Mathematical Geophysics

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Week - 06

Lecture – 30

Hello everyone. Welcome to the Swayam NPTEL course on Mathematical Geophysics. This is a continuation of module number 6, Wave Dynamics in Geophysics. This is lecture number 5 of this module, Internal Gravity Waves. In this lecture, the concepts covered are related to internal gravity waves.

First, we will look into the concept of internal gravity waves. Then, we will derive the internal gravity wave equation. Then, we will look at the dispersion relation, which governs the characteristics of the propagation of internal gravity waves. Finally, we will look into various geophysical applications that utilize internal gravity wave phenomena. So, let us begin.

First, what is an internal gravity wave? Internal gravity waves are oscillations that occur within a fluid medium. This fluid medium can be either the ocean, the atmosphere, or the liquid iron of the Earth's outer core. In internal gravity waves, the restoring force is the force of gravity. Just like we had The Coriolis force acts as a restoring force for inertial waves. Now, these are dependent upon gravity as the restoring force; thus, they are called internal gravity waves. Internal because these waves occur within the fluid rather than on the surface of the fluid. The surface waves are different from the internal gravity waves.

Now, two forces are important in internal gravity waves. First is the buoyancy force, which comes from gravity and stratification, which can be thermal or chemical composition. Now, these buoyancy forces interact with inertial forces of the fluid and generate the internal gravity wave. The inertial forces arise due to the acceleration of the fluid and its mass, while buoyancy forces are generated due to stratification. Thus, stratified fluids are a necessary ingredient for internal gravity waves.

Now, we will understand what is meant by stratification. Stratification means the density varies with depth, such as thermoclines in oceans. In oceans, at various depths, the density of the water is different. Similarly, in the atmosphere, the density of the air is different at different heights. Like in the troposphere of the atmosphere, in the first 10 kilometers from the surface of the Earth, the density of the air changes. Such change in the density of the medium which is fluid leads to stratification and this is useful to generate buoyancy forces which governs the production of internal gravity wave. Now look at this diagram here. This diagram shows the formation of internal waves at the interface of two density layers.

These are the two density layers. Below we have high density layer and above there is a low density layer. As we know low density layer lying on top of high density layer forms a stable interface which means that these immiscible fluids will not mix and there will be no upside down motion. Thus the high density layer will be always below the low density layer. Now if we perturb the interface of these layers it will generate a wave. These waves are a type of internal gravity waves which are occurring within the fluid domain.

These are not the same as the surface waves which are occurring at the top of the liquid. In this adjacent diagram, we can see the configuration of the system is such that we have one cold boundary at the bottom and hot boundary at the top. On both sides we have insulating boundaries. Now gravity is pointing downwards in this diagram and the setting is such that cold boundary at the bottom would lead to high density of the medium.

The density is denoted by ρ. Now, due to the low density being above the high-density material, this system is a stable system. This can give rise to internal gravity waves within the fluid if perturbed. So, we can see that if there is a perturbation such as this one, where we have alternating layers of fluid motion such that each layer is of a different temperature, with alternating hot and cold fluids moving in opposite directions. This leads to an internal gravity wave.

One can also see that the phase velocity is perpendicular to the group velocity. The phase velocity is the direction where the crests and troughs of the wave lie. Now, we can see that the crests, which are hot fluids, and the troughs, which are cold fluids, are aligned in the direction of fluid velocity. This is the direction where crest and trough follow one another. However, the fluid motion is perpendicular.

The fluid transfers energy and hence information across in the perpendicular direction to the phase velocity. Thus, the group velocity is in the direction of the propagation of information. These are a few typical illustrations of internal gravity waves. Now, we will look into the rigorous mathematical details of internal gravity waves, their generation, and the dispersion relation, which gives their characteristics. The internal gravity wave equation.

Now, to obtain the equation which governs the generation and propagation of internal gravity waves, we have to consider Newton's Reynolds decomposition equation. The Reynolds decomposition is given as follows. Here, P is pressure, ρ is density, $\mathbf{u}$ is the velocity field, and T is the temperature. The star denotes a steady, time-independent factor. Thus, any quantity such as pressure or temperature can be decomposed into its background state, which is steady, and the perturbation state, which is unsteady.

The background state is denoted by a star. And the perturbation state is denoted by a prime. Thus, we can say that the background state is a function of space only, while the perturbation state P' is a function of both space and time. Let us consider the x-z plane in a Cartesian coordinate system. In the x-z plane, which is also equivalent to y equals a constant, we have a continuously stratified and unbounded fluid.

Let's say we are looking at the $y = 0$ plane in the x and z directions, as shown here. The fluid is unbounded, which means there are no boundaries. Then, we can look at the linearized diffusionless perturbation state governing equations. These equations can be obtained from Newton's laws of motion for fluids. We have looked into such equations previously in the formulation of inertial wave dynamics.

We will also look into the heat transfer equation. These are the three equations. The first equation: $\nabla \cdot \mathbf{U} = 0$. If the divergence of any quantity equals zero, it means there is no input or output from the region under consideration, which implies the conservation of mass. Note that these equations are written per unit mass. Hence, mass does not appear on both sides of this equation. The second equation is Newton's law of motion, where if we multiply m on both sides, we will have mass times acceleration equals net force. If we remove mass from both sides, we have acceleration equals force per unit mass. Thus, the left-hand side is acceleration, which is given by $\frac{\partial \mathbf{u}}{\partial t}$, equals the right-hand side, which is force per unit mass, with the first term given by the pressure gradient. The second term is the buoyancy force.

The pressure gradient forces can apply mechanical forcing for the fluid motion, while buoyancy forcing can induce convective flows in the fluid. Now, the third equation is the heat transfer equation. The first term is the evolution of temperature. The evolution of temperature with time is affected by the advection of temperature by the flow field. The advection of temperature is just the motion of a fluid parcel, which has a particular temperature, along the flow field.

For example, if we have a particle like this of a certain temperature and the velocity field given by these arrows, then this particle will be moving in a path as shown by the dotted line. Now, as this parcel moves along this path, the temperature at various other regions will be changing because of the influx and outflux of temperature. Thus, the temperature at any point, such as this cross mark, will be enhanced or diminished based on the passage of this fluid parcel, which may have a certain high temperature. So, as this parcel crosses the crossing point, the temperature will rise and then it may fall.

That would be detected by the evolution of the temperature term here. Thus, we are tracking the evolution of temperature and the velocity field through this set of equations. So, we are tracking the evolution of $\mathbf{U}$ and T using these three sets of equations. Now, in the stratified region, the density is assumed to vary only in the vertical direction. Thus, the temperature variation can be considered only in the vertical direction.

Therefore, we have the thermally stable region defined as $\frac{\partial T}{\partial z} > 0$ and the thermally unstable region defined as $\frac{\partial T}{\partial z} < 0$. This can be understood as, let's say, we have hot and cold regions like this with gravity pointing downwards. Now Hot regions will induce lower density, while cold regions will induce higher density of the surrounding medium. Due to the condition that the low density lies below the high density, the consequence can be the high-density material taking the place of the low-density material. This overturning leads to instability. This unstable configuration can now be formulated as follows: considering this as the z-axis, and T_1 and T_2 as the temperatures of the cold and hot regions respectively, we have Now, z_1 and z_2 also represent the heights of the cold and hot planes.

We can see that as z points upwards, z_1 is higher than z_2. While T_1 is lower than T_2 because T_1 is the colder temperature. Thus, the numerator is negative while the denominator is positive. This gives $\frac{\partial T}{\partial z}$ as negative, which is an unstable configuration. The opposite configuration can be obtained by reversing the hot and cold planes.

This ensures that ΔT becomes positive and $\frac{\partial T}{\partial z}$ becomes positive. This is a stable configuration because the density now becomes like this: the hot region will induce low density, and the cold regions will induce high density. This is a stable configuration because low density will not have the tendency to overturn high density. Thus, this overturning is absent, making it a stable configuration. This is given by the thermally stable region. $\frac{\partial T}{\partial z} > 0$. Now, we look into the second term of the previous equation here. $\mathbf{u} \cdot \nabla T$ can be written as the dot product between the u_z and $\frac{dT}{dz}$ as there is no temperature variation along x and y axis. since the temperature varies only along the axial direction that is z. Thus, we have the heat transfer equation simplified to $\frac{\partial T'}{\partial t} = -u_z' \frac{\partial T^*}{\partial z}$. Note that only T' exists here as $\frac{\partial T^*}{\partial t} = 0$ because T^* is independent of time. On the right hand side we have u_z' multiplied by $\frac{\partial T^*}{\partial z}$ with a minus sign. Now for the equation number 2 as shown here we are going to operate two operations that are given by these operators.

The first operator is $\hat{z} \cdot \nabla \times$. The second operator is the axial component of double curl of equation 2. We have seen in previous lectures how to evaluate the curl. After evaluating the curl on this vector equation number 2, on both sides, we've obtained this equation where ξ is $\nabla \times \mathbf{u}$. which is the vorticity and taking the z component, we have ξ_z. This is the z component of the vorticity with prime as the perturbation quantity. The pressure gradient term vanishes due to curl because $\nabla \times \nabla \equiv 0$.

Thus, the third term $\nabla \times (\alpha g T' \hat{z})$ also does not have any z component. Since this equation is a z component equation, we do not have any component of the buoyancy term. Due to the operation of curl, the buoyancy term vanishes along the z-axis. Note that. Do not confuse between the buoyancy term along z and curl of buoyancy term along z. The buoyancy term exists along the z-axis.

But the curl of the buoyancy term does not exist along the z-axis. Hence, equation number 5 has 0 on the right-hand side. Now, the second equation. This equation is obtained after taking the double curl of the heat transfer equation. The double curl of velocity gives the Laplacian of u_z, and we have the z-component of the double curl of the buoyancy term given by $\alpha g \nabla_h^2 T$.

The horizontal Laplacian is given by $\frac{\partial^2}{\partial x^2}$, while the total Laplacian is $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Now, operating $\frac{\partial}{\partial t}$ on equation number 6 gives us the second derivative of Laplacian with respect to time equals $\alpha g \nabla_h^2 \frac{\partial T'}{\partial t}$. This we have obtained so that we can eliminate $\frac{\partial T'}{\partial t}$ from equation 7 using equation 5. This we have performed because our aim is to eliminate $\frac{\partial T'}{\partial t}$ using previous equations. Thus, we substitute equation number 4 to eliminate $\frac{\partial T'}{\partial t}$. This gives $\frac{\partial^2}{\partial t^2} \nabla^2 u_z = -\alpha g \frac{\partial T'}{\partial z} \nabla_h^2 u_z'$.

This is coming from this equation where we have substituted $\frac{\partial T'}{\partial t}$ here. Now, if we perform further algebraic derivations, we obtain equation 9 substituted into equation 8 gives. Now, substituting equation 9 into 8, we have $-N^2 \nabla_h^2 u_z'$. where N^2 denotes the Brunt-Vaisala frequency. This can be written in operatorial form as The operator $\mathcal{L}$ is $\frac{\partial^2}{\partial t^2} \nabla^2 + N^2 \nabla_h^2$ of $u_z' = 0$. Note that this is a wave equation. It has the components of a wave equation, such as the time derivative and the spatial derivatives as two terms for the velocity. This is the final form of the internal gravity wave equation. Now, the Brunt-Väisälä frequency, which we introduced in equation 10, is defined as $\sqrt{\alpha g \frac{\partial T'}{\partial z}}$.

We can understand that the Brunt-Väisälä frequency depends on the thermal stratification $\frac{\partial T'}{\partial z}$. Now, coming to the dispersion relation. The dispersion relation gives the properties of internal gravity waves that can be determined using normal mode analysis, similar to inertial waves. This is the normal mode expansion for the velocity perturbation u_z'. We have $u_z' = u_0 e^{i(kx+mz-\omega t)}$. Here, k and m are wavenumbers along the x and z directions, respectively, while ω is the frequency. Having the normal mode analysis in space and time as waves, we substitute the u_z' expression obtained in equation 3 into equation 12, which is the internal gravity wave equation.

Now, we have the simplified form as $\omega^2 = \frac{N^2 k_h^2}{k^2}$. The k_h^2 is the horizontal wavenumber vector, which is nothing but k^2. This is because the horizontal Laplacian is only in the x direction, that is $\frac{\partial^2}{\partial x^2}$. This gives $k_h^2 = k^2$, as k is the wavenumber along x. The total wavenumber vector k^2 is $k^2 + m^2$.

Thus, we have the frequency-wave number relation given by this. This means that for waves having different wave numbers, the frequency of oscillation is different for internal gravity waves. Now, we come to the various applications of internal gravity waves in geophysics. First, we consider the atmospheric applications. Internal gravity waves transport energy and momentum in vertical and horizontal directions in the atmosphere.

They affect atmospheric dynamics and weather patterns. As the atmosphere is layered, with higher density below and lower density above, it forms a stably stratified system. In this stably stratified system, internal gravity waves occur between the troposphere and stratosphere. The breaking of internal gravity waves can lead to the generation of turbulence, which contributes to mixing and affects weather patterns. Similarly, in the ocean, where stratification of the water exists with high-density water below low-density water, internal gravity waves can occur in the ocean's interior.

The interaction of internal gravity waves enhances the mixing of nutrients and gases in the ocean and leads to biological productivity and ecosystems. These waves are also useful in providing information about thermocline structures and vertical heat transport in oceans. Finally, in the Earth's core, where the fluid is liquid iron, internal gravity waves are generated. These internal gravity waves influence the geomagnetic field dynamics.

The geomagnetic field fluctuations, which can be observed as secular variations on the surface, are somewhat affected by internal gravity waves. Internal gravity waves also affect the propagation of P-waves, which propagate through the Earth's outer core fluid. In conclusion, we can say that internal gravity waves are a fundamental type of wave, very prevalent in various natural phenomena related to geophysical applications. They are very important for Earth's dynamic processes, which are time-dependent and interconnected. It is important to note that internal gravity waves are periodic oscillations within stratified fluids in the interior.

For internal gravity waves, the necessary ingredients are gravity, stratification, and a volume of fluid. The restoring force is the buoyancy force. Finally, internal gravity waves play a crucial role in various geophysical processes by influencing energy transfer, mixing, turbulence, and large-scale dynamics within geophysical fluid applications. One can look into the following references for more information and applications of internal gravity waves. Thank you.