Mathematical Geophysics

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Lecture – 24

Hello everyone, welcome to the SWAYAM NPTEL course on mathematical geophysics. Today, we continue with module 5: diffusive processes in geophysics. This is lecture number 4: solution of the magnetic diffusion equation. In this lecture, the following concepts are covered. The solution of the magnetic diffusion equation.

We have seen the derivation of the magnetic diffusion equation in the previous lecture. In this lecture, we will cover the following five components. Diffusivity limit, method of separation of variables, diffusion in one and two dimensions, and further, we will look into various applications in geophysical studies. So, let us begin. The diffusivity limit.

We have seen the derivation of the magnetic induction equation from Maxwell's laws. The electromagnetic induction equation is given as equation number 1. This equation states the evolution of the magnetic field B vector with respect to time. Now, on the right-hand side, there are two terms. One is curl of u cross b plus η Laplacian b. The curl of u cross b is dependent on the velocity u and the magnetic field vector b, and this is known as the induction term.

While the second term is η Laplacian b. η is the magnetic diffusivity. And V is the magnetic vector. Now, in general, these two processes occur simultaneously, but in the diffusivity limit, when η tends to infinity, the second term dominates. Also, if the velocity is very low—that means slowly moving dynamics of the fluid flow—then the first term becomes very small. Thus, either the velocity becomes small or the diffusivity becomes large.

We have the diffusion process dominating over the induction process. This leads to the non-advective induction equation as equation number two. This is $\partial B/\partial t = \eta\nabla^2 B$. We can see that this is a partial differential equation. Here, B is a function of x, y, z spatial coordinates in the Cartesian system, and t is the time. Thus, in the Cartesian system, the Laplacian is given by this expression.

Now, let's say the B vector is given by the modulus or the magnitude B, and the direction is the x vector. Thus, the one-dimensional non-advective induction equation is now simplified from 2 to 3, given by $\partial B/\partial t = \eta(\partial^2 B/\partial x^2)$. Here, B is only a function of x and t, and hence the one-dimensionality. Now, we are going to look into the processes and the methods that can be utilized from this equation. First, we will solve this equation and see what the consequences of the solution are and the characteristics of these solutions.

This diffusion of the magnetic field is a very useful concept in relation to geophysical studies, which we will also consider later. First, let us look at the method of separation of variables, which we will utilize for the solution of this equation. The function B, which is dependent on x and t, is separated into two functions. The product of x, which is a function of x only, and the product of the product of x and t. Here, x depends on the spatial coordinate x, and capital T is a function that depends upon the time t. Now, substitution of this equation into equation number three gives us $X(dT/dt) = \eta T(d^2X/dx^2)$.

This is the variable-separated form. Note that the variable separation form is just an assumption. In general, it may be that B is more general than the variable-separated form. But this type of analysis gives us an easy method of solution, which is also insightful. Now we have to divide both sides of equation 5 by the product XT. This gives $(1/T)(\partial T/\partial t) = (\eta/X)(\partial^2X/\partial x^2)$. Now look at this: the left-hand side is only a function of time, whereas the right-hand side is only a function of space. Now, since they are equal, it cannot depend on either time or x. Thus, these two parts, the left-hand side and the right-hand side, are equal to a constant. Thus, we can write both the left-hand side equals to k (equation 7) and the right-hand side equals to k (which is equation number 8). We can solve equations 7 and 8 separately to get the functions X and T separately.

And then we can put it back into equation 4 to get the solution for the magnetic induction equation in one dimension. Now, let us look into the solution. First, we have the time part. So from equation 7 we can proceed as $\partial T/\partial t = Tk$. So here T as a function of t will be replaced by only capital T for notational convenience.

Now we have By integrating both sides we have $\int(dT/T) = \int k\, dt$. This gives the natural logarithm of T equals kt plus c. This can be further derived into $T = e^{(kt + c)}$. Now e^c can be written as a constant capital P. So we have the function capital T equals a constant P multiplied by $e^{(kt)}$. Now k is known as the time constant since it is a coefficient of t under the exponent.

Now We can have two possibilities. One k is positive the other k is negative and the time solution would be $T = P e^{(+kt)}$ and $T = e^{(-\kappa t)}$ where $\kappa = -k$. Now, we can have two cases. Depending upon the sign of k, the solutions would be for k positive, $T = P e^{(kt)}$.

Here, k is positive. For k negative, we can use $-\kappa$ where κ is positive. And we can write $e^{(-\kappa t)}$. So here κ is positive. Thus we can have different behavior for different sign of k for the time solution. Now we look into the space part.

Here we will determine the function capital X. So from equation 8 for notational convenience we will be using capital X only in place of capital X as a function of x. We derive $d^2X/dx^2 = (k/\eta)X$. Now let us suppose we have $k/\eta = \lambda$. Thus we can get equation 11 as $d^2X/dx^2 = \lambda X$. This is an ordinary differential equation. So we assume the form of the function capital X equals $A e^{(mx)}$. This is the assumed form of the solution.

Now $X = A e^{(mx)}$ means that the solution X which is depending upon the space coordinate x behaves as an amplitude A multiplied by an exponential function e. Now the exponential function has an exponent mx. Here m is the coefficient of x which decides the rate of increase or decrease of the value of the function X with space. If we substitute this $X = A e^{(mx)}$ into equation 11, we will get $m^2 = \lambda$. This is because we have the double derivative of $A e^{(mx)} = A m^2 e^{(mx)}$.

So we have $m = \pm\sqrt{\lambda}$. Now remember, λ is capital k divided by η. So depending upon the sign of k and η, we can get different solutions. Nevertheless, we have $X = A e^{(+\sqrt{\lambda}\, x)} + B e^{(-\sqrt{\lambda}\, x)}$ as the superposition of two solutions. This is the general solution.

Here, A and B are constant coefficients. So we have $X = A\ e^{(\pm\sqrt{(k/\eta)}x)}$. These are the two components of the general solution in space. We can have a look at a special case.

For example, if $\lambda = 0$, then this equation reduces to $d^2X/dx^2 = 0$, for which we have a linear straight-line solution. This is a straight-line solution where X is equal to ax + b. This indicates the limit where η tends to infinity. Now, we will consider the solutions and the characteristics of these solutions as we have derived in previous slides. First, we look into the spatial modes. Next, we will consider the various characteristics of the solutions, i.e.

Diffusion in one dimension. The first solution is $X = e^{(+\sqrt{(k/\eta)}x)}$. This is a solution where the function rises with x. You can see the two solutions given in green and blue colors. The blue color rises much faster than the green color. The blue color denotes $\sqrt{(k/\eta)} = 5$, whereas the green color indicates $\sqrt{(k/\eta)} = 1$. Similarly, we have the next solution.

X as a function of $x = e^{(-\sqrt{(k/\eta)}x)}$. Due to the minus sign, this is a decaying solution. The green color indicates $\sqrt{(k/\eta)} = 1$, while the blue color denotes $\sqrt{(k/\eta)} = 5$. We can see that the rate of decay or the diffusion is controlled by the coefficients $\sqrt{(k/\eta)}$. The higher the coefficient, the higher the diffusion. Next, we have the time dependency.

The time dependency also gives a decaying or rising solution. $e^{(+kt)}$ gives a rising solution, whereas $e^{(-kt)}$ gives a decaying solution. Note that as time progresses, we have a change in the magnitude of the solution. For extension, we will look into the diffusion of various quantities in two dimensions. Here, we have just shown a schematic solution of the two-dimensional magnetic diffusion equation as obtained from numerical solutions.

The first at t = 0, if we have a concentrated spot in the x and z directions, Then this concentrated spot diffuses due to the gradient, and with progress in time, let's say t = 10, we have a diffused spot. Then, as time progresses, the concentration of the initial spot diffuses and becomes more and more uniform. This is diffusion in two dimensions. Next, we look into the various applications in geophysics.

The diffusion equation is used in the geomagnetic field, as we have seen for the magnetic field application, and thus it was very important to understand the evolution of the magnetic field with the diffusion process. The decay and growth of magnetic fields over time in highly conducting materials, such as liquid iron in the Earth's outer core, is governed by the diffusion equation. The diffusion equation also provides insights into the time scales for the magnetic field decay. For example, if the magnetic field is not generated by induction and we only have diffusion, then any strong magnetic field that exists will decay, just like we have the decaying solution in the previous slide. This indicates the reduction in the magnitude of T with time.

Thus, the magnetic fields decay over time if induction is not coming into play or the velocity is small. Next, we have magnetotelluric surveys. Magnetotelluric surveys are used to identify subsurface features using magnetic field and electric field measurements. In these surveys, the time-varying magnetic fields also diffuse. They diffuse through the Earth's crust and mantle.

Thus, these signals can only exist up to a certain depth, and depending on the frequency of the time variation of these magnetic fields, the diffusion acts. If the frequency of the magnetic fields is high, then they dissipate or diffuse out quicker. Hence, the depth of penetration in these surveys is limited. For deeper surveys, slowly time-varying magnetic fields, which exhibit very little diffusion, are used.

In core-mantle boundary dynamics, the diffusion equation is used to model the interaction of magnetic fields with heat transfer and mass transfer processes that occur across the core-mantle boundary. This is the boundary between the mantle and the core. So this is a sketch of the Earth's interior, and the boundary between the outer core and the mantle is called the core-mantle boundary. Here, the magnetic fields generated in the outer core interact with the various heat and mass transfer processes that occur at the core-mantle boundary.

To study this, the magnetic diffusion equation, mass diffusion, and heat diffusion equations are also very important. We also have variations in the electrical conductivity, or η, across the core-mantle boundary. Now, depending on the variations in η, the diffusion process varies from place to place on the core-mantle boundary. And it has a huge impact on the Earth's geomagnetic field. Not only the one-dimensional diffusion equation but also multiple dimensions, such as two- and three-dimensional diffusion equations, are very important for studying these processes.

And to solve higher dimensional diffusion equation, we need methods such as numerical methods for solving and getting the solution. Thus, the magnetic diffusion equation describes the magnetic fields and how they evolve over time in conducting media. Now, conducting media exhibit resistive effects and these resistivity effects enhance the diffusion of the magnetic fields. Now, we also conclude that the magnetic diffusivity that is $\eta = 1/(\sigma\mu)$ depends on the material conductivity and magnetic permeability. So this is a material dominated effect.

Various different materials will have a huge range for the value of the magnetic diffusivity. And this creates a whole different dynamics and time evolution of the diffusion process. Thus, we can summarize that the magnetic diffusion equation is a vital tool for understanding the behavior and evolution of magnetic fields in geophysical and planetary contexts. It bridges the gap between theoretical and practical applications in geophysical exploration studies and also fundamental geoscientific research. One can have a look at these references for more details of the magnetic diffusion equation and the solution strategies and their applications in various domains of geophysical and geomagnetic studies.

Thank you.