Mathematical Geophysics

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Lecture - 25

Hello everyone, welcome to the SWAYAM NPTEL course on mathematical geophysics. We continue with module number 5, *Diffusive Processes in Geophysics*. This is lecture number 5, *Decay of Magnetic Field*. In this lecture, the following concepts are covered. The decay of the magnetic field in general, with various other aspects related to the study of this decaying process.

Now, the decay of the magnetic field is based on the diffusion equation, which we have studied in previous lectures. Thus, first we will get introduced to spherical harmonics, which are used for magnetic fields, then the diffusivity limit, diffusion time scale, and applications in geophysics. So, let us begin. First, let us look at the concept of spherical harmonics. Now, spherical harmonics are functions of two coordinates, that is, θ and φ, in the spherical coordinate system. Recall that we have studied the spherical coordinate system, in which we have three coordinates. One is r, next is θ, and φ. r is the radial coordinate, where θ is the meridional angle and φ is the azimuthal angle. Now, the spherical harmonics are functions of the meridional angle or azimuthal angle.

In the geophysical parlance, we have θ equivalent to latitude while φ equivalent to longitude. So on the surface of the earth where we describe each location in terms of latitude and longitude, we describe the spherical harmonics as a function of these latitude and longitudes on any spherical surface. Now this spherical surface can be the surface of earth it can be at any height let's say 100 kilometers above the surface of the earth or it may be at a depth let's say 1000 kilometers below the depth of the earth still it is a spherical surface with a particular value of r thus A spherical surface in spherical coordinate system is denoted by r equals to constant. Thus, we have the spherical harmonics denoted by Y_l^m as a function of θ and φ.

Now, this is useful to obtain the solutions of Laplace equation in spherical coordinates. This obey very nice formulas and relations in spherical coordinates. Now this is the definition of the spherical harmonic $Y_l^m(\theta, \varphi)$. First we have $(-1)^m$ followed by square root of (2l + 1) multiplied by (l - m)! divided by 4π (l + m)!.

This is then multiplied by $P_l^m(\cos\theta)$. Then we have e^(i m φ). Now these are the four parts of a spherical harmonic. Here l is called the degree and m is called the order. Thus, L and M are the degree and order of the spherical harmonic.

The first part keeps track of the sign. The second part is a normalization factor such that the magnitude value of the spherical harmonic over the entire surface equals 1. The third part is the Legendre polynomial. Specifically, in geophysics and geomagnetism, the associated Legendre polynomial is used.

Thus, P_l^m is the associated Legendre polynomial. It is a function of θ in the form of cosine of θ. The fourth part is e^(i m φ), which is a Fourier function. So, e^(i m φ) in Euler notation can be written as cos(m φ) + i sin(m φ). Now we have the Legendre polynomial to cover the latitudinal dependence and the Fourier or sinusoidal functions to represent the azimuthal dependence.

This is because the azimuthal angle is a periodic function. It goes from 0 to π and then again from 0 to π because it represents the circular features of the sphere. The meridional angle θ moves from 0 to π. It has limits from 0 to π, which is a finite domain. Thus, the Legendre polynomials, which are defined over finite domains, are used.

Since φ is periodic, we have the periodic functions sine and cosine for its representation. Now, the peculiarity of spherical harmonics' order and degree is such that the order can only attain values from -L to +L. For example, if L = 5, then the permissible values for M, that is the order, are -5, -4, and so on till +4 and +5. Thus, the total number of harmonics for a particular value of L equals 2L + 1. Now, we will look into the various characteristics of these spherical harmonics.

So, have a look at the adjacent diagram. Here, for degree and order equal to zero, we have Y_{00}. This is a function that attains a constant value over the entire spherical surface. Next, we have Y_{10}, where the degree equals 1 and the order equals 0. Here, we can see that there is a latitudinal variance. There is variation along the latitudes, while there is no variation along the longitudes. In contrast, we have Y_{11}. Here, you can see that there is no latitudinal variation but only longitudinal variation, as indicated by the red line.

The colors vary from left to right and not from top to bottom. Next, we have higher harmonics such as Y_{20}. It has only latitudinal variations. Then we have Y_{22}, which has only longitudinal variations. Next, we have Y_{21}, the harmonic at the center.

This harmonic has variations both in longitude and latitude. You can see that from top to bottom, it changes color from blue to red, and from left to right, it also changes color from blue to red or vice versa. Thus, these are the overall aspects of the fundamental characteristics of spherical harmonics. Higher harmonics will display more complicated features. Now, it is important to understand the symmetry of spherical harmonics.

For example, if the difference in L and M—that is, degree minus order—equals an odd number, such as L - M = 1 for the case L = 3 and M = 2, it gives us an equatorially anti-symmetric structure, which means that above the equator and below the equator, the sign of this function is different. This is the equator, and above it, we have positive values for this function Y_{32}, whereas below it is negative. On the other hand, if L - M is even or zero, then we will have a symmetric equatorial structure. This means above and below the equator, there is the same sign. This entire zone is positive, which means above and below the equator, the sign of the function is the same.

Thus, it is the symmetry. So, these are the details of spherical harmonics, which are used for the solution of geomagnetic decay. Now, we have looked into the diffusivity limit in the previous slide. We will revise it quickly. Thus, in the diffusivity limit, where η tends to infinity or velocity tends to zero, we have the non-advective induction equation given by:

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}$$

Now, we will assume that $B = B(\theta, \varphi, t)$ at a particular radial location R in the spherical coordinate system.

For example, if this is the sketch of the cut section of the Earth's interior, at any depth r, let's say R_A, we can write this form for B. So, B, that is the magnetic field magnitude, depends only on latitude, longitude, and time. We assume the form as:

$$\mathbf{B} = \mathbf{B}_0(\theta, \phi)e^{\alpha t}$$

Now, we can assume the form of:

$$\mathbf{B} = \mathbf{B}_0(\theta, \phi)e^{-\alpha t}$$

The time dependence is chosen as an exponential form. Now, if we substitute this form of the solution (equation 2) into the governing diffusion equation (equation 1), we get:

The left-hand side $\partial B/\partial t$ gives us:

$$-\alpha\mathbf{B}_0(\theta, \phi)e^{-\alpha t}$$

The right-hand side gives:

$$\eta\nabla^2\mathbf{B}_0(\theta, \phi)e^{-\alpha t}$$

Now, we can get rid of e^(-αt) from both sides, leaving us with equation (3):

$$-\alpha\mathbf{B}_0(\theta, \phi) = \eta\nabla^2\mathbf{B}_0(\theta, \phi)$$

Now, since we know that the spherical harmonics are the most appropriate functions on the spherical surface, we expand the spatial dependency of $B_0(\theta, \varphi)$ in terms of spherical harmonics. Now, this indicates that the B_0 can be represented as a linear superposition of various spherical harmonics and their combinations.

We have A_{lm} as the coefficient for these spherical harmonics. With suitable combination, any function on the surface of sphere can be obtained. And thus we can study the evolution of the magnetic induction or the diffusion equation for such functions of the magnetic field. For example, if we limit the functional dependence to L = 1 and M = 0, we only have:

$$\mathbf{B}_0(\theta, \phi) = Y_{10}$$

considering $A_{10} = 1$.

Thus, we will proceed for the solution of the magnetic diffusion equation for the two-dimensional case of:

$$\mathbf{B}_0(\theta, \phi) = Y_{10}$$

In equation (3), we will substitute B_0 as Y_{10} to get equation (4):

$$-\alpha Y_{10} = \eta\nabla^2 Y_{10}$$

Now we have an identity for the spherical harmonics that is:

$$R^2\nabla^2 Y_{lm} = -l(l+1)Y_{lm}$$

That means for l = 1 and m = 0, we can have:

$$R^2\nabla^2 Y_{10} = -2Y_{10}$$

Here l = 1. Thus this factor becomes -2. Thus the above equation becomes:

$$-\alpha Y_{10} = -\frac{\eta}{R^2}(2Y_{10})$$

This is coming from equation number (4). Solving for α, we have:

$$\alpha = \frac{2\eta}{R^2}$$

Now, R can attain any value in the Earth's deep interior, such as this, which lies in the outer core of the Earth. Thus, R is any radius from the CMB, that is, the core-mantle boundary, to the center. Now, we have to substitute the value of α into the expression for B to get the general solution. Now, we can get, upon substitution of the value of α, the general solution of the magnetic field as:

$$\mathbf{B} = \mathbf{B}_0(\theta, \phi)e^{-t\left(\frac{2\eta}{R^2}\right)}$$

which can also be written as:

$$Y_{10}e^{-\frac{2\eta}{R^2}t}$$

Now, this gives us the idea that at what rate the magnetic field is going to decay once we have no induction effect. As we know, the coefficient of time dictates the rate of decay of any spatial quantity. For Y_{10}, this is $2\eta/R^2$. Now, for the Earth's core conditions, these values can be determined. This leads to the concept of the diffusion time scale.

Now, the diffusion time scale is the scale of time where the diffusion process reduces an initial magnitude to a 1/e factor. We have:

$$\eta = \frac{1}{\sigma\mu}$$

which is a material property for the Earth's outer core. The fluid is liquid iron. We have:

$$\mu = 4\pi \times 10^{-7}\ \text{H/m}$$

and:

$$\sigma \approx 1 \times 10^{6}\ \text{S/m}$$

This gives us a resultant value of the magnetic diffusivity:

$$\eta \approx 0.8\,\mathrm{m}^2/\mathrm{s}$$

Now, this is the magnetic diffusivity, which is estimated for the Earth's outer core. Now, note that there is no evidence of direct measurement of this magnetic diffusivity, as the Earth's outer core is inaccessible to direct methods. This is based on various experiments and mineral physical calculations to obtain this value of η. The radius is considered as 10^6 meters, which is 1000 kilometers. This is from the center of the Earth.

Taking this 1000 kilometers as a length reference, we can obtain the magnetic diffusion time as:

$$\tau = \frac{R^2}{\eta}$$

which is substituted as:

$$\tau = \frac{(10^6)^2}{0.8} \approx 40{,}000 \text{ years}$$

What does this mean? It means that if the Earth's magnetic field were to be a pure dipole (Y_{10}) and the induction effect shuts down—let's say U tends to 0—then what would be the time before this decays to a factor of 1/e? That is 40,000 years. In 40,000 years, an initial magnetic field of a given strength will reduce to a factor of 1/e, which is approximately 2.7 times smaller.

This refers to the gradual reduction in the strength or intensity of the magnetic field over time. This phenomenon can occur in various other contexts, such as astrophysical and stellar interiors. Also, in various materials where the magnetic field is involved. Thus, we have the diffusion time scale for Y_{10}. Note that if the harmonic changes to Y_{32}, then the time scale will change because this factor will change.

Once this factor changes, It changes this coefficient too, and thus we have a different time scale. So this will be changed. So for higher harmonics, the magnetic diffusion time scale is smaller. This means that more complicated functions of the spherical surface will decay faster for the magnetic field.

Now, various applications of the diffusivity limit and the magnetic field decay exist in geophysics. We have specifically looked into the Earth's liquid outer core, where the magnetic diffusion time competes with the convective processes to sustain the geomagnetic field. The magnetic field of the Earth continuously decays with various time scales for various harmonics inside the Earth's outer core. However, the convective processes, which give rise to finite velocity u, regenerate the magnetic field and maintain the geomagnetic field through the process called geodynamo. If the diffusion time is high, this indicates that the induction process dominates over the diffusion process because a high diffusion time indicates slow diffusion or a weaker effect.

Also, diffusion contributes to the long-term evolution of the geomagnetic field. This has consequences for secular variations and magnetic reversals, which are flips in the polarity of the Earth's magnetic field. Apart from the Earth's liquid outer core, magnetotelluric surveys are another important geophysical application where the decay of the magnetic field is of paramount importance. It uses natural variations in electromagnetic fields to probe conductivity at various depths. In this process, the magnetic diffusion length scale determines the depth of penetration for specific frequencies.

The length and time scales of magnetic diffusion are very important for magnetotelluric surveys. Overall, the magnetic diffusion time models the evolution of signals and the interpretation of various other applications, such as oceanic circulation patterns. We can conclude that the decay of magnetic fields in conductive materials is primarily driven by resistive dissipation. This is described as magnetic diffusion. The magnetic decay time scale is proportional to the square of the system size, that is L, and inversely proportional to the electrical conductivity.

Thus, the diffusion time scale can be given as:

$$\tau = \frac{L^2}{\sigma\mu}$$

Overall, the understanding of various decay processes of magnetic fields of different structures and different length scales determines the evolution of geophysical and geomagnetic processes. This is also important for modeling astrophysical magnetic fields, where the decay of the magnetic fields and various length scales are in operation. The history of the magnetic field evolution, modeling of co-dynamics, and the exploration of the interaction between electromagnetic and thermal processes are important applications of these magnetic field decay concepts. Thus, we conclude the present lecture in module number 5.

We have the following references for more details on magnetic diffusion and the decay of magnetic fields, as well as their applications to geomagnetism and geophysics in general. Thank you.