Mathematical Geophysics
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Hello everyone, welcome to the SWAYAM NPTEL course on mathematical geophysics. We will begin module number 4, which is mathematical modeling part 2. This is the first lecture of this module: system of linear differential equations. In this lecture, the concepts covered are related to the system of linear differential equations. The system of linear differential equations can be divided into four components.

First, the concept of differential equations, linear first-order equations, and then methods to solve linear first-order equations. Then we will look at various examples of first-order differential equations. Lastly, we will look into the application of differential equations in geophysical studies. This forms the overall aspects of this lecture. So let us begin.

The concepts of differential equations. What is a differential equation? A differential equation is an equation that contains derivatives. If it contains partial derivatives, it is called a partial differential equation. If it contains only ordinary differential components, then it is called an ordinary differential equation.

For example, Newton's second law, which we have seen earlier, F equals mass into acceleration, in differential equation form becomes:

$$F = m\frac{d^2r}{dt^2}$$

This is an ordinary differential equation. This is because the derivatives contained in this equation are taken with respect to only a single variable, that is t. The second is the wave equation. In the wave equation, it reads:

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

This is a partial differential equation. This is the expansion of the wave equation as written here. The Laplacian operator in Cartesian coordinate system is given by:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Recall that this has been discussed in detail in previous lectures. So, here we can see that this equation, which is a differential equation, contains derivatives with respect to four variables, namely x, y, z (spatial coordinates) and t (the temporal coordinate).

Hence, due to the presence of multiple variables with respect to which the derivatives are taken in this equation, this equation is called a partial differential equation. Now we look into various properties or notations related to differential equations. First is the order of a differential equation.

This is nothing but the order of the highest derivative in the equation. For example, in the previous examples, the highest derivative is taken with respect to t, which is of order 2. Hence, the order becomes 2.

Then comes linearity. Linearity means that the exponent of any derivative is limited to a maximum of 1. We can have the first derivative with respect to x or any order derivative, but the power of these derivatives should be 1. Such terms where the power of ∂/∂y is more than 1 are not included in linear differential equations. The general form of the linear differential equation with x as an independent and y as a dependent variable can be given as:

$$a_0(x)y + a_1(x)\frac{dy}{dx} + \cdots + a_n(x)\frac{d^ny}{dx^n} = b(x)$$

where prime denotes derivative with respect to the independent variable x. a_0 and b are constants or functions of x. The examples of linear and nonlinear differential equations can be shown as these two equations. The first example is of a nonlinear first-order equation, which has:

$$\frac{dy}{dx} + xy^2 = 1$$

The presence of an exponent 2 for the dependent variable y gives the nonlinearity.

This is an example of a linear first-order equation where the exponent of any quantity, which is derivative or y only, is maximum limited to 1:

$$\frac{dy}{dx} + p(x)y = q(x)$$

Note that the coefficient of the derivative is a function of x. This makes it a variable coefficient differential equation. If the coefficient is independent of x, then it would be a constant coefficient differential equation.

Next, we have the solution of a differential equation. The definition of the solution of a differential equation is given as the relation between the variables x and y, which, when substituted into the equation, provides an identity, which is nothing but the left-hand side equals the right-hand side.

For example, consider this equation:

$$\frac{dy}{dx} = e^x$$

If we have the function y(x) = e^x + C and we substitute this into the equation, then the left-hand side would be equal to e^x. That is the right-hand side. Then, the function y(x) would be the solution to this differential equation.

Next, we look into the linear first-order differential equation in more detail. A first-order differential equation, which is linear in nature, contains the first derivative y', which is equal to dy/dx, but no higher derivatives. The absence of any higher derivatives makes it first-order.

The general form of such linear first-order differential equations can be given as:

$$\frac{dy}{dx} + p(x)y = q(x)$$

Here, p and q can be functions of x in general. Our aim is to study a general method for the solution of such equations.

So, let us consider q = 0 in equation 1. This assumption is also called the homogenized form or homogeneous form of the first-order linear differential equation. With q = 0, we have:

$$\frac{dy}{dx} = -p(x)y$$

Writing it in operator form, we have:

$$\frac{dy}{y} = -p(x)dx$$

Upon integration of both sides, we get:

$$\ln|y| = -\int p(x)dx + C$$

This gives:

$$y(x) = Ae^{-\int p(x)dx}$$

where A = e^C.

We can denote ∫p(x)dx as I(x). This is the integrating factor. This can also be written as:

$$\frac{dI}{dx} = p(x)$$

obtained by differentiating both sides. Thus we have:

$$y(x) = Ae^{-I(x)}$$

or equivalently we can write:

$$y(x)e^{I(x)} = A$$

Next we look into the method to solve linear first order differential equation where q is not equal to 0. If q ≠ 0, we differentiate y e^{I(x)} with respect to x:

$$\frac{d}{dx}\left(ye^{I(x)}\right) = e^{I(x)}\left(\frac{dy}{dx} + p(x)y\right)$$

From the original differential equation, we know:

$$\frac{dy}{dx} + p(x)y = q(x)$$

Thus:

$$\frac{d}{dx}\left(ye^{I(x)}\right) = e^{I(x)}q(x)$$

Integrating both sides gives:

$$ye^{I(x)} = \int e^{I(x)}q(x)dx + C$$

Thus, the solution is:

$$y(x) = e^{-I(x)}\left(\int e^{I(x)}q(x)dx + C\right)$$

Now we look into various examples of first-order differential equations as they arise in various physics and geophysical problems. We have the first Bernoulli equation, which is a famous equation in fluid dynamics and geophysical fluid dynamics:

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

Here p and q are functions of x. The fact that y has exponent n on the right-hand side makes it a nonlinear equation. Since this is a nonlinear equation, we cannot solve it using the methods we have studied earlier. Thus, we have to find a way to reduce it to a linear equation for using the previous methods. We employ a change of variable. Let us consider:

$$z = y^{1-n}$$

Taking the derivative leads to:

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

Multiplying the original equation by (1-n)y^{-n} gives:

$$(1-n)y^{-n}\frac{dy}{dx} + (1-n)p(x)y^{1-n} = (1-n)q(x)$$

Substituting z and dz/dx gives:

$$\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x)$$

We have reduced the nonlinear equation to a linear form where the highest power for any derivative is 1. This is the highest power in this equation, and thus this equation becomes a linear differential equation. Note that the penalty which we have to keep in mind here is that dz / dy is no longer the form of derivative which was present in the original equation. In the original equation, the independent variable was x. Now with the substitution, we have introduced one more dependent variable z, which depends on y, which further depends on x. Essentially, we have introduced one more variable which is dependent on the dependent variable y in the removal of nonlinearity. So we have equation 4, which is a linear first-order differential equation, and the methods of first-order differential equations can be used to solve this equation. The solution to this differential equation will be in the form of z, and upon further substitution, we can get y as a function of x, which will be the solution to the original differential equation. Now let us consider various applications of differential equations in a nutshell, which occur in geophysics. We will be expanding on each of these topics in further lectures.

First, we have the seismological applications of differential equations. Differential equations describe the elastic rebound theory and the stress-strain relationships in seismological applications. The stress-strain relationships occur in fault areas, which are the regions where earthquakes originate. Also, we have the seismic wave propagation, which is nothing but a form of wave equation applied to seismic waves originating from the earthquake source:

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

The propagation of these waves through the Earth's various layers is governed by partial differential equations like the wave equation discussed earlier.

In geodynamical applications, the heat transfer equation helps us determine the temperature distribution in the lithosphere as well as the mantle:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

where T represents temperature, t represents time, and α is thermal diffusivity.

Lastly, we have the example from geomagnetism. Geomagnetism governs the evolution and dynamics of Earth's magnetic field through magnetohydrodynamic equations. One of these is the induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

where B is the magnetic field, u is the velocity field, and η is the magnetic diffusivity.

We also have the curl operator. We can see that this is a partial differential equation with various calculus operators, which will result in derivatives and partial derivatives, giving the differential equation. Thus, we arrive at the conclusion of differential equations for geophysics.

In geophysics, linear differential equations are an important tool for modeling and interpreting various geophysical processes. These processes range from wave propagation to fluid dynamics in the Earth's interior. The differential equation is widely used as a basic mathematical framework for modeling various governing equations. In particular, we have Maxwell's equations, which are very useful for mineral resource exploration and form an important and integral part of geophysical applications. Also, linear differential equations form the foundation of complex nonlinear models.

Nonlinear models are closer to realistic conditions, which are found in geophysical processes. However, the complexity of these nonlinear models makes it very difficult to gain insight into the processes, which makes it necessary to reduce them to linear forms before solution. Although nonlinear models have been recently used for detecting various geophysical processes through the use of computer simulations and other advanced methods. One can refer to the book 'Mathematical Methods in Physical Sciences' for further understanding.

Thank you.