Mathematical Geophysics

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Lecture – 29

Hello everyone, welcome to the SWAYAM NPTEL course on Mathematical Geophysics. We continue with module number 6, Wave Dynamics in Geophysics. This is lecture number 4, Inertial Waves. In this lecture, we will cover the following concepts. The concept of inertial waves in general, followed by the inertial wave equation.

We will look into the dispersion relation of the inertial wave next. Next, we will consider the phase and group velocity of inertial waves. Finally, we will look into various geophysical applications of inertial waves. So, let us begin. The concept of inertial waves.

The inertial wave is a type of oscillatory motion that occurs in rotating fluids. Rotation is an important and necessary criterion for inertial waves. The rotational motion generates the restoring force for the inertial wave. As we know, In rotational motion, the Coriolis force is a pseudo force that is generated when the observer is in a rotating frame of reference.

The Coriolis force acts as a restoring force for inertial waves. So, this is a pseudo force—the Coriolis force—acting as the restoring force for inertial waves. Inertial waves occur in fluids. Now, we know that fluids don't sustain shear. But rotation adds a restoring force and a certain amount of rigidity to the fluids.

This occurs because rotational dynamics enhance the vertical independence of the fluids. Vertical invariance means that any event occurring at various depths of a rotating fluid remains the same along the entire vertical axis. That is the axis of rotation. So, in rotating fluids, the most fundamental effect is that the velocity and accelerations—or these dynamics of the fluid—remain invariant along the axis of rotation. Now, this generates inertial waves, as inertial waves are triggered due to disturbances to this condition of no change along the rotation axis.

Inertial waves depend on the rotational rate of the fluid and their ability to propagate in directions determined by the fluid's angular velocity. Now, inertial waves also play a crucial role in the dynamics of geophysical and astrophysical systems alike, such as oceans, atmospheres, planetary interiors, and stellar interiors. We will look at a simple sketch that demonstrates one of the inertial wave types. This is an example from COUNF 2010 in Ocean Dynamics. Now, consider the equator of the Earth, where the motions are such that above the equator in the northern hemisphere, the Coriolis force tends to move the fluid in a clockwise manner.

This is occurring in the northern hemisphere for any fluid motion. Similarly, any fluid motion in the southern hemisphere experiences anticlockwise motion. This occurs because any fluid motion moving in the northern hemisphere is deflected to the right. Any fluid parcel moving in the southern hemisphere is deflected toward the left. This gives rise to the clockwise or anticlockwise circular motions of the fluid in the northern and southern hemispheres.

Now, this is occurring due to the Coriolis force, which is denoted by F. The Coriolis force is positive in the northern hemisphere and negative in the southern hemisphere. Now, why is this so? Suppose we look at the cross-section of the Earth. This is the equator.

These are the northern and southern poles. Let us consider two points above and below the equator. Now, note that the rotational direction is given by the $\boldsymbol{\omega}$ vector, which makes the Earth rotate from west to east. Now, at this location, we have the radial direction given by the rotation vector at these points.

For example, this is the $\boldsymbol{\omega}$ vector. The component of the rotation vector along the radial direction is this component. Now, this component points outward; hence, the Coriolis parameter F is positive. On the other hand, any point in the southern hemisphere will have the Coriolis parameter pointed inward because that is the projection of the $\boldsymbol{\omega}$ vector onto the radial line. Now, this inward-pointing Coriolis force makes F negative.

Now, this change in the sign of the Coriolis force causes deflection toward the right in the northern hemisphere and, oppositely, toward the left in the southern hemisphere. This, in turn, gives rise to clockwise fluid motion in the northern hemisphere and anticlockwise fluid motion in the southern hemisphere. Now, consider the fluid motion near the equator. Suppose a fluid particle is near the equator. At this location, let's say, just above the equator, it experiences clockwise motion.

It would turn like this, in line with the clockwise motion. Now, due to this, The particle motion is such that it crosses the equator from the northern to the southern hemisphere. Upon entering the southern hemisphere, it experiences anticlockwise motion. Now, this keeps repeating.

This gives an oscillatory motion of the fluid parcel. This is an example of an inertial wave, which is caused by the Coriolis force. One can notice that there is a displacement of the fluid parcel in such a manner that it resembles an oscillatory motion or an inertial wave. We can also have inertial waves due to Such oscillations of a disc placed inside a volume of fluid.

Now, this volume of fluid is rotating at an angular velocity $\boldsymbol{\omega}$. Now, this oscillation of the disc sets the nearby fluid into motion. In such a manner that, suppose the disc moves upward, it compresses the fluid above it and diverges or rarefies the fluid below it. Now, this compression and rarefaction travel along the axis of rotation, since the rotating fluid tries to maintain no change along the rotation axis. So, this compression-rarefaction disturbance tries to move in the vertical direction as well as in the downward and upward directions.

Thus, we have the inertial wave due to a vibrating disc inside a volume of rotating fluid. So, these are two typical examples of inertial waves from ocean dynamics and rotating fluids. Next, we consider the derivation of the inertial wave equation. The wave equation is governed by the balance between the inertial forces and the Coriolis forces. The inertial forces are nothing but mass times acceleration, and the Coriolis forces are the pseudo-forces that arise in a rotating frame of reference.

Let us consider an unbounded homogeneous fluid with density ρ. This is an unbounded homogeneous fluid. Its density is ρ. The rotation vector is $\boldsymbol{\omega}$, which is constant. Now, let us introduce velocity perturbations $\mathbf{U}$ about a state of rest in the rotating frame.

Now, the linearized momentum equation using Newton's laws gives us, per unit mass:

$$\frac{\partial \mathbf{U}}{\partial t} + 2\boldsymbol{\omega} \times \mathbf{U} = -\frac{1}{\rho}\nabla P$$

P is the mechanical pressure. Now, to proceed, we take the curl of the above equation. The curl operator is the vector calculus operator which we had discussed in the basic concepts. If we take the curl, we can have $\frac{\partial \boldsymbol{\xi}}{\partial t}$, where $\boldsymbol{\xi}$ is the curl of $\mathbf{U}$. Recall that the curl of $\mathbf{U}$ denotes the vorticity.

It denotes the rotational characteristics of the velocity field. Next, we also have the curl of $\boldsymbol{\omega} \times \mathbf{U}$. The curl of $\boldsymbol{\omega} \times \mathbf{U}$ equals $\boldsymbol{\omega} \cdot \nabla \mathbf{U}$ and other terms which go to zero due to the assumption of mass continuity, such as $\nabla \cdot \mathbf{U} = 0$ and other gradients. So one can look into the various vector identities which we have looked at in the previous lecture and try to derive this expression.

Thus, we have the vorticity equation:

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = 2(\boldsymbol{\omega} \cdot \nabla)\mathbf{U}$$

Now, again operating curl and a temporal derivative on equation 2, we get $\nabla \times \boldsymbol{\xi} = -\nabla^2 \mathbf{U}$. This occurs from the vector identity. This is also an interesting result which one must try to obtain using the vector identities we had discussed in previous lectures.

So curl and time derivative reduces this to this and u to del by del t of vorticity. So we proceed to get the inertial wave equation as

$$\frac{\partial^2}{\partial t^2}(\nabla^2 \mathbf{U}) + 4(\boldsymbol{\omega} \cdot \nabla)^2 \mathbf{U} = 0$$

This is the inertial wave equation. Here we have the time derivative del by del t square and the spatial derivative in the form of grad square. Now we proceed to obtain the dispersion relation.

Now we proceed to obtain the dispersion relation. The dispersion relation is a relation between the wave numbers and the frequencies of any wave. The wave number is nothing but spatial frequency, where λ is the wavelength and T is the time period of the wave.

Now we proceed to obtain the dispersion relation. We use the method of normal mode analysis. In normal mode analysis, we break down $\mathbf{U}$ in the form of normal modes, which are Fourier components expressed as $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$.

$e^{i\theta}$ can be written as $\cos\theta + i\sin\theta$, where $\theta = \mathbf{k} \cdot \mathbf{r} - \omega t$. Here we have used Euler's notation to represent the Fourier normal modes, which are nothing but sines and cosines. The $\mathbf{k}$ is the wave number, and ω is the frequency.

We thus have the disturbance with $\mathbf{U} = \mathrm{Re}\left(\widehat{\mathbf{U}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right)$. Note that the $\mathbf{k}$ vector is the wave number vector. This indicates that the wave number is along each of the three axes of the Cartesian coordinate system, and the vector is $k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$.

Now we substitute this normal mode into the previous equation of the inertial wave. We obtain:

$$\omega^2 \mid \mathbf{k} \mid^2 \widehat{\mathbf{U}} = -4(\boldsymbol{\omega} \cdot \nabla)^2\widehat{\mathbf{U}}$$

We can simplify and solve for ω as:

$$\omega = \pm 2\frac{\boldsymbol{\omega} \cdot \mathbf{k}}{\mid \mathbf{k} \mid} = \pm 2\omega\cos\theta$$

Here $\cos\theta$ is the angle between $\boldsymbol{\omega}$ and $\mathbf{k}$.

Thus, we can understand that for an inertial wave, the frequency is directly proportional to the rotation rate and the angle between the rotation rate and the direction of the propagation. It is important to understand that $\mathbf{k}$ vector is the direction of the wave number vector. The direction of the wave number vector is a property of the pattern of the inertial wave.

And this determines the frequency of the inertial wave. Thus, inertial waves of different patterns result in different frequencies. And this frequency is restricted from zero to 2ω. The minimum frequency is zero and the maximum frequency is 2ω. The maximum frequency occurs when $\theta = 0$, which means that the pattern of the inertial wave is such that the wave vector $\hat{\mathbf{k}}$ is aligned with the rotation axis. This alignment gives $\theta = 0$.

Next we look into the phase velocity which is the direction in which the individual troughs and peaks of the inertial wave propagate. This is given by the ratio frequency to wave number. Thus we have:

$$\mathbf{c}_{\mathrm{phase}} = \frac{\omega}{\mid \mathbf{k} \mid}\hat{\mathbf{k}} = \pm 2\frac{\boldsymbol{\omega} \cdot \mathbf{k}}{\mid \mathbf{k} \mid^2}\mathbf{k}$$

Next, we have the group velocity. The group velocity is the direction in which the energy of the inertial wave propagates. This is obtained as the derivative of ω, which is the frequency, with respect to $\mathbf{k}$, obtainable from the dispersion relation.

Thus, the group velocity, denoted by $\mathbf{c}_g$, equals:

$$\mathbf{c}_g = \frac{\partial\omega}{\partial\mathbf{k}} = \pm\frac{2}{\mid \mathbf{k} \mid^3}(\mid \mathbf{k} \mid^2 \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{k})\mathbf{k})$$

Now, this is an interesting expression where we can see that the group velocity is such that for very small $\boldsymbol{\omega} \cdot \mathbf{k} \ll 1$, which can occur for very low frequencies or misalignment of $\boldsymbol{\omega}$ and $\mathbf{k}$, this results in $\mathbf{c}_g$ being parallel to the rotation axis because this term goes to 0 due to the misalignment of $\boldsymbol{\omega}$ and $\mathbf{k}$, the dot product. We have only this term remaining, which makes the group velocity parallel to the rotation axis. This means that the information content or the energy of the inertial wave is directed along the rotation axis if the pattern of the inertial wave is such that it makes $\boldsymbol{\omega} \cdot \mathbf{k}$ nearly negligible.

In geophysics, we have various applications of the inertial wave, such as understanding heat transfer, energy transfer, and fluid motions in Earth's ocean, atmosphere, and the outer core, which is liquid iron. In geophysics, the applications of inertial waves range from ocean mixing, atmospheric circulation, and understanding the dynamics of planetary systems and turbulent motions of the fluid inside the Earth's core. Particularly in atmospheric studies, as we have seen earlier, the Rossby and gravity waves are examples of inertial waves. They impact large-scale circulation, which occurs from the equator towards the pole and from the pole towards the equator. We also have high-latitude jet streams, which are affected by inertial waves.

These jet streams are very important for navigation by planes across continents. They affect their speed and the duration of flights from one continent to another, as planes take advantage of these jet streams for fuel efficiency. Atmospheric dynamics are also influenced by inertial waves. In core dynamics, inertial waves play a very important role as they contribute to the growth and generation of the magnetic field, leading to an efficient geodynamo process. An efficient geodynamic process gives rise to a strong and stable magnetic field.

Inertial waves and disturbances contribute to variations such as the secular variation of the geomagnetic field. Thus, inertial waves are an important category of waves that are very useful for geophysical applications.

Thus, we come to the conclusion of this lecture. Inertial waves arise in rotating fluids. The restoring force is the Coriolis force, and these waves propagate at frequencies below the rotation rate.

A fundamental understanding of inertial waves can improve various models of geophysical phenomena, such as oceanic circulation, atmospheric patterns, and planetary interior dynamics. The dynamics and characteristics of inertial waves can aid in climate prediction and fundamental research in geophysics. In particular, inertial waves have vast applications in ocean mixing, inertial oscillations, length-of-day variations, energy transfer, and ocean currents.

Thus, we come to the end of the lecture on inertial waves. One can follow these references for more details on inertial waves.

Thank you.