Mathematical Geophysics
Swarandeep Sahoo
Department of Applied Geophysics
Indian Institutes of Technology (Indian School of Mines), Dhanbad
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Hello everyone. Welcome to the SWAYAM NPTEL course on mathematical geophysics. Today, we begin the sixth module, which is wave dynamics in geophysics. This is the first lecture of this module: the wave equation. In this lecture, the following concepts are covered.

We will have an overall derivation of the wave equation, followed by various applications in geophysics. The wave equation is derived for a fundamental understanding of its applications in geophysics, which will be dealt with in further lectures. So, let us begin. Now, let us consider the physical concept of the wave equation first. What is a wave equation?

The wave equation describes the propagation of waves. These waves can be either sound waves, which are mechanical compression and rarefaction waves. It can be light waves or waves on the surface of water, and the passage of these waves through a medium is governed by the wave equation. Now, have a look at this adjacent diagram. This gives the basic characteristics of a wave.

Now A wave can be characterized by its amplitude and its wavelength. This is the wavelength λ. The amplitude is denoted by y. Now, this is a typical displacement wave, whereas other kinds of waves can also exist. This displacement wave is acted upon by some restoring force such that the displacement varies with distance.

Mathematically, this can be represented as a partial differential equation. We denote displacement as $u(x, y, z, t)$ for generality. Now, we have the equilibrium point that is depicted as a dotted line. The wave is a displacement or disturbance from the equilibrium. The equilibrium is the average quantity over the distance of this displacement. Thus, if we average the red line over a large distance, we will get the equilibrium value.

The partial differential equation that governs the wave phenomena is given by:

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

This we have come across earlier. Now, we will look into the derivation of this wave equation and understand the fundamental processes that go into the derivation of this equation.

Here c is the speed of the wave in any medium, and this speed is controlled by the property of the material in which the wave will propagate. ∇^2 denotes the Laplacian operator. Here we have shown the Laplacian operator in Cartesian coordinates.

Next, we look into the derivation of a one-dimensional wave equation. To derive the wave equation, we will look into the adjacent diagram first. Now, this diagram is that of a string. We consider a small portion of the string which is denoted by the solid black line. Now this is a small portion which has its dimensions Δx which covers the distance and Δu which covers the displacement. Here $u(x, t)$ is the vertical displacement of the string from the x-axis.

θ is the angle between the string and the horizontal line at position x and time t. Now θ is this angle. It makes an angle with the length or the elemental length of the string with the x-axis. Now we have $T(x,t)$ as the tension in the string at position x and time t. Tension acts away from the line element. Thus it is shown by arrows diverging from the line element.

Finally, we have ρ as the mass density of the string at position x. Now ρ is the mass of the string per unit length. The length of the string can be given as $\sqrt{\Delta x^2 + \Delta u^2}$. Now multiplying this ΔL with ρ gives us the elementary mass as ρ is the line density.The forces acting on the tiny element of the string can be given as the tension which is pulling to the left and right having magnitudes $T(x)$ at $T(x + \Delta x)$.

Now this tension is acting at an angle θ at $x + \Delta x$ above the horizontal. Similarly, we have the tension pulling to the left and various external forces like gravity. Now gravity is assumed to be acting vertically. And this and other forces which may be acting will be denoted by $F(x,t)$. Multiplied by Δx gives the net magnitude of the external force acting on the elemental length of the string. This is the external force F. Now we will be applying Newton's law for the equilibrium condition and its displacement.

Now Using Newton's law which is nothing but mass times acceleration equals net force:

$$\rho\sqrt{\Delta x^2 + \Delta u^2} \cdot \frac{\partial^2 u}{\partial t^2}(x,t) = \text{Net force}$$

The net force is the sum of all the forces which is acting on this elemental string.

That is the tension on the right, the tension on the left and the external forces F.Now the tension towards left has a minus sign because it is directed towards negative x-axis. Now dividing by Δx and taking the limit $\Delta x \to 0$ for differential element we get the left hand side as:

$$\rho(x)\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} \cdot \frac{\partial^2 u}{\partial t^2}(x,t)$$

On the right hand side we have the derivative of the tension force. This is at the limit $\Delta x \to 0$ of the difference which we have written earlier. The difference between these two terms in the differential limit gives the spatial derivative of the tension plus we have the external force $F(x,t)$.

Note that the sign of the θ gives the component along the vertical direction. We have considered the vertical direction because the displacement is in the vertical direction. Now this wave is a displacement wave that is why this displacement is the variable in this wave equation and thus the force balance in the Newton's law is taken along the direction of the vertical axis.

Along the x-axis, the forces are balanced by the tension forces and no such wave exists along the x-axis. Thus, we have the vertical component along the y-axis. This boils down to:

$$\frac{\partial T}{\partial x}(x,t)\sin\theta(x,t) + T(x,t)\cos\theta(x,t)\frac{\partial \theta}{\partial x}(x,t) + F(x,t)$$

From the right angle triangle which is shown in the previous slide, the tangent of the angle θ is $\frac{\Delta u}{\Delta x}$ which in the limit $\Delta x \to 0$ becomes $\frac{\partial u}{\partial x}$. And also, we can have other quantities such as:

$$\sin\theta = \frac{\partial u/\,\partial x}{\sqrt{1 + (\partial u/\,\partial x)^2}}, \quad \cos\theta = \frac{1}{\sqrt{1 + (\partial u/\,\partial x)^2}}$$

The $\theta = \tan^{-1}(\partial u/\,\partial x)$, which comes from this expression, and $\partial\theta/\,\partial x$ is given as:

$$\frac{\partial\theta}{\partial x} = \frac{\partial^2 u/\,\partial x^2}{1 + (\partial u/\,\partial x)^2}$$

Now, with these expressions, we can now substitute into the earlier equation and get the simplified form of the wave equation. Substituting all these quantities, we have:

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial T}{\partial x}\frac{\partial u}{\partial x} + T\frac{\partial^2 u}{\partial x^2} + F$$

In the limit where $\theta \ll 1$, which means that the angle is very small, then we can proceed with further simplifications. For example, $\tan\theta \ll 1$, and also $\partial u/\,\partial x \ll 1$. This is only because θ becomes very small. Thus, the expressions can be simplified as:

$$\sin\theta \approx \frac{\partial u}{\partial x}, \quad \cos\theta \approx 1, \quad \frac{\partial\theta}{\partial x} \approx \frac{\partial^2 u}{\partial x^2}$$

With these simplifications, we can write the previous equation as:

$$\rho\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2} + F$$

Now we can go for another simplification by assuming that there are only transverse vibrations. Which means that our displacement Δu is always perpendicular to the x-axis. There is no vibration or there is no displacement along the y-axis. Thus this perpendicular fluctuation is known as transverse vibrations. The tiny string element moves only vertically.

Thus the net horizontal force on it must be zero. That is along the x-axis. The tension to the right and tension to the left are equal. Thus, dividing by Δx and taking the limit as $\Delta x \to 0$ gives the horizontal gradient $\frac{\partial}{\partial x}$ of the tension going to 0.

T_x cos theta while T sin theta is the vertical component. Now, coming to small transverse vibrations, we have the force balance simplified to:

$$\rho\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2} + m$$

This we can obtain by neglecting del u by del x as a function of x, t since it is much less than 1. Thus we are left with this, this, and this quantity, which gives us this equation. Now, If any external force is absent, then F can be neglected.

Thus we can obtain the simplified wave equation as:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{T/\rho}\frac{\partial^2 u}{\partial t^2}(x, t)$$

This gives us the c^2, that is, the square of the speed of the wave through this medium:

$$c = \sqrt{T/\rho}$$

Now, ρ is the material property, and T is the tension. Thus, the speed of the wave in a medium with density ρ and having a tension T applied across it is given by $\sqrt{T/\rho}$. Thus, we can understand that the wave speed depends both on the tension, which is the restoring force, and density, which is the material property.

Thus, more dense materials will have slower wave speeds. Although it may look like more dense material would have slower wave speed, the wave speed is also dependent on the restoring force. So the total wave speed can only be determined by $\sqrt{T/\rho}$. Now, there are various applications of the wave equation in geophysics.

The wave equation is used in seismology. In seismology, the propagation of seismic waves is governed by the wave equation. Seismic waves are triggered by earthquakes. Then, the vibration of the material displacement, which is triggered by this earthquake, propagates throughout the various layers of the Earth. The study of seismic wave propagation helps us understand fault dynamics and the energy release process during an earthquake.

It also helps us reconstruct the interior of the Earth and its properties, such as the velocity or speed of waves in the Earth's interior, giving us an idea about various material properties of the Earth's interior. The study of seismic waves interacting with various heterogeneous materials inside the Earth, such as rock or surrounding clay minerals, can give us an idea about the presence of anomalous structures, which are geologically very important, such as faults and magma chambers. Further applications of the wave equation in geophysics lie in oceanography. Wave energy from underwater earthquakes or landslides is studied. Ocean wave dynamics and their coupling with ocean currents are also studied using the wave equation.

The wave equation is also utilized in seismic wave surveys for geophysical exploration studies to understand the functioning of hydrocarbon reservoirs and further identification in the subsurface of the Earth. The wave equation also applies to controlled-source seismic methods to locate mineral deposits within the Earth's subsurface. Essentially, we can understand that the propagation of seismic waves inside the Earth's interior, where it is solid, and the various properties governing their propagation can give us useful information about the Earth's interior, which cannot be directly perceived. Thus, we come to the conclusion about the wave equation. The wave equation describes the propagation of waves, which can be seismic, acoustic, or electromagnetic waves.

It describes the passage of the wave through different geophysical layers.This includes seismic activity such as earthquakes, ocean waves, and ocean currents, which lie in ocean dynamics, and electromagnetic surveys, which are governed by electromagnetic waves.

We also have the wave equation as a cornerstone of geophysical analysis. Geophysical analysis, such as studying, predicting, and modeling wave phenomena - which are critical to our sciences, natural hazard assessment, and planetary exploration - is very much dependent on the utilization of the wave equation.

Thus, in summary, we can say that in geophysics, seismic waves for earthquake studies, surface mapping, and wave interactions in oceanography for tsunami modeling are various aspects that are directly connected to the wave equation and give us useful understanding of natural phenomena. One can go through the following references for various applications of the wave equation, its detailed derivation, and the logic behind the mathematical derivation of the wave equation.

Thank you.