Mathematical Geophysics
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Hello everyone. Welcome to the NPTEL course on mathematical geophysics. We continue with module 3, mathematical modeling part 1. This is the fifth lecture, upward continuation. In this lecture, the concepts covered include upward and downward continuation.
To understand upward and downward continuation, we have to look into the concepts of potential theory. These are the general concepts of potential theory. In particular, we will be looking into scalar and vector potential theories. This would help us to understand the method of continuation in relation to the gravitational field. So, let us begin.

The concept of potential theory. Potential theory is the study of force fields through the use of their potentials. Remember that potentials can only be obtained for force fields that are irrotational. That means the curl of the force field must be equal to zero over the entire domain, and hence the potential can be obtained such that the force field can be represented as the gradient of the potential. So if P is the potential, **F** is equal to minus the gradient of the potential P:

$$\mathbf{F} = -\nabla P$$

Instead of directly studying the force field **F**, we use P as a potential to study the various aspects of a gravitational field or any other field is what potential theory does. So, potential theory is meant for describing the energy of the field at any point in space using scalar functions, the potentials. Now, recall the concept of field and generator as we discussed in the previous lecture. A field is a region in space related to a physical quantity that varies in both space and time. The source is the origin of the field, and that source is known as the generator.

For example, for a gravitational field, the masses are the generators. The charges are the generators of an electric field, and so on. The relation between the field and potential is discussed next. Based on the nature of the quantity and the resultant coordinate transformations, the field can be divided into various categories, such as scalar, vector, and tensor fields. The corresponding potentials can also vary.

Thus, corresponding to a scalar field, we can have a potential. We can also have a potential corresponding to a vector field. In this case, the potential of a vector field is a scalar quantity. For a tensor field, the potential is a vector quantity.

Thus, we can have scalar potentials and also vector potentials. To further motivate us from the geophysical perspective, we examine various applications of potential theory in geophysical studies. Potential theory provides a powerful mathematical framework for interpreting gravitational, magnetic, electrical, and seismic fields, which are conservative in nature. Potential theory can also be used to understand subsurface structures and detect various anomalies, such as density anomalies, magnetic anomalies, and conductivity anomalies inside the Earth's surface. Potential theory also models dynamic processes that occur in the Earth, such as the motion of magmatic material in the mantle or the motion of iron in liquid form in the liquid outer core of the

Earth. Potential theory is used in a wide range of applications, from natural resource exploration to environmental monitoring and earthquake analysis.

In brief, we examine the two potential theories: scalar potential theory and vector potential theory. Scalar potential theory is associated with a scalar potential field, denoted as F. For clarity, it is stated that F is a scalar and represents a potential field. The conditions for the potential field F, which must be ensured, are as follows.

F exists only if the field corresponding to F is curl-free. That is, if the vector field $\mathbf{F}$ is the field, then the curl of $\mathbf{F}$ equals zero, which means it is irrotational:

$$\nabla \times \mathbf{F} = 0$$

This allows the field to be expressed as the gradient of the scalar potential F. The potential field typically satisfies Laplace's or Poisson's equations.

The governing equations for scalar potential field are as follows. In a region where the source of the field is present, then the scalar potential field F satisfies the Poisson's equation:

$$\nabla^2 F = -S$$

where S is the source. Devoid of any source, the Poisson equation reduces to the Laplacian equation:

$$\nabla^2 F = 0$$

This implies that scalar potential in source-free region are harmonic in nature as they satisfy the Laplace's equation.

Next, we consider the vector potential theory. The vector potential is denoted by a vector. The conditions are the field must be solenoidal. That means the vector field $\mathbf{F}$, its divergence, that is divergence of $\mathbf{F}$ must be equal to zero:

$$\nabla \cdot \mathbf{F} = 0$$

Being divergence-free is same as being solenoidal.

This allows the vector field $\mathbf{F}$ to be represented as the curl of the vector potential field $\mathbf{A}$:

$$\mathbf{F} = \nabla \times \mathbf{A}$$

Since $\nabla \cdot \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$, the above curl of the vector potential leads to the divergence-free field. This is also accompanied by the governing equations for the vector potential. $\mathbf{A}$ is determined by solving the Poisson-like equation such as:

$$\nabla^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{F}) = -\nabla \times \mathbf{F}$$

If the force field $\mathbf{F}$ is known, then $\nabla^2 \mathbf{A}$ can be easily obtained and further $\mathbf{A}$ can be determined.

In general, we have the Helmholtz decomposition of a vector field. This is the Helmholtz decomposition which is stated as:

$$\mathbf{F} = -\nabla f + \nabla \times \mathbf{A}$$

The first term, which is $-\nabla f$, is the irrotational part, whereas $\nabla \times \mathbf{A}$ is the solenoidal part of the vector field $\mathbf{F}$. The vector potential is widely used in geophysical fluid dynamics where the velocity can be expressed as the curl of a stream function. This occurs in three-dimensional fluid dynamics.

Using the various concepts and examples which we have discussed up till now, we next look into the application in gravitational field theory in geophysical context. We look into the particular concept of upward continuation. Upward continuation, as the name suggests, helps us to determine the gravitational field at a point which is above or upward from its measurement location, that is the Earth's surface. Consider this diagram. The Earth's surface or the measuring location is shown here.

On the Earth's surface, the gravity can be measured using various instruments. Consider that we have the gravitational field which is measured on the Earth's surface. The idea is to obtain the gravitational field at any upward location. For example, this point which is above the surface of the Earth. Alongside this aim, we also have an alternative objective. This objective is to reduce the influence of geological noise from the target deep-seated structures. Now, these are shallow geological structures which may have various density differences among them. The aim is to reduce the noise obtained in the measurement due to these geological structures. The original signal is coming from the target deep structure, which is located at further depths inside the Earth's surface.

So, the deep structure is the target, which is the useful signal. And the shallow subsurface, with its laterally varying rock density and other geological noises, has to be removed. This can be achieved using the upward continuation method. In the upward continuation method, since we are obtaining the gravitational field at a point above the surface of the Earth, the signature of the geological noise structures becomes smaller and smaller at a faster rate than the targeted deep structure. Let us understand why this happens.

If A is the distance between the measurement location and the location where the gravitational field is to be obtained, and B is the difference between the location where geological noise is present and the point where the gravitational field is to be obtained, and C is the distance between the target deep structure and the point where the gravitational field is to be obtained. As the gravitational point moves further upward, the influence of the geological sources diminishes at a faster rate than the influence of the target deep structure. This occurs because $\frac{\Delta B}{B}$ is much greater than $\frac{\Delta C}{C}$.

ΔB is the change in B if the point of estimation moves upward. If ΔC is also measured, then that is equal to ΔB because the change is equal. But since B is much smaller than C, $\frac{\Delta B}{B}$ becomes much larger than $\frac{\Delta C}{C}$. This reduces the contribution of geological noise at a much faster rate than the targeted deep structure contributions, which essentially remain unaffected. So, through the use of upward continuation, we can reduce geological noise.

Now, what is upward continuation in terms of mathematical formulation? In terms of mathematical formulation, we intend to obtain the field at any point above the Earth's surface in terms of the field at the surface. To calculate the field and potential in such a manner, we consider a half-space. Consider this diagram where the Earth's surface is shown. Point S is below the Earth's surface.

The hemispherical dome represents the upper half-space. R is the radius of the upper half-space. P and Q are two position vectors where the gravitational force is being determined. The distance between them is L_{QP}. Now, as we have seen, since the gravitational field is conservative and irrotational, we can have the potential U satisfying the Laplace equation:

$$\nabla^2 U = 0$$

This is valid for the region which is upper half-space since there is no mass within this upper half-space. On the Earth's surface, the gradient of the potential which gives the component of gravity is given by the partial derivative of U along the z-direction gives the z-component of the gravitational field:

$$\frac{\partial U}{\partial z} = g_z$$

We also have that the potential tends to zero as r tends to infinity:

$$U \to 0 \quad \text{as} \quad r \to \infty$$

It means that the potential decays away to zero as the upper half-space becomes infinitely large. To obtain the upward continuation equation, we take help of the Green's formula, which is given here.

Recall that Green's formula relates volume integrals to surface integrals. It states:

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, dV = \oint_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) dS$$

Both integrated over the depicted volume and surface enclosing that volume, respectively. Both the functions ϕ and ψ are continuous, and their first derivatives exist. The **n** vector is the unit vector which is directed outside the volume.

Now, let us assume that $\phi(Q) = U(Q)$ and $\psi(Q) = G(Q)$. This means at this point Q, ψ and ϕ are defined such that ϕ is the vector potential or the gravitational potential and ψ is another potential which is G. Thus, the Green's formula reduces to:

$$\int_V U \, \nabla^2 G \, dV = \oint_S \left(U \frac{\partial G}{\partial n} - G \frac{\partial U}{\partial n} \right) dS$$

We have used the condition or the result that the potential obeys Laplacian equation at the point Q, which is outside the Earth's surface and included within the half-space.

To obtain an explicit expression for the potential U, we assume the following. First, the Laplacian equation for G equals zero:

$$\nabla^2 G = 0$$

This is valid everywhere inside the half-space except at the observation point P. Now, as Q approaches P, G must follow a singular-like behavior, which is equivalent to $\frac{1}{L_{QP}}$, where L_{QP} is the distance between Q and P. Thus:

$$G(QP) \to \frac{1}{L_{QP}} \quad \text{as} \quad Q \to P$$

As the radius R of the hemisphere of the half-space increases, the function G decreases at least inversely proportional to the square of L_{QP}. Thus, substituting this into the previous equation, we get:

$$\oint_S \left(U \frac{\partial G}{\partial n} - G \frac{\partial U}{\partial n} \right) dS = 0$$

Now, this equation includes the vector potential for the gravitational field U, and the potential G over the entire closed surface area, which is the upper half-space. Now, we intend to obtain the potential function for the gravitational field, that is, $U(P)$, from this equation (3). This involves detailed calculus and algebraic derivation, which is available in the referred book.

The relevant equations are 2.98 to 2.101. Thus, we have the gravitational potential U at any desired point P equals:

$$U(P) = \frac{1}{4\pi} \int_S \left(G(QP) g_z - U(Q) \frac{\partial G(QP)}{\partial z_P} \right) dS$$

where z_P is the axial location of the point P where the potential is to be estimated. The axial component of the gravitational field $g_z(P)$ can be obtained by the partial derivative of U with respect to z:

$$g_z(P) = \frac{\partial U}{\partial z} = \frac{1}{4\pi} \int_S \left(g_z(Q) \frac{\partial G(QP)}{\partial z_P} - U(Q) \frac{\partial^2 G(QP)}{\partial z_P \, \partial z_Q} \right) dS$$

Now, recall that $G(QP)$ is the distance between Q and P. Hence, also $G(QP)$ tends to a singularity if Q and P become closer and closer. But if Q and P become farther and farther, then $G(QP)$ goes to 0. Thus, this equation gives the equation for upward continuation.

With the help of the equation of upward continuation, we can obtain the gravitational field at any desired point P above the surface of the Earth using the location of P and the location of Q and the potential U. Also, we need the gravitational field at Q. If Q is the Earth's surface, then the measurements of the gravitational field made on the Earth's surface are required.

Next, we come to the topic of reduction of geological noise. The reduction of geological noise can be obtained in the limit $\frac{\partial G}{\partial z_P} \to 0$, which means that as the location of estimation of the gravitational field, that is P, moves further and further up, the derivative of G with respect to z_P, that is the axial height of this position P, becomes 0.

This essentially means that the effect of the geological sources near the surface of zero becomes negligible. This simplifies the potential for the gravitational field as:

$$U(P) = \frac{1}{4\pi} \int_S G\,(QP) g_z(Q)\, dS$$

And with the partial derivative with respect to z, we obtain the gravitational field at P in terms of the gravitational field at Q:

$$g_z(P) = \frac{1}{4\pi}\int_S g_z\,(Q)\,\frac{\partial G(QP)}{\partial z_P}\,dS$$

This gives the upward continuation for the geological noise reduction. So these are the equations of upward-downward continuation.

From this discussion, we can conclude that upward continuation smooths the gravitational field, reducing the influence of short-wavelength anomalies. These short-wavelength anomalies are caused by near-surface geological noise structures. The upward continuation is also used in relation to signal attenuation. Signal attenuation occurs when elevation increases. As elevation increases, the magnitude of the gravitational signal decreases with greater distance from the source, emphasizing large-scale anomalies. Upward continuation is commonly used in exploration geophysics, planetary sciences, and gravity anomaly detection to understand subsurface features and refine satellite gravity data.

One can refer to the following book for more details. Thank you.