Mathematical Geophysics

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Lecture – 23

Welcome to the SWAYAM NPTEL course on mathematical geophysics. We continue with module number 5: diffusive processes in geophysics. This is lecture number 3: magnetic diffusion. In this lecture, we cover the general concept of magnetic diffusion. In the previous lecture, we discussed thermal diffusion.

Now, we are looking into magnetic diffusion phenomena. Magnetic diffusion can be understood in terms of Maxwell's equations. We will look into a schematic representation of Maxwell's equations for better understanding. The magnetic induction equation is the culmination of magnetic diffusion and magnetic induction. The derivation of the magnetic induction equation provides the basis for magnetic diffusion phenomena.

The geophysical applications will be considered finally. So, let us begin. Here, we have listed the various Maxwell's equations. There are four Maxwell's equations: Gauss's law, Gauss's law of magnetism, Faraday's law of induction, and the Ampere-Maxwell law. On the left-hand side, we have the differential form of Maxwell's equations, and on the right-hand side, we have the integral forms.

Now, the differential forms are expressed in terms of vector calculus properties, while the integral forms are represented in terms of integral theorems. These mathematical basics we have discussed earlier. First, we have Gauss's law, which states that the divergence of E equals ρ divided by ϵ_0 or, equivalently, the flux of the electric field is the integral of the charges over the entire volume. The second Gauss's law of magnetism states that the divergence of B equals zero.

This implies that the flux of the magnetic field across a closed surface is equal to zero. Next is Faraday's law of induction. Faraday's law of induction states that changes in the magnetic field with respect to time can give rise to an electric field, which has a curl equal to the change in the magnetic field. Thus, the electric field develops a curl when the magnetic field changes. So essentially, the electric field becomes rotational and loses its conservative property because of the influence of changing magnetic fields.

In other words, the circulation of the electric field across a closed loop is equal to the rate of change of the magnetic flux over time. We have Ampere-Maxwell's law as $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$. Here, the curl of B is finite, which represents the rotational characteristics of the magnetic field in the presence of current. The second term is the displacement current, which is essentially the time rate of change of the electric field. In integral form, Ampere-Maxwell's law essentially reads as The integral of $\mathbf{B} \cdot d\mathbf{l}$, which is the circulation of the magnetic field around a closed loop, equals the flux of the current plus the rate of change of the flux due to the electric field. Here is the list of notations to help understand Maxwell's equations. These are the properties. These are the field quantities. So essentially, Maxwell's equations will be used in understanding the induction equation, which contains magnetic diffusion and magnetic induction.

Now, let us have a look at the schematic representation of Maxwell's equations for better understanding. First, we have Gauss's law. The divergence of the electric field is equal to the charge density divided by ϵ. You can see that this is the divergence of the electric field from a source ρds. If we consider a surface S with the normal vector $d\mathbf{S}$ and the electric field $\mathbf{E}$ vector passing through it, the flux is equal to the total charge divided by ϵ_0.

Next, we have Gauss's law of magnetism. Gauss's law of magnetism states that the total flux across any surface equals zero. This means that the magnetic field lines passing through the surface are such that the number of lines entering is equal to the number of lines exiting the surface. Now, contrasting this, we have the difference between electric and magnetic fields. The electric field is not divergence-free because of the presence of charges or point sources of charges.

There is no such point source of magnetic charges or equivalent magnetic poles for the magnetic field. Hence, the integral of $\mathbf{B} \cdot d\mathbf{S}$, or essentially the divergence of $\mathbf{B}$, equals 0. This essentially means the non-existence of magnetic monopoles or point sources of the magnetic field. This is in contrast to the existence of point charges or sources of the electric field. This is the difference between electric and magnetic fields.

Now we look into Faraday's law. Faraday's law rests on the occurrence of time variance of the magnetic field. Across any loop that encloses an area, the fluctuations in the magnetic field can give rise to a current driven by the electric field $\mathbf{E}$, which is induced in the loop due to this time variance of the magnetic field. Thus, we have the circulation equals the time rate of change of the magnetic flux. If the flux changes, the circulation is non-zero, which means current flows through the loop.

Next, we have the Ampere-Maxwell law. This is related to the circulation of the magnetic field. Thus, the circulation of the magnetic field depends on two factors. First is the rate of change of the electric field flux. If we have a loop that contains some area, then through this area, the passing electric field lines, if they change over time, induce a magnetic field around this loop.

The second component of the circulation of the magnetic field comes from the presence of a current. The flow of current through a conductor with area $d\mathbf{S}$ induces a magnetic field across the loop that encloses or circles the conductor. Now, we have two equivalent sources of magnetic field induction. The passage of current as a current density through a unit surface area and the change in electric flux. These two phenomena contribute to the formation or generation of magnetic field circulations, which are indicated by this circling loop of the magnetic field lines.

Thus, we have the schematic representation of Maxwell's equations. Comparing the electric and magnetic field lines in terms of circulations, we see that the circulation of the electric field can be generated from a changing magnetic flux, and the circulation of the magnetic field can, in turn, be generated from a changing electric flux. But additionally, the passage of current also generates a magnetic field circulation. Next, we move on to the magnetic induction equation. The magnetic induction equation is based on Maxwell's equations.

It represents the rate of change, advection, and diffusion of the magnetic field. The magnetic field evolves based on the induction equation. Here, we have the magnetic induction equation given by $\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$. It is derived using Maxwell's equations and Ohm's law. The $\frac{\partial \mathbf{B}}{\partial t}$ indicates the rate of change or evolving magnetic field.

The first term on the right-hand side, $\nabla \times (\mathbf{u} \times \mathbf{B})$, indicates the advection of the magnetic field by fluid motion. Here, $\mathbf{u}$ is the velocity field of the fluid, which is conducting by nature and allows the passage of electric currents and the induction of magnetic fields. The third term is the diffusion equation for the magnetic field. The diffusion equation is also caused by ohmic dissipation. Here, η is the magnetic diffusivity.

Now, let us consider the case when η tends to zero. This means that the resistance to the motion of electric current is zero. This is a perfectly conducting material. The induction equation reduces to the first term on the right-hand side only. This is usually very applicable for the dynamo mechanism, which is the conversion of mechanical energy from fluid motion ($\mathbf{u}$ vector) to magnetic energy ($\mathbf{B}$ vector).

This is the induction effect. Now, on the other extreme, we have the diffusivity limit, where η tends to very high values. This makes the material essentially electrically insulating. In this limit, the first term on the right-hand side is neglected. In the phase of $\eta \nabla^2 \mathbf{B}$. Thus, we have the diffusivity limit giving the diffusion equation for the magnetic field. This reads as $\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}$. This is applicable for highly resistive systems. Thus, we have the general magnetic induction equation giving us two limits: one is the induction limit and the other is the diffusion limit. The induction limit is useful for dynamo action, whereas the diffusion limit explains the decay or destruction of magnetic field gradients. Now, we look into the detailed derivation of the magnetic induction equation as we had outlined in the previous slide.

We begin with Maxwell's equation, namely Faraday's law of induction. In differential form, Faraday's law of induction states that the curl of the electric field develops due to a change in the magnetic field. We then use Ohm's law for moving conductors. We have seen in previous lectures that the current density $\mathbf{J}$ is driven by two factors in the presence of moving conductors. First is the applied electric field.

Second is the cross product of $\mathbf{U}$ with respect to $\mathbf{B}$. σ is the electrical conductivity. From this Ohm's law, we get $\mathbf{E} = \frac{1}{\sigma}\mathbf{J} - \mathbf{U} \times \mathbf{B}$. Thus, the electric field is $\mathbf{J}$ per unit conductivity and the induction $\mathbf{U} \times \mathbf{B}$. Substituting this equation into the first equation for curl of $\mathbf{B}$, We also use $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, which is Ampere's law without the change of electric field with respect to time. The time rate of change of the electric field is neglected as it is not considered in the derivation of the induction equation. Thus, we have Ampere's law.

Substituting the equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, the Ampere's law into the first equation gives us $\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) + \nabla \times (\mathbf{u} \times \mathbf{B})$. Next using the Ampere's law we have the final form given by $-\frac{1}{\sigma \mu_0} \nabla \times (\nabla \times \mathbf{B}) + \nabla \times (\mathbf{u} \times \mathbf{B})$. Now we utilize A vector identity that is $\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$. Next, we use the vector identity which states $\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$. So using this vector identity and using the Maxwell's equation that is $\nabla \cdot \mathbf{B} = 0$,

Essentially, the non-existence of monopoles, we have simplified the above equation to $\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\rho \mu_0} \nabla^2 \mathbf{B} + \nabla \times (\mathbf{u} \times \mathbf{B})$. The $\nabla^2 \mathbf{B}$ results in Laplacian of $\mathbf{B}$. The induction remains as it is. Now $\frac{1}{\sigma \mu_0}$ is equivalent to η which is the magnetic diffusivity. Thus we have the final form of the induction equation which reads $\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$. In geophysics, there are various applications of the induction equation in particular and the Maxwell's equations in general.

First, we have the geo dynamo process. The geo dynamo process is a magnetohydrodynamic process which involves magnetic fields, flow fields and heat transfer to convert mechanical energy in the form of fluid motion to electromagnetic energy in form of magnetic fields and electric currents. The geo dynamo process occurs in the outer core of the earth where the fluid is liquid iron, which is a conducting fluid. The magnetic induction equation can explain the generation and evolution of the earth's magnetic field in this fluid outer core by the dynamo process. Also, the induction equation is used to understand the geomagnetic secular variation.

The induction equation helps model the slow changes in the Earth's magnetic field. These changes in Earth's magnetic field occur over decades or centuries, and the time evolution of the magnetic field can be understood with the help of the induction equation. We also have the space-weather interaction, where Earth's magnetic field interacts with the surrounding space, which contains the magnetic fields of other planetary and stellar bodies. For example, the Sun's solar wind radiation interacts with Earth's magnetosphere, which is the engulfing magnetic field of Earth. The interaction of the solar wind induces time-varying changes in Earth's magnetic field, including phenomena like magnetic reconnection and geomagnetic storms.

These geomagnetic storms and magnetic reconnection are essentially large variations in the magnetic field, triggered by incoming solar radiation in the form of solar wind. This helps predict and monitor space weather for essential warning systems to prevent satellite failures or other interspace disruptions. Thus, we conclude the discussion on magnetic diffusion as covered in this lecture. The magnetic induction equation describes the evolution of magnetic fields in conducting fluids. It combines the effects of three processes:

advection, diffusion, and induction. The magnetic induction equation also explains the generation and sustenance of magnetic fields in various planetary and stellar bodies. In particular, Earth's geomagnetic field is produced in the fluid outer core, governed by the induction equation. The induction equation includes the diffusion process of the magnetic field, which removes unwanted or drastic fluctuations and provides global magnetic field coverage for Earth's geomagnetic field.

Fundamentally, the magnetic induction equation provides a theoretical foundation for exploring various magnetic phenomena in astrophysical and geophysical contexts. There are also engineering applications that involve magnetohydrodynamics, such as the melting of liquid iron, its mixing, and other turbulence-based flows. One can refer to the following references for a better understanding of the magnetic diffusion effect and geophysical applications. Thank you.