕ term. Thus, we can introduce the corresponding functions by dividing the entire equation 3 by RTP. This gives equation number 4. We have also multiplied $r^2\sin^2\theta$ such that there are no r terms in the denominator; this factor is r.

In the denominator, $r^2\sin^2\theta$ is removed in equation 4 by multiplying $r^2\sin^2\theta$. The factor such as $\sin\theta$ is also removed from the denominator of equation 4 by multiplying $r^2\sin^2\theta$. Now, the purpose of multiplying this and removing r and $\sin\theta$ from the denominator is to avoid any singularity. Singularity can occur in equation 3 for $r=0$ and $\theta=0$ coordinates, which is the center of the sphere. To remove the singularity, we have the regular equation 4 instead of the singular equation 3.

We can simplify equation 4 to equation 5 in a manner such that the left-hand side contains r and θ, and on the right-hand side, we have only a function of ϕ. Now, since the left-hand side is a function of r and θ and the right-hand side is a function of ϕ only, this means that both the left-hand side and right-hand side are constant. That constant is assumed to be m^2. Now, we move on to obtain the ϕ component solution for the Laplace equation.

We consider the ϕ part:

$$\frac{\partial^2 P}{\partial\phi^2}+m^2P=0.$$

This is an ordinary differential equation whose solution we have discussed in previous lectures. Assuming P as an exponential form given by $Ae^{p\phi}$ and substituting into equation 7, we can obtain the ϕ solution as:

$$P = a\cos m\phi + b\sin m\phi.$$

These are sinusoidal functions. Equation eight represents the ϕ solution of the Laplace equation. Now we move on to find the r and θ solutions of the Laplace equation. First, let us consider $m = 0$. If $m \to 0$, we have this equation. We can separate out the two terms as equation number 9. The left-hand side depends only on r, while the right-hand side depends only on θ. Now, this variable separation form can be treated such that each individual left-hand side or right-hand side becomes equal to some constant. Now, this constant is $L(L+1)$, where L is a constant quantity.

We will soon understand the purpose of choosing such a form for the right-hand side constant factor. Now, let us look into the R solution. The R solution can be proceeded as follows:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = L(L+1).$$

The capital R is a function of r only. This gives:

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - L(L+1)R = 0.$$

Now, this is an ordinary differential equation which we can solve by assuming the form of the solution as Cr^{α}. Here, we have a polynomial expansion such that the functions of R are polynomials. Substituting it into equation number 12, we get the algebraic equation number 13:

$$(\alpha - L)(\alpha + L + 1) = 0.$$

This gives $\alpha = L$ or $\alpha = -L - 1$. One can try to obtain the intermediate steps between 12 and 13 for practice. Just by substituting Cr^{α}, it can be easily done. Having the two solutions L and $-L - 1$ gives two solutions for R.

Note that the combination of these two solutions gives the general solution for R:

$$R(r) = Cr^{L} + Dr^{-L-1}.$$

We can understand these two components as the rising component and the falling component because r^{L} increases in its magnitude with increase in r, while r^{-L-1} falls or diminishes in its magnitude as r increases. The rising part is the first and the falling part is the second component.

These are the two ways in which the radial variation of the Laplacian equation can be understood. Now we come to the θ solution. Similar to the R solution, we proceed with the constant $-L(L+1)$. It gives the following equation number 14. Now, consider $x = \cos\theta$.

We are substituting x in place of $\cos\theta$ because we want to represent the solution in the form of polynomials. Thus, our aim is to convert the trigonometric form to polynomial form. Now, substituting x as $\cos\theta$ and $\sin\theta$ appropriately given by $\sqrt{1-x^2}$, we have equation number 14 given by:

$$\frac{1}{\sqrt{1-x^2}}\left[-\sqrt{1-x^2}\frac{d}{dx}\left((1-x^2)\frac{dT}{dx}\right)\right]+L(L+1)T=0.$$

This can be simplified into equation number 17 and finally equation number 18:

$$(1-x^2)\frac{d^2T}{dx^2}-2x\frac{dT}{dx}+L(L+1)T=0.$$

Equation number 18 is the celebrated Legendre equation. And the solution for T is the celebrated Legendre polynomial series. The Legendre polynomial series are given by P_L. This $P_L(\cos\theta)$ or $P_L(x)$ is the Legendre polynomial. Now we can see that Laplace's equation can lead to Legendre polynomials, with the subscript L denoting the degree of the polynomial.

This can be related to the spherical harmonics, where the Legendre polynomial is used in θ, and in ϕ we have the sinusoidal components. Thus, these functions—sinusoidal in ϕ and Legendre in θ—make up the spherical harmonics. Thus, the spherical harmonics are nothing but the combination of θ and ϕ solutions of Laplace's equation in a spherical coordinate system. Thus, we arrive at the general solution, which is the product of R, T, and P. Therefore, the function f can be written as the sum of all the linear combinations of the r part, the ϕ solution, and the θ solution.

This is the most general solution of Laplace's equation in a spherical coordinate system. We now examine Poisson's equation, which is the inhomogeneous form of Laplace's equation. We have:

$$\nabla^2 F=S,$$

where S is a non-zero quantity. The need for Poisson's equation arises when we have fields or systems where sources are present. To account for the source, the introduction of the term S on the right-hand side is necessary.

Thus, the term S on the right-hand side of Poisson's equation accounts for the presence of source or sink terms in systems that otherwise consist of conservative fields. To clearly differentiate between Laplace's equation and Poisson's equation, we can say that Laplace's equation has no sources or sinks, whereas Poisson's equation includes a non-zero source or sink term. It represents the generation or absorption of various field quantities. Now, we examine the application of Laplace's and Poisson's equations in geophysics. These equations are widely used to understand various phenomena in Earth's subsurface, atmosphere, magnetic field, and gravitational fields.

In magnetic and gravitational fields, which are conservative fields, there may be the presence of anomalous sources of gravity and magnetic field in the form of high density material or magnetic field generators. The presence of such sources would require the use of Poisson's equation while the understanding of these fields without any sources will require the use of Laplace's equation. Specifically, gravitational field modeling requires the Poisson's equation to be used for calculating the potential in regions where the mass density acts as a source. Laplace's equation is used to model gravitational field in free space where the source is upset. Such regions are above the Earth's surface.

While below the Earth's surface, Poisson's equation is more appropriate for gravitational field modeling. Next, we have the magnetic field modeling. Similar to the gravitational field modeling, Poisson's equation is appropriate for understanding subsurface magnetization. Below the surface of the earth, high electrical conductivity bodies may exist which can act as source or sinks of magnetic field or electric current when geophysical methods based on induced polarization or resistivity methods are used. Above the earth's surface, the magnetic field is devoid of any magnetic sources and in such regions Laplace's equation can be used.

Note that in regions such as ionospheric regions where magnetic or electric currents can exist, the Poisson's equation may be more appropriate. Thus, we can see that the combination and interplay of various systems in geophysical applications would require the appropriate choice of Laplace's equation or Poisson's equation to model the phenomena. Thus, we come to the conclusion for this lecture. The Laplace's equation describes steady state or equilibrium conditions without sources and sinks. It governs phenomena which involve conservative fields such as gravitational potential, electric potential and temperature in source-free regions.

The solutions of Laplace's equation are very important for geophysical modeling. These solutions are called harmonic functions, and the solutions for the θ component result in Legendre polynomials, while the solution in the ϕ component results in Fourier components or sinusoidal functions. Overall, these two combined give the spherical harmonic functions, which are very appropriate for modeling surface-related functions on a sphere. The Laplace equation is more appropriate for modeling fields in free space, voids, or equilibrium systems. On the other hand, Poisson's equation, which is the inhomogeneous form of Laplace's equation, can accommodate the existence of source or sink terms.

It extends the applicability of the Laplace equation but it comes at a more challenging cost, as the solution of Poisson's equation is more intricate than that of the Laplace equation. This is because the solution of Poisson's equation depends on the nature of the source term. Thus, the solution of Poisson's equation varies from case to case. Overall, it can be said that Poisson's equation describes how physical quantities are influenced by localized disturbances or distributed sources.

One can refer to the following references for more details on Laplace's equation and Poisson's equation. Various applications related to geophysical phenomena are also discussed in these references. Thank you.