Mathematical Geophysics

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Lecture - 20

Hello everyone, welcome to the Swayam NPTEL course on mathematical geophysics. We continue with module number 4, mathematical modeling part 2. This is the fifth lecture: electric current, conductivity, and Ohm's law. In this lecture, we will cover various concepts such as electric current, conductivity, and Ohm's law. We will look into the fundamental aspects of electric current, and then the concept of electrical conductivity, which is a property of matter. Next, we will look into Ohm's law for electric circuits and then Ohm's law for an electric field. The applications of these concepts in geophysics will be discussed last. So, let us begin. Electric current.

The concept of electric current is the flow of electric charges. These electric charges move through a conductor or a medium with finite electrical conductivity. The mathematical concept can be stated as follows. Here, Q is the charge, t represents time, and I is the current. The rate of flow of electric charge through a given cross-sectional area is termed the electric current.

The unit for electric current is the ampere, and it is denoted by capital A. Q is the electric charge, whose unit is the coulomb. Time is represented by t, and its unit is the second. We will look into the most important concept of current density. It is denoted by the vector $\mathbf{J}$. Now, current density is a vector quantity, while current is a scalar quantity.

The current density is related to the current as well as the area through which the current is flowing. Since the area can be represented by its normal vector, the current density is a vector quantity. The current density is the amount of charge passing through an elementary surface $d\mathbf{S}$ in unit time. This is the elementary current $dI = \mathbf{J} \cdot d\mathbf{S}$. Now, $\mathbf{J} \cdot d\mathbf{S}$ is nothing but the flux of charges.

Since an elementary area is concerned, the current is elementary. Now, $\mathbf{J}$ is the current density, which is a vector quantity. However, if $\mathbf{J}$ is tangential to the surface, then it results in zero amount of charge passing through the surface. Hence, the current passing through becomes zero, even if the current density is finite. Now, we apply the divergence theorem to this flux of charges.

The integral of the charge flux over a given volume dV equals $-\frac{d\rho}{dt}dV$ where ρ is the volume charge density. This just states that for a given volume V, the charge distribution ρ. The flux of

the charges through the bounding surface S can be related by this formula. This equation states that the flux of charges through the surface S plus the rate of change of the charge density ρ is conserved.
It can be made more clear if we write this equation as:

$$\int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_V \rho \ dV = 0.$$

This gives the conservation of charges within the volume. Now this means that the net charge content in the volume V is controlled by two sources. One is the flux of charges and the second is the material change in the volume charge density ρ which means the charge per unit volume itself changes within the volume. Thus equating the integrals we have:

$$\nabla \cdot \mathbf{J} + \frac{d\rho}{dt} = 0.$$

This is the continuity equation for the volume charge density ρ. Hence, we have the differential form of the conservation of charge. Next, we look into the second concept which is electrical conductivity. Now, electrical conductivity is not a field. It is a property of matter.

It is the measure of a material's ability to conduct electric current. Electrical conductivity refers to the presence of conducting electrons or ions in a material, which facilitates the motion of electrical charges. Hence, it allows the flow of electric current through the material. The mathematical concept behind electrical conductivity is as follows: Electrical conductivity is the reciprocal of electrical resistivity.

We know that electrical resistivity is the resistance to the motion of electrons or charged particles inside a medium. The reciprocal of electrical resistivity is electrical conductivity. Electrical conductivity is denoted by σ, which has units of Siemens per meter. Whereas, R is the electrical resistivity with units of ohm-meter. In geophysics, this property is very important.

We have explored the concept of various geophysical methods, such as the resistivity survey, where the resistivity in the subsurface is measured. We also have the magnetotelluric method, in which the motion of magnetic telluric currents within the Earth's subsurface is tracked. This depends on electrical resistivity. In this process, the apparent resistivity of the Earth's subsurface is determined.

Thus, electrical resistivity, or in other words electrical conductivity, is an important property of matter for geophysical applications. Thus, here we present the resistivity of certain geomaterials, which are materials found in geophysical applications. Essentially, they are categorized into two parts. One is rocks and sediments. The second is ores.

Rocks and sediments are found closer to the surface, whereas ores are found in deep mines. Now, let us have a look at the various examples of rocks and sediments. We have limestone, commonly known as marble. It has a resistivity of greater than 10^{12}. Next, we have quartz, which has a resistivity greater than 10^{10}.

These are very high-resistance materials, which means that electrical conductivity through them is very low. Next is rock salt, which is commonly used. It also has a high resistivity of the order 10^6 to 10^7. Commonly used granite, rocks, sandstones, etc. have a moderate resistivity.

These are of the order of thousands to ten thousands. Then we have the least resistant rock which is clay. It has a very small value of resistivity of 1 to 120. Note that the unit of resistivity is ohm-meter in all these values. Next, we look into the various ores.

Ores contain metals, minerals within them which can be extracted. The first pyrite ore has a resistance of 10^{-5} to 10^{-3} ohm meters. Whereas the zinc blend ore has a very high resistivity with values as high as 10^4. These ores magnetite and hematite are magnetic ores.

They are also used in various paleomagnetic and geomagnetic studies. These have a very less electrical resistivity which means that these are conducting materials. These ores which are dominated by metallic content have very less resistivity compared to the surrounding rocks and sediments which occur in geological strata. Thus, we can understand from this that in the subsurface, the surrounding area which consists mostly of rocks and sediments has a high resistivity which means lesser electrical conductivity. Whereas the presence of ores has a low electrical resistivity.

This low electrical resistivity is what geophysical methods are after. Now, using various electrical field-based methods, these ores are detected. The measurements used in geophysical methods help detect these ores due to their anomalously low electrical resistivity compared to the surrounding high-resistivity rocks and sediments. For example, if this is the surface of the Earth, we have highly conductive bodies, which are ores, surrounded by material that is mostly rocks and sediments.

Using various geophysical methods, these anomalous structures are detected, and this is the heart of mineral exploration in geophysical applications. Next, we look into Ohm's law for electric circuits. Now, in Ohm's law for electric circuits, we have the fundamental principle of potential or voltage V. This can be related to the potential function for the electric field. Now, in electric circuits, we can have the flow of charges, which can be represented by the current I and the material property of resistance, R. Mathematically, Ohm's law for electric circuits can be written as:

$$I = \frac{V}{R},$$

which means that the electrical current is proportional to the voltage and inversely proportional to the resistance. This is famously known as Ohm's law.

There are certain conditions for Ohm's law. The material must exhibit ohmic behavior, meaning the resistance remains constant regardless of voltage or current. In the material, when current flows, the electrons or the conducting elements themselves become perturbed, which may lead to a change in the material's resistance. However, ohmic behavior assumes that the resistance remains constant regardless of the passage of current or changes in voltage. Additionally, the temperature and physical conditions must remain constant.

such that the resistance does not vary with these factors. The temperature conditions maintain the state of the matter also the energy levels of the participating electrons in current flow whereas other physical conditions for example pressure also determines the ohmic behavior condition. Whereas other physical conditions such as pressure if changed to a large extent may affect the ohmic behavior. Now we look into the Ohm's law for an electric field. We have seen that the electric field can be represented using the potential function V. The Ohm's law for electric field is a fundamental relation between the current density and force per unit charge.

The current density is represented by **J** vector whereas the force per unit charge is represented by **F** vector. **F** vector is the electric field. Now when a conductor which is a body having finite conductivity is moving with a certain velocity **u** through a magnetic field **B**, the electromagnetic force generates electricity. By electricity we mean the potential difference or electromotive force. This electromotive force drives the charges.

Hence the current density is given as:

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}),$$

this is the current density, this is the electrical conductivity, this is the electromotive force field which is already applied plus we have the additional electromotive force field due to the motion of the charges themselves. This is given by the cross product of the velocity field with the magnetic field. Now this is the induced field due to the motion of charges inside a conducting medium.

For stationary charges, velocity goes to zero, and we obtain Ohm's law in a field form. This is Ohm's law, which states that the current density equals conductivity multiplied by the applied electric field:

$$\mathbf{J} = \sigma\mathbf{E}.$$

This is equivalent to the current being equal to the potential voltage V divided by R, since R is the resistance, which is inversely proportional to conductivity.

The equation $I = \frac{V}{R}$ is equivalent to Ohm's law $\mathbf{J} = \sigma\mathbf{E}$, as the current density can be represented by I, whereas the electric field **E** vector can be represented by its potential function V, and conductivity is inversely proportional to resistance, hence $\frac{1}{R}$. Thus, we have Ohm's law for the electric field. Next, we look into the details of applications of these concepts in geophysics. The concepts of electric current and resistivity are extensively used in geophysics for subsurface exploration. As discussed earlier, we have resistivity methods and magnetotelluric methods, which are various techniques in geophysics for subsurface exploration and monitoring. They provide insights into the Earth's interior.

Also, electric current and resistivity help determine fluid flow in the Earth's outer core. In Earth's outer core, the electric current is induced by the motion of charges, and it generates the magnetic field. The mineral composition is part of surface and near-surface geophysical exploration. In these methods, electrodes are placed at two different points. Between these electrodes, an electric current is induced into the ground.

Now, this current is induced due to an applied potential difference or voltage across these electrodes. Ohm's law is used to calculate the resistivity of the subsurface by using the relationship between current, voltage, and the resistance of the material in the subsurface. This helps to understand the resistivity field or the resistivity distribution in the subsurface. Any anomalous resistivity detects the presence of highly conducting materials, which are the ores, and these ores are then located and mined for valuable resources. We also have the induced polarization method as one of the most widely used geophysical methods for mineral exploration.

Ohm's law helps to interpret the measured voltage and current, which flows in the subsurface, to identify zones of high chargeability or essentially the regions where electric flow is easy. This indicates the presence of ore bodies, such as disseminated sulfides. As ore bodies have higher conductivity, their presence is detected by high chargeability.

Thus, we come to the conclusion of this present lecture. First, electric current is a fundamental quantity that models the flow of electrical charges. This is used in various resistivity surveys, induced polarization, and cell potential methods to explore subsurface properties. The electrical current is also induced in the outer core, where fluid motions in conducting iron in liquid form give rise to magnetic fields. These magnetic fields also give rise to electric current, and hence this continuous loop helps to magnify the electromagnetic field and results in the geomagnetic field of the Earth.

The electrical conductivity is also utilized in resistivity-based techniques such as electrical resistivity tomography. This is used for detecting groundwater, mapping mineral deposits, and monitoring contamination in groundwater. Finally, we have the application of Ohm's law. Ohm's law is a principle that states the relationship between current, voltage, and resistance. It provides insight into the high or low chargeability of current flow in a particular region. In geophysical methods such as resistivity surveys and induced polarization, Ohm's law is applied to calculate resistivity and interpret subsurface features. Thus, it can be said that these concepts—such as electric current, electric potential, Ohm's law, and electrical conductivity—are useful tools in understanding various geophysical methods and their applications for mineral and resource exploration. One can refer to the following references for more details. Thank you.