Mathematical Geophysics
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Lecture – 19

Hello everyone. Welcome to the SWAYAM NPTEL course on Mathematical Geophysics. We continue with Module 4, Mathematical Modeling Part 2. This is the fourth lecture: Electric Potential. In this lecture, the concepts covered are electric potential and various applications and derivations of the electric potential.

The basic concept of electric potential is covered first. This is followed by Poisson's and Laplace's equations. Next, we consider the electric field outside a solid sphere. Next, we have the electric potential inside and outside a spherical shell. We will be looking into both the field and the potentials.

Next, we have the boundary condition on the electric field and the boundary condition on the electric potential. So, let us begin. First, the electric potential. We have discussed in the previous lecture that the electric field is a conservative field. This is because the curl of the electric field equals zero, $\nabla \times \mathbf{E} = 0$, having been obtained from the Stokes theorem applied to the circulation.

This gives the line integral of $\mathbf{E}$ around any closed loop equals to 0, $\oint \mathbf{E} \cdot d\mathbf{l} = 0$. Thus, we can have a potential function for this field. This potential function is called the electric potential. The electric potential relative to a reference point is V, our vector, given as $V(\mathbf{r}) = -\int_0^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$. Now this is with respect to the point 0 such that $V(\mathbf{r})$ is the work done by a unit charge at 0 to reach the location $\mathbf{R}$. We can obtain the mathematical representation of this potential function.

Let us look into the electric potential as the difference between two points A and B. If A and B are as shown here with the location reference as 0, we can have the electric potential at B and A given by $V(B) = -\int_0^B \mathbf{E} \cdot d\mathbf{l}$ and $V(A) = -\int_0^A \mathbf{E} \cdot d\mathbf{l}$ where the $\mathbf{r}$ is substituted by $\mathbf{b}$ and $\mathbf{a}$ respectively. The integral from 0 to A plus the integral from 0 to B. This results in $\int_A^B \mathbf{E} \cdot d\mathbf{l}$ with minus sign instead of plus sign by reversing the integral exponents. Now the combining this we obtain $V(B) - V(A) = -\int_A^B \mathbf{E} \cdot d\mathbf{l}$.

This is nothing but the circulation from A to B or the line integral of $\mathbf{E}$ along any path joining A to B. Since this is a conservative field, the path is not a factor. As $\mathbf{E}$ can be written as gradient of V, where V is the potential, we can have $\int_A^B \nabla V \cdot d\mathbf{l} = \int_A^B \mathbf{E} \cdot d\mathbf{l}$. This is possible as $\mathbf{E}$ is a conservative field which can have a potential function and $\mathbf{E}$ can be written as a gradient of the scalar potential V. Now, using this potential function formulation, we can look into the equations that govern the potential function V. First, we will use the Gauss law.

The Gauss law for electric field states that the divergence of the electric field equals $\frac{\rho}{\epsilon_0}$, $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$. Replacing $\mathbf{E}$ by $-\nabla V$, we can obtain the Laplacian of V equals $-\frac{\rho}{\epsilon_0}$, $\nabla^2 V = -\frac{\rho}{\epsilon_0}$. This follows from the vector identity that is divergence of a gradient of any scalar function V equals Laplacian of that scalar function. This is the Poisson's equation. In the case where there is no charge that is $\rho = 0$, the Poisson's equation becomes the Laplace's equation, $\nabla^2 V = 0$.

Next, we look into the electric field which is outside a solid sphere which is having a charge distribution. Now, this solid sphere is considered to be having a uniformly charged distribution. The sphere has a radius R. The dark grey circle represents the sphere with radius R. Now the total charge distribution on the surface is equal to Q. To measure the electric field outside such a uniformly charged sphere of radius R is the aim of the next exercise. For that, let us consider another hypothetical concentric spherical surface located at a distance r which is greater than R. This sphere is colored lighter gray.

Now at any point on this sphere we can have the outward vector $d\mathbf{S}$ vector representing the elemental surface area dS. We also have the electric field along this $d\mathbf{S}$ vector. Now Gauss law states that electrical field dotted with $d\mathbf{S}$ and integrated over the entire surface area is equal to the enclosed charge by ϵ_0, $\oint \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{enc}}}{\epsilon_0}$. In this case the enclosed charge equals Q. We also use the symmetric property of the sphere.

At any point on the sphere, the electric field is going to be the same because such points are all equidistant from the charged sphere. Thus, we can conclude that the electric field can have a similar magnitude for all these points. The difference is that the $d\mathbf{S}$ vector points in different directions at each of these points. Thus, the integral $\mathbf{E} \cdot d\mathbf{S}$ simplifies to the magnitude of the $\mathbf{E}$ vector and the integral of dS. This equals $\frac{Q}{\epsilon_0}$, where Q is substituted in place of the Q_{enc}.

The surface area of the sphere is $4\pi r^2$. This is the integral of dS. Now, substituting that, we have $E = \frac{Q}{4\pi\epsilon_0} \cdot \frac{1}{r^2}$. This is the magnitude of the electric field. The direction of the electric field is always outward in the radial direction.

That is the direction of the electric field at any point on the surface of the sphere. Next, we consider the electric potential inside and outside a spherical shell. This diagram shows a spherical shell of radius R. The point P is the test point, which is located outside the spherical shell. Now, this point P is at a distance r from the center of the spherical shell. The r is greater than R for the case where P is outside the spherical shell.

This case is considered first. The electric field is given by $\frac{Q}{4\pi\epsilon_0} \cdot \frac{1}{r^2}$. This is because the electric field expression has been obtained for a charge distribution as $\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{dQ}{r^2}$. This integral dQ becomes equals to Q for this case. This means that the total charge distribution on the sphere is equals to Q. Now, the potential for the points outside the sphere is the work done by the charges to take a unit charge from the reference point 0 to r. Thus, $V(r) = -\int_0^r \mathbf{E} \cdot d\mathbf{l}$ which is equals to $\frac{1}{4\pi\epsilon_0} \int_\infty^r \frac{Q}{r^2} dr$. This gives $\frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r}$. The reference point is chosen at infinity. Thus the potential is essentially bringing a point charge from infinity to r.

The field inside the shell is now considered. Since inside the shell the field is equal to 0, we can derive the potential for points inside the sphere as $V(r) = -\int_{\infty}^{r} \mathbf{E} \cdot d\mathbf{l}$. Now this equals $-\frac{1}{4\pi\epsilon_0}\int_{\infty}^{R} \frac{Q}{r^2} dr - \int_{R}^{r} 0 \cdot dr$. This comes from the absence of any charge inside the spherical shell. This gives the $V(r) = -\frac{1}{4\pi\epsilon_0}\int_{R}^{r} \frac{Q}{r^2} dr$. Remember that r is less than R for a point like this. Thus we have the final expression for the potential as $\frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{R}$. Now this is a constant quantity as long as the Q remains same. Thus we can say that the potential for all the points inside the sphere is constant. There is no potential difference inside the spherical surface.

This indicates that the gradient of the potential inside the spherical surface is equal to zero. This is the same as the field being zero inside the shell. Thus, we reinforce the idea about the absence of a field and hence a constant potential, which is equivalent to the case of the field inside a spherical shell. Next, we will discuss the boundary conditions on the electric field. The boundary conditions on the electric field refer to the conditions that may be imposed on the boundary which encloses the region where the electric field exists.

For example, let us consider a thin sheet which extends over the edges in both directions. This is shown in this diagram. σ is the charge density. We have the surfaces which are above this thin sheet and the surface which is below this thin sheet. Thus, this thin sheet acts as a boundary between the space above and the space which lies below this thin sheet.

Consider an area A on this thin sheet. The charge density, which is the surface charge density σ, is uniformly distributed on the surface of the sheet. Thus, Gauss's law gives the integral of $\mathbf{E} \cdot d\mathbf{S}$, which is nothing but the flux of the electric field, equals $\frac{Q_{\text{enc}}}{\epsilon_0}$, that is equal to $\frac{\sigma A}{\epsilon_0}$. σA is the total charge.

Now we will look into the boundary condition in two directions. First is the normal component. Second is the tangential component. The normal components are directed along this direction, which is perpendicular to the thin sheet. The sides of the thin sheet contribute nothing to the flux because the sheet is extremely thin with ϵ that is the thickness of the sheet going to zero. We have the electric fields above and below forming the flux. This is because the electric field above and below is the difference in the flux which is crossing through this thin sheet. From Gauss law this difference can be equal to $\frac{\sigma}{\epsilon_0}$. This also comes because integral dS equals A.

One can write integral dS as the difference multiplied by A upon equating this to this expression upon substituting this into this expression. We have this expression where the difference in the electric field above and below this thin sheet equals $\frac{\sigma}{\epsilon_0}$. This implies that the normal component of the electric field is discontinuous across a thin sheet carrying a charge density σ. This discontinuity is by an amount $\frac{\sigma}{\epsilon_0}$ at the boundary. Next, we consider the tangential components.

We will consider the line integral of the electric field around the rectangular loop as shown here. This is the same thin sheet with charge density σ. Here we are considering a rectangular loop that goes above and below the thin sheet. This rectangular area has a length L and a thickness slightly greater than the thin sheet, such that the rectangle extends above and below the thin sheet.

If we consider the integral $\mathbf{E} \cdot d\mathbf{l}$ across this entire rectangular loop, it goes to zero because the loop has arms in the space above and below the thin sheet where no charges exist. Remember that the charges are only located on the surface of the thin sheet. The sides of this loop do not contribute to this integral in the limit as ϵ tends to zero. Thus, the tangential components, indicated by the parallel superscript, are obtained as $E_{\text{above}}^{\parallel} \cdot L - E_{\text{below}}^{\parallel} \cdot L = 0$.

This provides a better understanding where the loop encloses the thin sheet with a charge distribution σ. These regions are taken into account for the integral, but they do not contain any charges. Hence, $E_{\text{above}}^{\parallel} - E_{\text{below}}^{\parallel} \cdot L$ goes to zero. Thus, we have our boundary condition for the tangential component, which shows that the tangential component of the electric field is always continuous across any surface with a charge distribution σ. Thus, the total boundary conditions on an electric field are $\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$.

Here, $\hat{\mathbf{n}}$ is the unit vector normal to the surface. Since the tangential component is absent, the resultant boundary condition is directed along the normal to the surface area. This gives the overall idea of the behavior of the electric field on boundaries. Next, we consider the corresponding boundary conditions for the electric potential. We have the same thin sheet with charge distribution σ.

We consider two points, A and B, which are located slightly above and below the thin sheet. The potential distribution is such that the difference of potential between these two points that is, $V_{\text{above}} - V_{\text{below}}$, can be given as $-\int_A^B \mathbf{E} \cdot d\mathbf{l}$. This is nothing but the circulation of the electric field from A to B because the potential difference is such that the potential is the work done in bringing a charge from infinity to point A. Hence, the difference is just taking the charge from A to B, which is nothing but $\mathbf{E} \cdot d\mathbf{l}$ integrated from A to B. Now, for thin sheets, ϵ goes to zero.

That means the length or the distance of separation between A and B shrinks to zero. Thus, A and B become coincident points. which makes the potential equal. Thus, the boundary condition for the electric potential implies that the potential is continuous across any boundary. Even if the boundary has a charge distribution of density σ.

Now we consider both boundary conditions on the electric field and the electric potential. The electric field boundary condition and the electric potential boundary condition, when combined, can give the potential gradient difference equals $-\frac{\sigma}{\epsilon_0}$. This is obtained just by substituting the $-\nabla V$ in place of $\mathbf{E}$. This is written as the normal derivatives of the potential above and the normal derivative of the potential below, the difference equals $-\frac{\sigma}{\epsilon_0}$. In other words, the gradient of V is discontinuous across the boundary.

Thus, we can conclude that the electric potential, which results from the conservative property of the electric field, can be widely used in geophysical applications. The geophysical applications that use electric potential for their understanding and interpretation are locating groundwater, mineral deposits, hydrocarbons, and geothermal reservoirs. The electric potential is also instrumental in techniques like magnetotellurics, which explore shallow depths to deep crustal structures. Essentially, the potential of the electric field is induced due to artificial methods in geophysical systems to understand subsurface properties.

The mapping is obtained from the distortion of this potential due to the presence of anomalous bodies having different material properties, which are electrical in nature from the surroundings. Thus Electric potential can be used for various geophysical applications that depend on subsurface bodies having anomalous changes in the electrical properties from the surroundings. One can refer to the following references for more details on the electric field, electric potential, and other applications in geophysics. Thank you.