Mathematical Geophysics

Swarandeep Sahoo

Department of Applied Geophysics

Indian Institutes of Technology (Indian School of Mines), Dhanbad

Week - 01

Lecture - 02

Hello everyone, welcome back to the SWAYAM NPTEL course on mathematical geophysics. We continue with module 1, the basic concepts, and this is lecture number 2 titled Coordinate Systems. The concepts covered in lecture number 2 are as follows. We will be looking into number lines, followed by the Cartesian coordinate system, the polar coordinate system, and then we will move into curvilinear systems such as the spherical and cylindrical coordinate systems. Now, coordinate systems are the way to represent the spatial and temporal distributions of various geophysical quantities, vectors, and scalars, as we have discussed in the previous lecture.

We start with the first concept, which is number lines. Now, number lines are the simplest representation of a coordinate system. The number lines simply denote the magnitude and the sign of a given physical quantity. You can see the number lines given below in this figure. The number lines consist of the positive quantities denoted by blue and negative quantities denoted by red. As you can see, number lines are best suited for representing scalar quantities, which have only a magnitude but no direction. So, this is an example of a coordinate system that is suited only for scalar quantities. So, the coordinate of any point P, which is defined on this number line, is the distance from the origin O towards the right or towards the left. So, any point P, let's say, can be considered as Like this, where the distance can be given by the arrow indicated. So, each point is given a unique coordinate, which consists of a single number, and that number represents the coordinate of the position of the quantity. Next, we look into the coordinate systems which are suitable for vector quantities. The simplest of the coordinate systems suitable for vector quantities is the Cartesian coordinate system. The Cartesian coordinate system is prevalent among various physical and scientific phenomena and is used to write the equations governing these phenomena.

The Cartesian coordinates specify a point's location in two or more directions. And hence, they are two or three-dimensional space coordinate systems. The coordinates of a two-dimensional Cartesian system are represented as the abscissa and ordinate set. Usually denoted by x, y. And the distance of any point P from the origin is given by the $D = \sqrt{x^2 + y^2}$.

This can be seen from the adjacent diagram. So here is the origin, and any point P is given by a set of two numbers, in this example, -3, 2. This is the -3 abscissa. This is the +2 ordinate.

So, this is the two-dimensional representation of any point P in a Cartesian coordinate system. We will look into some other properties of the two-dimensional Cartesian coordinate system as shown in this diagram. The two-dimensional Cartesian coordinate system can be divided into four quadrants. Quadrants 1, 2, 3, and 4 are shown in this figure. You can see the sign of various quantities in different quadrants.

For example, In the quadrant on the bottom left, numbered as quadrant 3, the abscissa as well as the ordinate is negative. While on the bottom right, in quadrant 4, the abscissa is positive while the ordinate is negative. Moving on to a three-dimensional version of the Cartesian coordinate system, we have the representation of any point as a triplet x, y, z. Then, the corresponding distance of any point from the origin is given by the sum of squares of these abscissa and ordinates followed by a square root. This is clearly observable from the adjacent diagram.

$$D = \sqrt[2]{x^2 + y^2 + z^2}$$

This is the origin of the three-dimensional Cartesian coordinate system. Any point is given as a triplet XYZ with the three axes denoted by the X, Y, and Z axes. The planes are shown in light green color. These planes are defined such that one of the coordinates remains constant over the entire plane. Moving on to the next coordinate system, we have the polar coordinate system.

Now, in geophysics, Cartesian coordinate systems are used for local approaches where a small area is under study and for systems which are models of relevant geophysical phenomena. However, due to inherent geometrical features such as curvature and other characteristics, curvilinear coordinate systems are more preferable. The polar coordinate system is usually one of the choices for representing geophysical quantities.

As the name suggests, it has wide utility in demonstrating or representing polar-based quantities. In geophysics, the polar coordinate system is useful in representing various phenomena. But first, let us look at the basic building blocks of the polar coordinate system. This diagram shows the polar coordinate system and the associated notations. Suitable for vector quantities, the position vector in the polar coordinate system is represented as P(r, θ) coordinates for this point P. Now, r is the distance from the origin, and θ is the angle from the x-axis. Now, the x-axis is actually a part of the Cartesian coordinate system, while in the polar coordinate system, it is represented as θ = 0. So, any point P has two identities, which are the distance from the origin and the angle it has to be rotated from the $\theta = 0$ axis. The $\hat{r}$ and $\hat{\theta}$ are unit vectors which are perpendicular to each other at any point P. Here, the box highlights the relation between the polar coordinate system and the Cartesian coordinate system.

The distance from the origin is the r = $\sqrt{x^2 + y^2}$ and The angle is the $\theta = \tan^{-1}\left(\frac{x}{y}\right)$.

So, in the polar coordinate system, The radial coordinate measures the distance from the pole, while the angular coordinate, also known as azimuth, measures the angle.

Now, it is important to understand the range of values of these coordinates. In the partition coordinate system, the range of each of x, y, and z coordinates varies from 0 to ∞, while in the polar coordinate system, r goes from 0 to ∞, however, the range of the angular coordinate is from 0 to 2π. In geophysics, the polar coordinate system is used in various aspects. Some of them are listed here.

The polar coordinate system is widely used in seismic wavefront analysis, which is used for exploratory activities. It is also used in groundwater and fluid flow studies, which are very important in geophysical applications. The polar coordinate system is also used in radar and remote sensing applications for various surveying and sensing purposes. Now, we look at the spherical coordinate system. The spherical coordinate system is the most appropriate and relevant for use in geophysical applications.

This is Obvious from the structure of the coordinate system, as can be seen in this figure. Since the planet Earth and other planets are spherical bodies, many of the phenomena are governed by the sphericity or the curvature of the planet. Many geophysical applications are affected by the geometrical shape of the Earth.

The resulting phenomena can be more appropriately and conveniently expressed in a spherical coordinate system rather than Cartesian or polar coordinate systems. Although other coordinate systems are also used for some specific applications. But in general, the spherical coordinate system is preferable for geophysical applications. This diagram shows the origin 0. Any point is represented by three quantities. The point P has the coordinates r, θ, and $\emptyset$. Where r is the radial distance, θ is the polar angle, and $\emptyset$ is the azimuth angle. Now the range of these quantities are r goes from 0 to ∞ , while the polar angle θ is restricted to 0 to π. The azimuth angle ranges from 0 to 2π. Now take a look at the diagram.

This is the unit vector r. This is the unit vector of the polar triangle. This is the unit vector of the azimuth angle. The vector representation of any point P can be given in terms of its components as shown here. The $\vec{p}$ is $\vec{p} = P_R\hat{R} + P_\theta\hat{\theta} + P_\emptyset\hat{\emptyset}$ So this is the vector representation of a point.

Corresponding infinitesimal displacements are given by dL_r, dL_θ,and $\mathrm{dL}_\emptyset$, where these are given by a differential in the radial coordinate, a differential in the polar angle, and a differential in the azimuth angle. It is important to note the difference in the infinitesimal displacements as represented in the spherical coordinate system compared to the Cartesian coordinate system. In the Cartesian coordinate system, the infinitesimal displacements would be dx, dy, and dz. But in the spherical coordinate system, the infinitesimal displacements are given by dr. For the polar angle, the displacement is d_θ weighted by r. This occurs because at different distances from the radius, the size of the spherical surface becomes larger and larger.

So compared to the Cartesian coordinates, the planes which were shown in previous slides, these planes upon displacement along any of these axes do not change their area. However, in the spherical coordinate system, as we move outward from the center of the origin of the spherical coordinate system, the spherical surfaces change their area. Such changes in the area as one moves from the center of the spherical coordinate system outward are evident in the infinitesimal displacements. The rd_θ and rsin $\theta\ d_\emptyset$. Now, similarly, the infinitesimal area at particular radial locations is given by dL_θ,$\mathrm{dL}_\emptyset$, which becomes $r^2 \sin\theta\, d_\theta d_\emptyset$. Corresponding Cartesian systems will have an infinitesimal area that can be represented by dx dy or dx dz. So, it can be understood that the spherical coordinate system takes care of the changes in the spherical surfaces centered at the origin.

As one moves from the origin of the spherical coordinate system. Likewise, the infinitesimal volume can be given as the product of all three infinitesimal distances or displacements, resulting in $r^2 \sin\theta\, d_\theta d_\emptyset$. This is an elemental volume in the spherical coordinate system, which, in a Cartesian coordinate system, would have simply come from the product dx dy dz. So, it is important to recognize this difference, as this will be very useful in understanding geophysical applications. In geophysics, the spherical coordinate system is widely used because it describes the physical phenomena and the resulting dynamics that occur on the spherical surface of the earth.

Also, in the deep interior of the earth, as we had seen in the previous lecture, the structures are governed by the sphericity of the outer core or the mantle. So, you can see here that the inner core, as well as the outer core and other layers of the earth, are nothing but spherical concentric shells. And phenomena which are occurring in these layers are very much affected by the sphericity of the earth. So, this propels one to use the spherical coordinate system instead of the Cartesian coordinate system for geophysical applications. In particular, the earth's gravitational field is very well described by the spherical coordinate system instead of the Cartesian coordinate system.

A related structure is the geoid, which describes the shape of the Earth. Also, the Earth's magnetic field is described using spherical coordinate systems rather than Cartesian coordinate systems. We will look into more details of each of these geophysical aspects in upcoming lectures. Now we will go through the vector operations in the spherical coordinate system.

Remember, we have looked into various vector operations briefly in the previous lecture. Now we are going to look into the vector operators in terms of the spherical coordinate system, which is very important for the calculation of different geophysical quantities as they arise in nature. First, let us look at the relation between Cartesian coordinates and spherical coordinates as given here. So, x is equivalent to $r\sin\theta\cos\emptyset$ y is represented by $r\sin\theta\sin\emptyset$, and the z coordinate in Cartesian can be represented by r multiplied by the cosine of the polar angle θ. In geophysical applications, it is imperative to understand the following operators: the gradient, divergence, curl, and Laplacian. The gradient of a scalar function is a vector defined as the partial derivatives of the scalar f with respect to r,θ, and $\emptyset$ displacements, appropriately weighted by the infinitesimal displacements. Similarly, the divergence of any $\vec{U}$ along the three spherical coordinates is given as below.

This results in a scalar quantity. The curl of a vector in the spherical coordinate system is given as below. The curl of U equals the right-hand side, which gives the individual radial, polar angle, and azimuthal angle components of the curl of the $\vec{U}$. Interestingly, we can see that the curl of the vector is weighted by $r\sin\theta$ for the radial component, while the polar and azimuthal angles are weighted by simply r. Now, the Laplacian of any vector is given by the Laplacian of u as defined in this expression. Note that the curl of a vector is always a vector, while the Laplacian is a scalar quantity.

So, the gradient, divergence, curl, and Laplacian are very useful concepts and are commonly used in geophysical applications. Next, we move on to the cylindrical coordinate system. Like the spherical coordinate system, the cylindrical coordinate system is a curvilinear coordinate system that is useful for various geophysical applications. Similar to the spherical coordinate system, the angular coordinate $\emptyset$ is represented as shown in the figure. You can see that the cylindrical coordinate system is somewhat an amalgamation of the Cartesian and spherical systems.

It has the characteristics of the Cartesian coordinate system along the z-direction and the characteristics of the spherical coordinate system in the azimuth and radial directions. We have the resulting planes similar to the Cartesian coordinate system and cylindrical surfaces that change in area along the radial direction. So, the vector representation of any quantity is shown here. And in the Cartesian coordinate system, the x, y, and z coordinates are related to the corresponding cylindrical coordinate systems as

$$x = \rho \cos \theta, \text{ y} = \rho \sin \theta \text{ and z equals z}$$

Here, ρ is used instead of r for clarity. The infinitesimal displacements are shown here.

Note that only the azimuth angle is weighted with ρ, while the radial and axial coordinates are similar to the Cartesian coordinate systems. The infinitesimal area at any particular radial location is given DA_{ρ}. This is the product of infinitesimal displacements along the angular and the axial directions. Given like this. The infinitesimal volume is given by the product of the three infinitesimal displacements, that is ρ, d$\emptyset$, dρ, and dz. The origin of the cylindrical coordinate system is denoted by O, and any point P can be denoted by the three components of the cylindrical coordinate systems: ρ, $\emptyset$, and z.

Now, the cylindrical coordinate system is also commonly used in geophysics, such as modeling volcanic structures, subsurface fluid flows, and earthquake source studies. So these applications, in some way or other, have a geometrical resemblance to a cylinder or are dominated by the cylindrical geometry of the occurrences. Thus, these geophysical aspects are known. More suitable for a representation in cylindrical coordinate systems. The corresponding vector operators in the cylindrical coordinate system are shown below. As we have seen in the previous slide, the relation between Cartesian and cylindrical coordinates is given. The axial direction is identical to the Cartesian system. The gradient of a scalar function is given as this expression, which becomes a vector quantity. Similarly, we have the divergence, curl, and the Laplacian operators of a vector expressed in cylindrical coordinate systems. So note that the cylindrical coordinate system is similar to the Cartesian coordinate system along the axial direction and similar to the spherical coordinate systems in other coordinates, such as the angular and the radial directions.

Thus, from this lecture, we can conclude that The geometrical representations involving measurements and directions in space and time need to use appropriate coordinate systems. These geometrical and mathematical aspects have to be considered while representing any geophysical applications, which may be either local or global. Local geophysical applications are best suited for representation in Cartesian coordinate systems, while global phenomena that span the entire planet are more suited for representation using spherical coordinate systems. The cylindrical coordinate system offers a middle path with lesser complexity than the spherical coordinate system.

But it is more relevant for geophysical applications than the Cartesian coordinate systems. Thus, we end the lecture on coordinate systems, and we will look into more details in upcoming lectures and applications on geophysical aspects. Thank you.