Mathematical Geophysics
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Lecture – 18

Hello everyone. Welcome to the Swayam NPTEL course on Mathematical Geophysics. Today, we will continue with Module 4: Mathematical Modelling, Part 2. This is the third lecture: Electric Fields. In this lecture, the concepts covered pertain to electric fields.

The five components of this lecture are as follows. First, we will look at the basic Coulomb's Law. Second, we will discuss the electric field for a discrete charge distribution. Third, the electric field for a continuous charge distribution. Next, we will discuss the divergence of the electric field and the curl of the electric field.

Now, these concepts have their corresponding applications in geophysics, which we will also discuss. So, let us begin. Coulomb's Law Coulomb's Law describes the electric field due to a point source charge, Q. The test charge, Q, and the point source charge, q, are placed such that the distance between them is r. Here, q and Q are the two point charges shown in red and blue. Our aim is to determine the force on the test charge Q due to the single source charge q at rest and at a distance r away.

The force is found to be proportional to the product of the two charges and is also found to be inversely proportional to the square of the separation distance, which means that Force is proportional to Q and Q and inversely proportional to the separation distance. This is mathematically shown here. Now, this is based on the experiments conducted by Charles Augustine de Coulomb in the 1780s. The force is given by **F** vector which is the force field. equals $\frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\hat{\mathbf{r}}$. This is a constant quantity where ϵ_0 is the electrical permittivity of free space. Its value is 8.85×10^{-12} $C^2/N{\cdot}m^2$. Its value is 8.85×10^{-12} Coulomb^2 Newton^{-1} meter^{-2}. This is in SI units.

Now here we have the product of the two charges. This is the separation between the charges. This is the unit vector pointing along the separation distance. Having understood the Coulomb's law, now we move on to the electric field that is formed by a discrete charge distribution. By discrete charge distribution, we mean that the charges are point sources which are located at distinct locations without any connection between them.

The several point charges are denoted by Q_1, Q_2 up to Q_n. These are at corresponding distances R_1, R_2, etc. from the field point P. Now, this is clarified in this diagram. We have the Cartesian coordinate system with origin at O and the XYZ axes.

The field point P is the point where the electric field is to be estimated. The red dots are discrete charges. One among them is q_i, that is the source charge highlighted here. In general, these charges are denoted by q_i. Now, for the general charge q_i, the distances are subscript i. With this in mind, we can look into the total force as $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots$.

These are the individual electric fields or electric force fields, which are produced by the individual point source charges on the field point Q. We have used P and Q interchangeably here. The **F** vector now corresponds to the sum of the individual source fields. Having taken the component, which is constant outside, as a common factor, we have the rest of the expression given by $Q_1 \frac{Q}{R_1^2}$, which is R1 + $Q_2 \frac{Q}{R_2^2}$, etc.

This simplifies to $\mathbf{F} = \frac{Q}{4\pi\epsilon_0}\left(\frac{Q_1}{R_1^2}\widehat{\mathbf{R}}_1 + \frac{Q_2}{R_2^2}\widehat{\mathbf{R}}_2 + \cdots\right)$. Now, having performed the vector addition, we get the resultant direction of the electric field **E**. Thus, the force is given as Q multiplied by the electric field, that is, $\mathbf{F} = Q\mathbf{E}$. Thus, **E** can be summarized as $\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{R_i^2}\widehat{\mathbf{R}}_i$. Upon performing the vector addition, we will get the resultant direction of the electric field **E** vector.

Now, the **E** vector, as can be seen from this expression, is the electric field due to the source charges. It depends only on the source charges. It does not depend on the test charge Q. Thus, the electric field is proportional to the source charge distribution and inversely proportional to the square of the separation distances between the source charge and the test charge. Now, we will have a look at some special cases of the electric field distribution for discrete charges. For simplification, let us consider that the total number of point charges is limited to two. Now, these two charges are placed at a certain distance between them. They form a charge couple. Note that the charges are of the same sign. Both are positive charges. The field lines are shown here.

Recall that we have looked into the field line concept in the geometric representation lecture, and we have seen that the field lines interact with each other and get distorted from the original individual and independent field lines, which can be caused by the individual point charges. This results in distortion such that the effective field lines become as shown here. Now suppose the charges are of opposite sign. We can see the dramatic change in the geometric representation of the field lines. This is a dipole configuration where we can see the field lines emanating from the positive charge.

and converging on the negative charge. Whereas in the case of the coupled charges, the field lines tend to move away from the opposite charge field lines. We can also analyze various other forms, such as the curl and divergence of the field lines, as we will look into later in this lecture. Next, we have the electric field of a continuous charge distribution. A continuous charge distribution means that we have charge at every point in space.

So, to denote such a charge distribution, we need some concept of continuous charges. Thus, we have the assumption that charge is distributed continuously over some region. Now, we look into the electric field, which is obtained for such continuously distributed charges. With the help of these diagrams, it would be very simple to understand this. The gray-shaded area is a continuous distribution of the charges, where an elementary charge is shown as dq.

Now, this continuous distribution is at a distance $\mathbf{r}$ from the test point P. Our aim is to find the electric field generated due to the continuous distribution. As we have seen earlier, this electric field does not depend on the test point P. It only depends on the charge distribution and the separation between the charges and the test point, that is R. Along the same lines as a point source charge distribution, we can obtain the electric field for a continuous distribution of charges. Now, that is given by $\frac{1}{4\pi\epsilon_0}\int \frac{dq}{R^2}\hat{\mathbf{R}}$. Now, we can understand this from a simple analogy.

Have a look at this expression. We have the common factor $\frac{1}{4\pi\epsilon_0}$, followed by a summation. Now, this summation occurs because the charges are discrete points. For a continuous charge distribution, the summation sign is replaced by the integral sign. This is because now the charges have essentially become a conglomeration of elemental and infinitesimal charges dq.

Thus, the integral of $\frac{dq}{R^2}$ is essentially a representation of a discrete source charge distribution, as shown here. We can go further to examine some simplified and special cases. First, a line charge distribution. A line charge distribution can be depicted in diagrammatic form as follows: here we have a line where an elemental part is shown by dl. Now, the continuous charge distribution can be represented in the form of the line charge density.

The line charge density is nothing but the charge per unit length which is λ. It means for a unit length the total charge along this line is λ. Then the elementary charge distribution in dl becomes $dq = \lambda dl$. Thus at the test point P which is located at a distance r away the total electric field due to this line charge distribution is $\frac{1}{4\pi\epsilon_0}\int \frac{\lambda dl}{r^2}\hat{\mathbf{r}}$. All we have done is replacing dq by λdl.

Next, we move on to the distribution that is a surface charge distribution. The surface charge distribution is depicted as this diagram. Here we have a surface where an elemental surface is shown as ds. Now the density of this surface charge distribution is given by the charge per unit area that is equals to σ.

Thus, the elementary charge distribution in the ds elemental area now becomes σds. Replacing dq by σds in this expression, we get the electric field due to a surface charge distribution at a test point P, which is located at a distance R from the source charge distribution. Finally, we will look into the most general form of a volume charge distribution. The volume charge distribution is depicted here.

Here we have the volume V in which an elemental volume is dV. Now the charge density is given by the charge per unit volume, that is ρ. Thus, the elementary charge distribution becomes ρdV. Now the elementary charge distribution in dV becomes ρdV. The electric field of the entire volume charge distribution can be obtained by simply replacing dq by ρdV in this expression. Thus, we obtain the electric field for a volume charge distribution as $\frac{1}{4\pi\epsilon_0}\int \frac{\rho dV}{r^2}\hat{\mathbf{r}}$. Thus, we now understand how to obtain the electric field for any charge distribution, whether it be continuous or discrete. We can also use both expressions for obtaining the electric field for a mixed charge distribution, which contains point sources as well as continuous charge distribution, by just the summation of this expression and this expression.

Now we look at the divergence of the electric field. To understand the concept of divergence of the electric field, we look into the flux of an electric field. As we had discussed in earlier lectures, the flux actually means the extent or magnitude of the field lines crossing through a particular area. Here we have the total area denoted by S and an infinitesimal area $d\mathbf{S}$. The $d\mathbf{S}$ vector is normal to the surface S and is shown here.
The electric field is $\mathbf{E}$. We can see the crossing of these field lines through this surface. Now the flux is given as $\int \mathbf{E}\cdot d\mathbf{S}$. This is the flux of the electric field. Now this flux of this electric field is equals to the $\frac{Q_{\text{enclosed}}}{\epsilon_0}$.

Here Q_{enclosed} denotes the total charge distribution which is enclosed within the surface. This can be obtained using the Gauss law. This also represents the integral form of the Gauss law or the divergence law for the electric field. We can also understand this from the differential form of the Gauss law. Applying the divergence theorem on the left hand side of the above equation we obtain $\int (\nabla\cdot\mathbf{E})dV = \int \mathbf{E}\cdot d\mathbf{S} = \int \frac{\rho}{\epsilon_0}dV$, where ρ is the volume charge density and dV is the volume element. So ρdV provides the Q_{enclosed}. Upon equating the integrals We obtain

$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$. This is the differential form of Gauss law.

Next, we consider the curl of the electric field. The curl of the electric field can be obtained as follows. We will be using the line integral of the electric field from point A to point B. This is in the same vein as obtaining the curl of the gravitational field, which we have seen in the previous module. Have a look at this picture, where we have a loop starting from A and ending at B. Here, the Cartesian coordinate system is used, with Q located at the origin.

Now, the distance between this charge Q and any point on this loop is given by R_A to R_B. This is the variable R, which gives the distance between the charge Q and a general point along this loop. Thus, the circulation The integral $\oint \mathbf{E}\cdot d\mathbf{l}$ along this loop is given by $\frac{1}{4\pi\epsilon_0}\int \frac{q}{r^2}dr$, integrated from A to B. This simplifies to $\frac{1}{4\pi\epsilon_0}\left(\frac{q}{R_A}-\frac{q}{R_B}\right)$. To obtain the circulation around a closed path implies A and B becoming coincident. This completes the loop but also means that A and B are equal.

Hence, R_A becomes equal to R_B. Now, we have the right-hand side of this equation going to zero. Thus, the integral from A to B of $\mathbf{E} \cdot d\mathbf{l} = 0$. This means that the circulation of the electric field around any closed loop equals zero. Applying Stokes' theorem, we can conclude that the curl of the electric field equals zero.

This is because the integral $\oint \mathbf{E} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{E}) \cdot d\mathbf{S}$. This integral must be obtained over the entire surface enclosed by this loop. By the property of curl-free fields, we can conclude that the electric field is a conservative field. Thus, we can conclude that The electric field can be used for various geophysical applications as it is a conservative field.

In particular, it can be stated that certain geophysical applications make use of electric fields. For example, the resistivity survey. The resistivity survey is a geophysical application for understanding the subsurface resistivity distribution. This is obtained by injecting electrical current into the ground through electrodes, which are nothing but points where the electric current is measured. The resulting voltage helps to determine the underground electric field distribution.

The distortions in the electric field distribution help map groundwater aquifers, locate mineral deposits, and assess soil contamination. This is possible because all these factors affect the electric field. Also, in induced polarization, which is another geophysical technique, the delayed response or polarization of subsurface materials to an electric field is used. The subsurface material distorts the response of the electric field. This is then utilized to detect possible anomalies that can be substituted for possible minerals.

This helps in extracting minerals, especially in clay-rich zones. One can look into the following references for more details. We will look into further applications in the next lectures. Thank you.