Mathematical Geophysics

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Week - 05

Lecture - 21

Hello everyone. Welcome to the SWAYAM NPTEL course on mathematical geophysics. Today, we will be starting the fifth module, which is diffusive processes in geophysics. This is the first lecture of this module, titled 'Diffusion.' In this lecture, we will cover various concepts related to the phenomenon of diffusion.

We will derive the diffusion equation and then examine various aspects of its geophysical applications. This covers the entirety of this lecture on diffusion processes. First, let us consider the phenomenon of diffusion. We know diffusion from our daily life experiences.

For example, consider this picture. This is molecular diffusion. Suppose we take a container containing some liquid and inject some particles or dye. The red color represents dye molecules, while the blue color represents water molecules. Diffusion is a process that dictates how particles move from one region to another, driven solely by the gradient in concentration.

Now we can see that if particles or some dye are injected at a particular location in water, at that point the concentration of the dye becomes very high. And the surrounding water has a low concentration of the dye. Now there is this gradient of the concentration of dye. This is the region of high concentration of dye, and the surrounding region has a low concentration of dye. So there is this difference in concentration, which drives this outward motion of these particles.

After some time, due to this outward motion or essentially diffusion, the molecules will spread over a larger area. With time passing, the concentration of the dye will become uniformly distributed in the entire volume of the water. Now this is the equilibrium condition, which has been achieved after a long time due to the diffusion process, which moves particles or dye from where it is concentrated to regions where it is less concentrated. This is the process in time. So, based on this idea, we come to the physical concept of diffusion.

The diffusion equation expresses the balance between the rate of change of the diffused quantity in time. In the diagram we have just seen, the diffused quantity is the particles or the dye, and the medium is water. We have looked into the diffusion process, which occurs both in time and space, as we have seen that the concentration changes from a localized concentration region to a uniformly distributed concentration over time. We also have changes in space because the gradient which is present in the initial distribution is no longer present in the final distribution.

Thus, in space, the diffusion process takes the medium from high spatial gradients to negligible spatial gradients. Thus, physically, the diffusion equation represents the diffusion of the quantity from high-concentration regions to low-concentration regions. This process is driven by gradients. Now, we look at another example. This is the movement of air molecules, or essentially wind, due to pressure differences.

This is a pressure-driven flow. Here, the gradients are present in the pressure. These are the regions of low pressure and high pressure. When air molecules or parcels of air move from high pressure to low pressure, it creates wind. But this motion is quite different from the diffusion process.

The diffusion process is to be understood only when the concentration is driving the diffusion. The motion of fluid or materials due to pressure differences is a different process. It is a pressure-gradient-driven phenomenon. Thus, it is important to differentiate between diffusive processes and pressure-gradient or force-driven processes. Thus, we have diffusion as a concentration-gradient-driven process, whereas wind is a pressure-gradient or force-field-driven process.

This should be kept in mind. Now, mathematically, the diffusion equation is one of the famous partial differential equation types. It's a canonical form of the partial differential equation that describes the time-dependent process of diffusion. It also represents the gradient spreading or gradient destruction of a quantity. These quantities can be heat, mass, or chemical compositions. Essentially, the diffusion process or molecular diffusion occurs due to the motion of the particles or molecules of the medium, which are in random arrangement, and the random motion of these surrounding molecules drives this gradient-driven diffusive process. Now, as we know that the diffusion equation is a partial differential equation, we have to represent the equation as a partial differential equation given here. This equation reads: the partial derivative of u, which is a function of x and t, with respect to time equals D times the Laplacian of u. Here, u is the quantity which is being diffused. The quantity being diffused may be a dye or a certain foreign substance which is being introduced into the surrounding medium.

t represents time. D is the diffusivity. Now, D is a property of the surrounding medium. Note that this is not the property of the suspended particles or the dye. Now, based on what the quantity is that is being diffused, the diffusivity is mentioned.

For example, in the case of heat diffusion or thermal diffusion, we have the diffusivity equal to the thermal conductivity. In the case of mass diffusion, we have the diffusivity as mass diffusivity. Or, in the case of momentum diffusion, we can have viscous diffusivity or essentially viscosity. As we know from previous classes, the Laplacian operator in Cartesian coordinates is represented by the partial derivatives with respect to individual coordinates and their sum. The Laplacian operator, as you can see, is given by $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

Now, imagine what is the effect of such a Laplacian operator on any quantity which has a gradient. The Laplacian exists and is higher for larger gradients. This is because it is essentially a double derivative of any quantity. So, if the quantity varies very fast over space—for example, in the x-direction—then the double gradient or the double derivative becomes higher. This is reflected as the first term on the right-hand side.

So, if a quantity has very large gradients, then the Laplacian will be very high, and thus the rate of change of that quantity with respect to time will also be high. For example, we can see two regions with two different gradients. The left region has a smaller derivative because the function varies slowly. While to the right, we have a sharp decrease of the function such that the gradient becomes very high. Now, the x-coordinate is towards the right, and the function is u as a function of x. Now, consider the Laplacian of this quantity, which is essentially $\frac{\partial^2 u}{\partial x^2}$, since y and z are absent in this example.

For this region, the Laplacian would be moderate. While for this region, the Laplacian will be high. If we denote these two regions as 1 and 2, We can see that the Laplacian for region 1 will be less than the Laplacian for region 2. Now, this indicates that the quantity u will change over time at a faster rate in region 2 than in region 1.

Thus, region 2 will have a greater change in u. So with time, these gradients will be destroyed, and the diffusion process will make the concentration, or essentially u, uniformly distributed or near-uniformly distributed with time. These are the profiles of u with time. We can see that the initial gradient, which existed, is now destroyed, and a more or less uniform distribution of u with respect to x is achieved with the passage of time. So that is the action of the diffusion equation, which is also controlled by the diffusivity value D. Now we will see how this D affects the diffusion equation and the process.

The derivation of the diffusion equation. We take the quantity u as a function of x and t. It can be either concentration, temperature, heat, mass, or momentum, etc. This quantity changes over time due to diffusion. Thus, we have the expression $u(x, t)$. Now, the conservation principle for the quantity u states that the rate of change of u within a small region is equal to the net flux of u into that region. The conservation principle stands because the total amount of the introduced dye or any trace of quantity Is constant, so in this region, for example, the total amount of the dye which is introduced at the center remains constant even if it disperses. So, this is before and after the diffusion process. Even though the concentration or the gradients have disappeared, the total quantity remains the same, and that is the essence of the conservation principle. Now, the conservation principle says the rate of change of u equals the net inflow of flux. For example, a concentration is introduced at the center O, and we have a closed loop C. Now, this closed loop initially does not have any of this introduced dye. Now, with time, the dye becomes dispersed or diffused.

Now, we have the diffused condition. Here, the original region C now includes some amount of the dye. This has arrived here through diffusion and has entered this loop through the diffusion process in this direction. Thus, the rate of change of u in any area is equal to the net inflow of flux of the material which is being diffused. For simplicity, let us consider a one-dimensional domain. In this one-dimensional domain, consider a small element of the length dx. Now, the total quantity here is $u\,dx$. Now, the rate of change of u in this region is given as $\frac{\partial u}{\partial t}\,dx$, which means the change of u in time over the small area dx. These arrows represent the diffusion of u from neighboring regions into this region.

That is represented by the flux $J(x,t)$. $J(x,t)$ represents the flow of u per unit area per unit time. Now, the flux at the left-hand side of the segment is $J(x,t)$, while at the right-hand side it is $J(x+dx,t)$. Since the length of this small elemental segment is dx. So, the flux difference is the net inflow of material or dye, which is $J(x,t)-J(x+dx,t)$. This is the net inflow of u into this elemental length dx. This expression can be simplified by using a Taylor expansion of $J(x+dx,t)$.

The Taylor expansion states that any quantity, such as $J(x+dx,t)$, can be expanded into a series of terms involving the quantity and its derivatives. The first term is the value at the nearby location x. The second term is the change from the nearby location, which is x. This is the base quantity. This is the linear term with the first derivative, and it denotes the change from the nearby quantity to first order. There are other terms in this expansion which represent the change from the nearby term to higher orders, but they are small since they involve dx^2 or higher powers. Since dx is a small quantity, they can be neglected.

So, we restrict ourselves to a Taylor expansion using two terms: $J(x,t)+\frac{\partial J}{\partial x}dx$. On back substitution, we get the net inflow equals $-\frac{\partial J}{\partial x}dx$. This can be derived by substituting the Taylor series expansion into the net inflow equation. We again take the help of the conservation principle. So, we combine conservation and flux.

Which states the rate of change of u equals net inflow of flux. We have the rate of change of u equals net inflow of flux. Equating the derivative coefficients, we get $\frac{\partial u}{\partial t}=-\frac{\partial J}{\partial x}$. For simplicity, one can understand that dx is a small finite length which can be neglected since it is not zero, which can be omitted from both sides since it's not equal to zero. Next, we use the constitutive law. The constitutive law for diffusive processes states that the flux J is proportional to the gradient of u. Now, constitutive laws are obtained from experimental measurements.

These are physical laws which are obtained through extensive experiments and then the proportionality is formulated. We have the proportionality constant as $-D$. Thus the flux J becomes equal to $-D\frac{\partial u}{\partial x}$. Here we term the quantity D as the diffusivity which is a positive quantity. We can use this constitutive law into our previous relation which is combining the conservation principle with the conservation and flux. We thus obtain $\frac{\partial u}{\partial t}=-\frac{\partial}{\partial x}\left(-D\frac{\partial u}{\partial x}\right)$. Since J is substituted as $-D\frac{\partial u}{\partial x}$.

This results in $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$ and D being a constant quantity is kept in the front as a constant coefficient. Now, if we generalize this to multiple dimensions, we will have the derivatives with respect to y and the derivatives with respect to z as well. Now this will be equivalent to $\frac{\partial u}{\partial t} = D\nabla^2 u$. Thus we have the Laplacian operator representing the sum of second order partial derivatives in all spatial dimensions in the diffusion equation. Thus we have the final form of diffusion equation. We next consider various application of this diffusion equation in geophysics.

First, the heat conduction in earth's crust. The diffusion equation is used to model the heat transfer through the earth's crust and mantle. Now if we consider this as a cross section sketch of the Earth's interior where we have the core, we have the lower and upper mantle and the crust. In the crust there is a heat transfer from mantle to the outside surface of the Earth. Here the temperature is not uniform.

Hence the diffusion process kicks in which is essentially heat conduction. Also we have heat conduction throughout the mantle. Now closer to the surface of the earth and in practical applications in geophysics we have the geothermal gradients. This is the gradient which drives hot water springs etc. We also have the cooling of magmatic intrusions which are smaller versions of deeper We also have heat flow in tectonic settings. All these are examples of diffusion of heat from localized concentration hotspots to the surrounding cooler medium. Next, we look into the groundwater flow. The groundwater flow contains many solutes and substances which are concentrated due to the presence of suitable rocks or minerals in certain regions. For example, let us consider the presence of groundwater below a subsurface strata.

Below the surface, there are sediments, and below that, maybe we have a reservoir of groundwater. The presence of certain rock or mineral-rich content, such as this, can concentrate solutes at a particular location. Now, these solutes, due to the gradient in concentration, will diffuse into the groundwater and may contaminate it. Thus, diffusion processes model aquifer behavior, pollutant dispersion, and groundwater management in geophysical applications. Next, we have seismic wave attenuation.

The diffusion process also attenuates seismic waves, which occurs when earthquakes emerge. Seismic waves are nothing but huge gradients in the displacements of material in the subsurface. Now, these displacements travel as waves, but diffusive processes tend to destroy these concentration gradients or displacement gradients, resulting in energy loss of the seismic waves. That is modeled as a diffusive process in geophysics. Finally, we have magnetic diffusion.

Magnetic diffusion governs the decay and evolution of the Earth's magnetic field. Since the Earth's magnetic field is a quantity that is not similar everywhere around the globe, it has gradients. Now, these gradients will involve a diffusion process and eventually lead to a decay of this geomagnetic field. Thus, from the discussion on diffusion, we obtain the following conclusions. Diffusion is the movement of particles, energy, or substances from regions of higher concentration to lower concentration.

This diffusion is driven by the random motion of the surrounding molecular media. The diffusion equation is a widely applicable partial differential equation governing various phenomena in physics and geophysics alike. The diffusion equation also provides a foundational framework for analyzing transport phenomena. These transport phenomena occur not only in geophysics but are also widespread in physics, chemical, and biological applications. Additionally, engineering systems are designed keeping diffusion processes under control.

We can refer to the following references for further understanding of diffusive processes, the diffusion equation, and the associated partial differential equations. Thank you.