Mathematical Geophysics
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Week - 01
Lecture - 03

Hello everyone, welcome to the Swayam NPTEL course on Mathematical Geophysics. This is Module 1: Basic Concepts, Lecture Number 3: Solid Angle. The concepts covered in this lecture are angles and their properties, and how these angles are used in various coordinate systems, as we have seen in previous lectures. Recall that in previous lectures, we have looked into four types of coordinate systems. The first one is the Cartesian coordinate system.

Then we looked into the polar coordinate system, the spherical coordinate system, and finally the cylindrical coordinate system. In this lecture, we are going to use the solid angle concept in various coordinate systems. First, let us look into the basic concepts of an angle. What is an angle? An angle is a quantity that measures the divergence between two intersecting lines or surfaces.

The angle is typically described in terms of rotation from one line to another or from one surface to another. It can exist in both two-dimensional and three-dimensional surfaces. The angles are of two types. One is the plane angle, and the second one is the solid angle. This is compared in the following table.

On the left-hand side, we can see the properties of the plane angle. The plane angle has a vertex O. And the lines marked as A and B. The arms OA and OB denote the lines which intersect at O, making an angle denoted by the angle between A, O, B. The plane angle is a 2D angle. It is analogous to triangles, squares, and circle angles.

It is analogous to 2D features, geometrical structures such as triangles, squares, and circles. The unit of a plane angle is radians or degrees. Now coming to the solid angle, which is the focus of this lecture on the right-hand side. Now, a solid angle has an origin or intersecting point marked by O. As you can see, the solid angle is a three-dimensional structure which is analogous to various other geometrical features or structures such as the cone, cube, or sphere. Its unit is steradians, marked here. You can see the solid angle is denoted by a circular line. We will discuss more detailed aspects of the solid angles further. Now, here are some schematic diagrams which represent this concept of solid angle. A solid angle has four major components.

First is the closed contour. A closed contour is denoted by L. It can be any closed line, such as this one or this one. Here, in the first figure, the closed line L looks like an ellipse. Next, we have the closed contour surface, denoted by S, and it is shown by the grey area. This surface is bounded by the closed contour L. It is like a membrane stretched on a rigid surface. The third component is the apex or the point of observation, denoted by P. So, P is similar to the vertex of a plane angle. In a solid angle, it is called an apex. Sometimes, it is also called the point of observation because it appears like a person is sitting at point P and looking outward, and the coverage of their vision is the solid angle. Finally, we have the lines which join the apex point P to each of the points on the closed contour L. These lines, as well as the other dotted lines, form part of that conic surface, denoted by S_C. Now, it is not necessary that a solid angle is only subtended by a conic surface. It can be a triangular conic, or it can be a rectangular conic. The first picture on the left is the most common representation of a conic, which is a circular conic. All these conic surfaces subtend a solid angle at the apex point P. We also have the surfaces which look like a prism in the case of a triangle and rectangular conic, respectively. Next, we will consider how to measure a solid angle. The primary idea behind the measurement of a solid angle is the quantification of the internal part of the space confined by the conic section. In other words, we are looking at the space which is covered or included inside this conic section. Now, consider this figure on the right. We have the closed contour represented by L. We have the apex point P. So, every cone can be divided into two parts. First, the internal part, which is denoted by D_I, and the external part, which is denoted by D_E. Now, D_I and D_E represent the volume which is included and excluded by the conic section. Now, let us consider drawing a spherical surface of radius R with apex P as the center of the sphere. Now, this is the sphere which is required here. The area of the sphere inside the cone is denoted by S_I. And the area of the sphere which is outside the conic section is denoted by S_E.

Now, to measure the solid angle, we need to have a quantification of these volumes inside and outside the conic surface. That is D_I. And D_E. But as we can understand, D_I and D_E amount to an infinite volume if the conic section is extended further to infinity. So we look at a surface on the sphere, which we have drawn just now, to quantify this solid angle.

This surface is denoted by S_C. Now, S_C is the surface on the sphere which is included inside the conic surface. So the internal part of the cone, D_I, can be quantified by the ratio ω, which is subtended from the apex point P. To measure the internal part of the cone, that is D_I, we can consider the ratio of the area to the square of the radius. The area is S_I, and the radius is R. This ratio is called ω_P.

The function ω_P is called the solid angle, and this is the characteristic of the cone. ω_P can also be understood as the visual angle under which the surface S_I is seen from the point P. Now you can see that this surface S_I is dependent upon the radius of the sphere. Suppose we have larger and larger concentric spheres; the area becomes larger and that is proportional to the square of the radius. Hence, the solid angle, as defined here, becomes independent of the radius due to the division by R^2. So the solid angle becomes a constant quantity as long as The cone is fixed. Now we will look at some special cases of the conic section and the subtended solid angle. For example, $S_I=0$ indicates the conic section when the solid angle goes to zero. So this indicates when the conic section is reduced to a strip that can also be expressed in terms of $D_I=0$. This also means that the conic section is squeezed onto a line, and hence the solid angle goes to zero. The second case is when the solid angle encompasses the whole space, that is, D_I becomes the whole space. So if we diverge the conic section to cover the whole space, then D_I encompasses the whole space, and hence the surface area of the sphere which is included inside the conic section becomes $4\pi R^2$. Thus, the solid angle as defined here becomes

$\frac{4\pi R^2}{R^2}$, that is, 4π. So from this, we can summarize that the range of the solid angle is from 0 to 4π. This is because the surface area which is getting included inside the cone varies from 0 to $4\pi R^2$.

We can also consider the case when the conic surface is expanded to form a plane. So this is an example when a conic section tends to become an infinite plane. So it becomes similar to a disc. So the area of the sphere which is included inside the conic surface is the hemisphere, that is equivalent to an area of $2\pi R^2$. This results in the conic surface area of 2π.

Also, if the conic surface includes a quarter or quadrant of the space, then the area included is πR^2. This results in the solid angle of π, which is derived using the previous formula. So, from here, we have learned how to determine the solid angle from the space that is confined by the conic section. Next, let us consider the part of the spherical surface which is seen from the apex and use it to determine the conic surface and the solid angle.

Consider this figure, which has the apex point P and the distance from P to the surface Q is L_{QP}. This is denoted by a vector which is pointing from P towards the center of this infinitesimal surface, which has an area of *dS*. The *dS* is part of the spherical surface *S* shown by the shaded region.

So, *S* is part of the spherical surface whose radius is L_P, which is also the distance between the point *Q* and the apex point *P*. Now, the projection can be defined as dS^*, which is *dS* multiplied by the cosine of the angle between the vector $\vec{L}_{PQ}$ and the local perpendicular $n^\wedge$. So, $n^\wedge$ is the perpendicular unit vector to the infinitesimal area *dS*. Now, the elemental solid angle can be defined as the ratio of this infinitesimal area *dS* to the square of the distance from *P* to *Q*. Now, the infinitesimal area *dS* is given by this, as we have seen earlier. This can be simplified to the ratio of the dot product between *dS* and $\vec{L}_{PQ}$, which is also the scalar product of the two vectors divided by the cube of the distance between them.

Now, this is very important to understand that the solid angle is nothing but the projection of the infinitesimal area *dS* dotted with the distance between the apex point and the point where the *dS* is being measured from. So, the total solid angle subtended by the surface *S* can be obtained by an integration of this infinitesimal solid angle $d\Omega$ over the entire surface area, that is *S*. So, we have the surface integral *S* over which the infinitesimal solid angle is being integrated. So, this is the expression for the solid angle in terms of the area which is included inside the conic surface at a distance L_{PQ} from the apex point. Based on the previous result, we can look into some special features of the solid angle. Now, consider the case when the surface *S* is a sphere. The special feature of the sphere is that the $\vec{L}_{PQ}$ vector will be perpendicular to the surface of the sphere, which is also coinciding with the local normal $n^\wedge$, which means that this factor becomes equal to 1.

Therefore, $dS \cdot \vec{L}_{PQ}$ is simply the multiplication of the magnitudes of dS and $\vec{L}_{PQ}$. We can use this to evaluate the solid angle as this. $\vec{L}_{PQ}$, being independent of the surface *S*, is taken outside and the total surface area is *S*. This gives the total solid angle to be $\Omega_P = \frac{S}{L_{PQ}^2}$. S divided by $L^2{}_{PQ}$.

Now consider the second case, that is, suppose *S* is an arbitrary closed surface. For simplicity, we consider this surface as a sphere, or you can also consider it as any other surface which is closed. For example, this complicated shape. Now, as long as *P* is located inside this volume, which is included within this closed surface, the solid angle which is subtended at the point P is equal to 4π.

If the local perpendicular vector becomes inward, then the sign of the solid angle becomes minus. Now, we will also consider the case when the point P is outside the volume, which is included inside the closed surface. So, based on the location of P, whether it is inside V or outside V, we can have the solid angle as plus 4π or zero. The solid angle is plus 4π when P is located inside V because the conic section is completely encompassing all the space.

When P is outside V, the conic section reduces to a strip. So, the conic section becomes a straight line, and hence the angle subtended is equal to zero. The final example which we are considering here is that suppose S is an infinitely extended plane. Then the conic surface subtends an angle of 2π when Z is negative and minus 2π when Z is positive.

So, these, the Z negative means the local normal points outward, and Z positive means the local normal points inward. In geophysics, solid angle is very widely used and is of paramount importance. The major areas in which the solid angle is used are discussed here. For example, the gravitational field of the Earth. In order to calculate the gravitational attraction, which uses forward modeling of irregularly shaped bodies, solid angle is used.

These irregularly shaped bodies range from mountains, mineral deposits, or other topographical features. And solid angle is used to quantify the area or the volume subtended by these features at the desired point where the gravitational field is to be calculated. Similarly, the geomagnetic field. In the geomagnetic field, the contribution of distributed magnetic rocks to crystal magnetization is quantified using solid angle. Also, spherical harmonic expansions have been used to demonstrate the global and regional variations of the geomagnetic field, which are based on solid angle.

Another widely discussed concept in geophysics is that of seismological applications. In seismological applications, the energy is released when an earthquake occurs, and that energy spreads from the focus or the hypocenter of the earthquake. over the entire volume of the Earth. Now, considering that hypocenter to be the apex point, the propagation of these waves occurs in a manner similar to the solid angle. The intensity of the seismic waves becomes inversely proportional to the square of the distance, similar to the solid angle, and hence is appropriately studied using various properties of the solid angle.

Also, in other geophysical applications such as terrestrial radiation, which involves energy flux, solid angle is used to calculate the radiative flux and the involved energy transfer in applications such as thermal radiation from the Earth's surface or solar radiation, which is incident on the Earth's surface, and also the measuring of radioactive emissions from various geological sources using this solid angle. So, in this lecture, we conclude that the solid angle is a three-dimensional counterpart of the plane angle. In geophysical applications, which are predominantly based on solid spherical systems, the use of solid angles is of paramount importance. And understanding the solid angle theory helps in modeling and interpreting various geophysical aspects, which gives a better and improved understanding and characterization of the features which are inside as well as outside the Earth's surface. Overall, it can be said that the solid angle theory provides a framework for geometrically and theoretically describing the behavior of radiation and various fields which are in three-dimensional space, with significant implications for geophysical exploration and research.

Further details of the themes in geophysics will be discussed in upcoming lectures. These are a few references which you can use to get more ideas of the topics which are discussed in this lecture. Thank you.