Mathematical Geophysics

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Week - 06

Lecture – 28

Hello everyone, welcome to the SWAYAM NPTEL course on Mathematical Geophysics. This is continuing module number 6, Wave Dynamics in Geophysics. Today we are going to look at lecture number 3, which is Free Oscillation of the Earth. In this lecture, the following concepts are covered. We will be looking into the concept of free oscillation in general.

Then we will look into some fundamental mathematical and geometrical aspects of three-dimensional standing waves in a sphere. Next, we will look into the various types of free oscillation: radial oscillation, spheroidal oscillation, and tangential oscillation. Then we will look into the comparison with surface waves, which we have discussed in the previous lecture. So let us begin. The concept of free oscillations.

Now, oscillations mean periodic displacements. In the context of geophysics, very large earthquakes can set the whole Earth into vibration. Now, this is not only a smaller region or a smaller section of the Earth. The entire Earth, as a spherical body, vibrates.

Now, the vibration occurs with certain natural frequencies of the earth. This is determined by the elastic properties and the internal layer structure of the earth. We can understand the basics of free oscillations through this simple example. A one-dimensional vibration can be represented by the superposition of a number of vibrations.

Now Consider these figures. The first one on the left is the fundamental mode. This is the diagram of a string fixed at both ends to rigid supports. Now, the string, upon disturbing it at the center, vibrates in various modes.

These modes are nothing but the patterns shown here. The pattern which is the most fundamental and simplest is given here. This pattern has two nodes at the ends, which are due to the constraint that the string is fixed. Otherwise, there is no node at any other point in the interior of the string. A node means the displacement of that location is zero.

Next is the first overtone. In the first overtone, we have one internal node. This is the one internal node of the first overtone. Similarly, in the second overtone, we have two internal nodes. Now, these are the first three modes of vibration.

They are often called normal modes of vibration. These normal modes are associated with zero-displacement points known as nodes or nodal points of the vibration. In the Earth's context, the free oscillations involve three-dimensional deformations. Now, this deformation being three-dimensional represents the spherical deformations of the Earth.

Now, from this one-dimensional example which we have seen here, we can look into the fundamental mathematical aspects. Here, in one dimension, to understand the motion or displacement, we need only one variable. Suppose this is the x-axis; then the displacement u is only a function of x. It does not depend on the y-axis or z-axis. Thus, this is a one-dimensional oscillation. For two-dimensional oscillations, we would have u as a function of x, y, and so on. So in a spherical shape, we have the vibrations or displacements u as a function of all the three coordinates of the spherical system. Thus, we would be dealing with three-dimensional functions on a spherical body. Now, remember that in previous lecture we have discussed about functions on a spherical surface. Those were called spherical harmonics. So functions on spherical surface are spherical harmonics.

Now we have to look into the function on the radial axis which was not considered previously. As only θ and ϕ are covered under spherical harmonics. we remain, it is important to look into other functions which can represent the radial variations. Now a vibrating sphere which is displacement on the entire r-θ-ϕ axis can be resolved into the superposition of normal modes. The normal modes are three-dimensional normal modes. They may have nodes or non-zero displacements which are nodal surfaces.

Nodal surfaces are surfaces where the displacement goes to zero. Since it is a three-dimensional mode, the nodal points or nodal surfaces are two-dimensional. Just like we had in this example, a one-dimensional vibration has a nodal point as zero-dimensional points. So here we have the free oscillation function which is a function of r θ ϕ given by S. This function is accompanied by three indices. The first index n is the overturned number.

Now, this overturned number indicates the number of internal nodal surfaces. Thus, the number of nodal surfaces is equal to n. Then, we have m as the longitudinal order, which is the number of nodal lines on the sphere that are great circles or longitudes. Thus, the number of nodal longitudes is m. l is the order, which is equal to the number of nodal lines that are latitudes. Thus, the number of nodal latitudes is l. This l and m are nothing but the indices of spherical harmonics Y_l^m. Thus, we have the rotationally symmetric oscillations, which are considered as $m = 0$. Thus, we can understand that a simple particular case, $m = 0$, is the rotationally symmetric oscillation, which is given by And thus, the oscillation function becomes S_{nl}. Now, we look into the geometrical representations of these free oscillation functions. We have them separated as Y_l^m. which account for latitudinal and longitudinal functions of the spherical surface. And we have the spherical Bessel functions, which are given by $j_n(kr)$.

It is a function that depends on the radius. Now, we have looked into the spherical harmonics previously. If this denotes the spherical surface, then the vibration is given by the black lines. Now, this spherical harmonic is S_2^0. Now, $l = 2$, $m = 0$, and $n = 0$.

The $n = 0$ means that this function remains the same for all the radii as given by the $n = 0$ pattern. And $l = 2$ means there are two longitudinal nodes. Now we look at the S_3^0. Here $L = 3$ and $M = 0$, which means there are three longitudinal nodes. Note that we are looking at these functions from the top.

Similarly, S_4^0 has four longitudinal nodes. Now we look at the spherical Bessel functions. The spherical Bessel functions are described as $j_n(kr)$. Here n is the overtone number having values 0,1,2, and so on. For $n = 0$, the function is given by the black solid line, and this is the radial axis.

One can see that there are no nodes with respect to the radius since the displacement is always positive. The first overtone, $n = 1$, is given by this function. Here we can see that we have one radial nodal surface.

Now this R_1 is the first and only radial nodal surface. The $R =$ constant is a surface inside the spherical body, and over this surface, the displacement always remains zero. Above this surface and below this surface, the displacement has opposite signs. Now we come to the second overturn. $N = 2$ gives the following Bessel function. It has two radial nodal surfaces.

So $r = r_1$ and $r = r_2$ are the two spherical surfaces where the displacements go to zero. Hence, these are nodal surfaces. And the sign changes across the nodal surfaces. Similarly, three nodal surfaces for $n = 3$. Now, having combined this Y_l^m with j_n, we can get a function which is dependent on r, θ, and ϕ. For each value of n, l, and m, it represents a normal mode which can be used for representing the complicated three-dimensional standing waves, which are free oscillations on the spherical body. Now, we look into the various types of free oscillations. The free oscillations which occur for spherical bodies, such as Earth and other terrestrial planets, can be divided into three categories. First is the radial oscillation.

We have other oscillations, such as spheroidal and toroidal oscillations as well. The radial oscillation is purely radial displacement. The displacement u is only a function of r. There is no dependency on θ and ϕ. Next, we have spheroidal oscillation. The spheroidal oscillations are partially radial and partially tangential.

Now, this denotes a radial oscillation where the mean position is given by the surface of the Earth, then the surface of the Earth converges to the red and blue. Surfaces indicate inward and outward radial oscillations. Note that this oscillation is independent of latitude and longitude since the oscillations remain the same for all values of latitudes and longitudes. Whereas, tangential displacements are shown here. The tangential displacements are displacements on the surface of the spherical body.

Now, these displacements tend to deform the surface. The tangential displacements are a function of r, θ, and ϕ as well. This is the radial part of the spheroidal oscillation. And these are the tangential parts of the spheroidal oscillation. Thus, spheroidal oscillations are functions of all the coordinates: r, θ, and ϕ. Finally, we have the toroidal oscillations, which are purely tangential displacements. Here One can imagine that the upper hemisphere, which is the northern hemisphere, moves toward the left, and the southern hemisphere moves toward the right. So, this indicates that there is a shearing displacement with opposite signs above and below the equator. There are no radial oscillations.

The radius of the sphere remains fixed. Only the surfaces are displaced in a tangential manner. These are the various types of free oscillations. Next, we have the radial oscillations, which we will discuss in more detail. The radial oscillations indicate that only radial motions are allowed.

There are no tangential motions. The shape of the Earth remains fixed because the displacements are only in the radial direction and are invariant over latitude and longitude. Thus, all the particles vibrate only in the radial direction. These indicate radial motions where the particles are moving outward and inward only. This is the fundamental mode where the entire Earth expands and contracts in unison.

The higher mode of radial oscillation is shown in the adjacent diagram. Although we can see that the particles are vibrating radially, the direction of the motion of the particles may depend on r. This is the second normal mode. Consider this surface. At this surface, you can see that the displacements are either away from the surface or toward the surface, while there is no displacement of this radial surface. This radial surface, denoted by $R = R_1$, is a nodal spherical surface.

Here, the displacement equals zero. If there is only one nodal surface, this is the first overtone or the second normal mode. This is given by the $n = 1$ Bessel function, which has only one single nodal surface, that is, $R = R$. So, along the radial direction, we have one surface across which the sign of the displacement changes. Next, we have spheroidal oscillations. Now, spheroidal oscillations, as we have seen earlier, have two parts: the radial part and the tangential part.

This can be described by the spherical harmonics Y_l^m. Remember that the details of spherical harmonics can be obtained from lecture number 5 of week 5. Now, we look into some aspects of the spherical harmonic functions as appropriate for understanding spherical or spheroidal oscillations. The spherical harmonic functions are defined through an axis through the Earth at the point of the earthquake epicenter. The spherical harmonic functions are also defined through a great circle that contains that axis.

Now to understand this let us look into the adjacent diagram. Here the function s_0^0 is shown. $n = 0$ means there is no radial nodal surface and $l = 2$ means there are two longitudinal lines of nodes. These are the two longitudinal line nodes. The period of this is 53.9 minutes which means that in spheroidal oscillation it takes around one hour for the oscillation to complete.

Now this is the symmetry axis. The symmetry axis means that it is perpendicular to the both longitudinal nodal lines. The radial part of the spheroidal oscillation are indicated by these arrows which indicate outward and inward motions whereas the tangential part can be obtained by such displacements. Now we have the spherical harmonic functions with respect to the reference frame.

These spherical harmonics describe the latitudinal and longitudinal variations of the surfaces from the sphere. The general spheroidal oscillation function is denoted by S indexed by n, l and m. The second example of a spheroidal oscillation is S030. It indicates three latitudinal nodal lines. Now in practice, rotationally symmetric oscillations are considered as $m = 0$ is associated with these harmonics. $m = 0$ is mentioned like this.

Oscillation of the order $l = 1$ does not exist. $l = 1$ would mean that a single equatorial nodal plane. If the nodal plane is at the single equatorial point, then any vibration would mean the displacement of the center of gravity. Since the center of gravity of such spherical bodies, which are planetary systems, they cannot be displaced, thus oscillations of the order $l = 1$ are prohibited. Radial oscillations is a special type of spheroidal oscillation with $l = 0$.

This indicates the absence of any tangential modes and thus retaining only the radial part, the spheridal oscillation reduced to purely radial oscillations. Now let us discuss the toroidal oscillations in more detail. The toroidal oscillations are purely tangential displacements. The spherical shape and volume of the earth are unaffected by a toroidal oscillation because the tangential displacements are restricted to the surface or any spherical surface at any radius of the earth. There is no deformation in and out of the surface.

Thus, the spherical shape and the volume remains fixed. The amplitude of the longitudinal displacement varies with latitude. To understand this, let us have a look at the following diagram. These are the two modes of toroidal oscillation.On the left hand side is the most fundamental mode of toroidal oscillation.

Here, $n = 0$, which indicates the first vessel function's dependence on the radius. $l = 1$ means one latitudinal nodal plane is indicated by this, and $m = 0$ means there are no longitudinal nodal planes. Thus, this shearing motion, having opposite signs in the northern and southern hemispheres, remains invariant with longitude. The higher overtone of the toroidal oscillation is given here. Here, $n = 0$, $l = 2$, and $m = 0$. $l = 2$ indicates there are two nodal planes. The amplitude of the longitudinal displacement depends on the latitude, which means that as we move across latitudes, the displacement changes. This is how the amplitude of the longitudinal displacement varies along the latitude. These are the longitudes and latitudes for reference. The latitude with zero longitudinal displacement is the nodal plane.

Here, across the nodal plane, the sign of the longitudinal displacement changes. The toroidal function is described by T. The amplitude of toroidal oscillations inside the Earth can vary with depth, depending on functions of the radius given by the Bessel function. We have looked into various Bessel functions and seen that the number of nodes is the number of overtones, which is n. Now, we have an interesting comparison of the body waves or the three oscillations with surface waves, which we discussed in the previous lecture. Let us consider the Rayleigh wave. As we know, in a Rayleigh wave, the particles describe elliptical motion.

Similarly, in spheroidal modes, it is the particles which undergo both radial and tangential motion. In particular, we can note that the particle motions in Rayleigh wave are fixed in the vertical plane. They are radial and tangential components to this. These are the two components of the Rayleigh wave. Similarly, for three-dimensional waves, this parallel oscillations, we have both the radial and tangential components.

These are the radial component and the tangential components are on the surface. The spheridal oscillations are equivalent to standing wave patterns that arise from the interface and interaction of long period relay waves. which travel in opposite directions of the earth. Next, we consider the love wave. Recall that in the case of love wave, the particle motions are polarized in the transverse horizontal direction.

Similarly, the tangential displacements for the toroidal oscillations may be regarded as equivalent to the standing wave pattern that occur due to the interference of oppositely traveling love waves. Now, have a look at this diagram. The tangential displacements are indicated by the red and blue color. These are tangential displacements similar to the tangential displacement of the love wave. Thus, we can conclude that free oscillations of the earth which are natural vibrational modes are important aspects and tools which can be accounted for the vibrations which are generated in earthquakes.

Now, we come to the conclusion. First, we can note that the free oscillations of the Earth are natural vibrational modes or natural frequencies of the Earth. These occur after any major seismic event, such as an earthquake. These seismic events occur without any continuous external forcing and are impactful events that set the Earth ringing. Also, the various categories of free oscillations, such as radial, spheroidal, and tangential oscillations, help us fundamentally understand and simplify the complex vibrations of the Earth.

The oscillations typically have frequencies between 0.3 mHz and 10 mHz. These free oscillations complement the surface waves and refine the models of Earth's interior, helping us understand the various material properties and layers within the Earth's interior. One can refer to these references for a more detailed understanding of free oscillations and their fundamental physics. Thank you.