Mathematical Geophysics
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Hello everyone, welcome to the SWAYAM NPTEL course on Mathematical Geophysics. We continue with module number 3, *Mathematical Modeling 1*. This is lecture 4, *Equations of the Gravitational Field*. In this lecture, the concepts covered are related to the equations governing the gravitational field. These are the equations that state the properties of the gravitational field from various perspectives.

These perspectives are the integral form, the differential form, and the gravitational potential. The application of these equations of the gravitational field to various geophysical aspects forms the fourth component of this lecture. Thus, this lecture focuses on the various forms in which the gravitational field can be expressed mathematically and the construction of a gravitational potential, which is very useful in understanding different aspects of the gravitational field in a simplified mathematical manner. So, let us begin. First, we look at the integral form of the equations for the gravitational field.

So, the first integral equation is the line integral of $\mathbf{G} \cdot d\mathbf{l}$ equals zero. What does it mean? This is the first equation of the gravitational field in integration or integral form. So, this is the first equation of the gravitational field in integral form. That is, the line integral of $\mathbf{G} \cdot d\mathbf{l}$ equals zero.

We derive the derivation. We look into the derivation of this integral form of the equation. Consider an arbitrary loop. This is a differential line element denoted by $d\mathbf{l}$ vector. This loop is a closed loop.

Consider this as an elementary mass located at the position vector $\mathbf{q}$. We consider the integral $\int_a^b \mathbf{G} \cdot d\mathbf{l}$ as the partial circulation of the gravitational field. Now A and B can be any two points on this closed loop. We denote the location of this line element $d\mathbf{l}$ vector as P. The distance between P and Q is L_{QP}. Thus, the circulation can be given by $-G\ dm$, which is the elementary mass, multiplied by the integral from A to B of $\frac{dl}{L_{QP}^2}$.

Now, L_{QP}^2 is a function of the location P. As the location varies from A to B, the distance L_{QB} changes. Hence, the integral can be simplified and evaluated as $G\ dm$ multiplied by $\left(\frac{1}{L_{QB}} - \frac{1}{L_{QA}}\right)$, since the gravitational field depends on the end points as it is a conservative field. One can also obtain this using regular integral calculus. Now, we have the expression of the circulation from point A to B. But since this is a closed loop and the theorem is for a closed loop, we have to estimate what happens when the circulation is evaluated over a closed path.

Over a closed path, A and B will extend and coincide. This means L_{QB} will be equal to L_{QA}, and this makes the bracketed term equal to zero. This gives us the proof for the first equation of the gravitational field, which is the line integral of $\mathbf{G} \cdot d\mathbf{l}$ equals zero. This renders the gravitational field circulation-free. Now, we move on to the second equation in integral form.

The second equation, or the integral form of the equation, reads as the integral of $\mathbf{G} \cdot d\mathbf{S}$ equals $-4\pi Gm$. Physically, it means that the flux of the field over an entire closed surface equals $-4\pi Gm$, where m is the mass inside this closed surface. Now, have a look at this diagram. Consider this as a closed surface area enclosing a volume V. An elementary surface area on this surface can be given by the vector $d\mathbf{S}$.

Thus, the flux through this vector $d\mathbf{S}$ is given by $\mathbf{G} \cdot d\mathbf{S}$, where $\mathbf{G}$ is the gravitational field. Now, we consider the mass m. This is located at point Q, and $d\mathbf{S}$ is located at point P. Thus, the distance between these two points is L_{QP}. From Newton's law of attraction, we get $\mathbf{G} \cdot d\mathbf{S} = -G\, dm \cdot \frac{\mathbf{L}_{PQ} \cdot d\mathbf{S}}{L_{QP}^3}$. This is equivalent to $-G \frac{m}{L_{QP}^2} \cdot dS$. The $d\mathbf{S}$ vector is the area. It's also useful to represent this as $\mathbf{G} \cdot d\mathbf{S} = -G\, dm\, d\Omega$. $d\Omega$ is the elementary solid angle subtended by the elementary surface area dS at the point Q.

For a better understanding, let us have a separate diagram. The elementary surface area dS subtends the elementary solid angle $d\Omega$ at point Q. The $\mathbf{n}$ vector is the vector perpendicular to the surface area. Now we consider the total flux through this arbitrary surface area, which is given by the integral of $\mathbf{G} \cdot d\mathbf{S}$. This is nothing but the integral of the solid angle for the entire closed loop. We have Ω.

Now Ω, as we can recall from previous lectures, is the area divided by the distance between the point Q and the surface area squared. This has been used in the previous derivation. $\frac{\mathbf{L}_{PQ}}{L_{PQ}^3}$ is essentially $\frac{1}{L_{PQ}^2}$ multiplied by dS, which is then substituted as $d\Omega$ and integrated over the entire surface area to get Ω. Thus, the total flux, that is the integral of $\mathbf{G} \cdot d\mathbf{S}$, is $-\Omega G\, dm$. Thus, the flux through an arbitrary closed surface surrounding the elementary mass is given by $-4\pi G\, dm$.

The circle on this integral represents that this integral is being taken over a closed surface area. Now we know that if the solid angle is measured such that the point lies within the closed surface area, then the total solid angle is always 4π. Thus, the expression. This 4π is irrespective of the surface shape and the position of Q inside the volume surrounded by the surface S. Even if the point Q is at any of these locations, the total solid angle will be 4π. Now we use the principle of superposition and, assuming that the volume V has an arbitrary distribution of masses, we get the integral of $dm = m$, where m is the total mass in the volume V.

So this expression is for an elementary mass, and this expression is for all the elementary masses inside that volume. So we obtain the second integral equation for the gravitational field, which is the flux equals $-4\pi Gm$. Now we turn to the differential form of the equations of the gravitational field. These two equations are the differential form of the equations for the gravitational field. The first is the curl of $\mathbf{G}$ equals 0.

The second is the divergence of $\mathbf{G}$ equals $-4\pi G\delta$. The first indicates that the gravitational field is irrotational or curl-free, and the second equation shows that the divergence of the gravitational field is negative, which means that the field lines point inwards toward the location of the source of the gravitational field. So let us look into these in more detail. First, the derivation of $\nabla \times \mathbf{G} = 0$. Recall the Stokes theorem, which links line integrals with surface integrals.

The Stokes theorem says that the circulation is equal to the flux of the curl of a gravitational field. From previous discussion, we have obtained that $\oint \mathbf{G} \cdot d\mathbf{l} = 0$. And the curl of $\mathbf{G}$ being a non-zero quantity and hence this expression of $\int (\nabla \times \mathbf{G}) \cdot d\mathbf{S} = 0$ leads to $\nabla \times \mathbf{G} = 0$. This is because for any arbitrary surface if the integral goes to zero then the integral quantity is zero. This is the first equation of gravitational field in differential form.

This is valid at points inside and outside masses where the first derivative of the field can exist. Next we look into the derivation of $\nabla \cdot \mathbf{G} = -4\pi G\delta$. Here δ is the density. We have $\oint \mathbf{G} \cdot d\mathbf{S} = -4\pi Gm$ from the integral form of the gravitational field which we have discussed just previously. Denoting m as $\int \delta \, dV$, which is nothing but density into volume, we can obtain

$\oint \mathbf{G} \cdot d\mathbf{S} = -4\pi G \int \delta \, dV$. This can be given in terms of $\int (\nabla \cdot \mathbf{G}) \, dV$ equals the right-hand side. Now $\nabla \cdot \mathbf{G} \, dV$ comes from the Gauss theorem. This, you can recall that Gauss theorem connects surface integrals to volume integrals where the divergence of the field is involved. Because these integrals are equivalent or equal to each other for any arbitrary volume, we can equate their integrands.

This gives $\nabla \cdot \mathbf{G} = -4\pi G\delta$. This is the differential form of the second equation of the gravitational field. And it is valid for all regular points where the first derivative of the field exists. Now, what happens if the mass is located outside the enclosed area?

Since this expression is valid for δ or m, which is located inside the enclosed surface, thus if the mass is taken out of the enclosed surface, this goes to zero. Thus, we have $\nabla \cdot \mathbf{G} = 0$. Now we move on to a quick summary and further discussions. The gravitational field is an irrotational field, which we can obtain from the differential form of the gravitational equation, that is, $\nabla \times \mathbf{G} = 0$. Next, we look into the gravitational field and its potential function. This is the result of the irrotational field property of the gravitational field.

Since $\nabla \times \mathbf{G} = 0$ and $\nabla \times (\nabla U) = 0$, $\mathbf{G}$ can be represented as $-\nabla U$, which is the potential function. This is the property of curl, and this has been discussed in previous lectures relating to identities of curl of vector fields. This gives us the Poisson's equation formulation for the gravitational potential. Since $\nabla \cdot \mathbf{G} = -4\pi G\delta$, we can substitute $-\nabla U$ in place of $\mathbf{G}$ to obtain $\nabla^2 U = 4\pi G\delta$. And if the mass is outside the volume, then the gravitational potential obeys the Laplacian equation.

This is Laplace's equation, applicable for the gravitational potential without any internal masses. Having discussed the various forms of the gravitational equations, we now move on to their application in geophysical studies. The gravitational potential and Earth's gravitational field are very interrelated concepts. The Earth's gravitational field is very appropriately modeled in terms of gravitational potential. The variations in density within the crust and mantle make it suitable for representation as a gravitational potential.

The gravitational potential is also used to compute the shape of the geoid and detect various anomalies in gravity, which are caused by subsurface structures. This means that if this is the surface of the Earth, there may be structures that are of higher density than the surrounding. This leads to gravity anomalies, which are nothing but variations in the gravity readings made by surface measurements. If these are the locations where gravity is measured, these measurements will be affected by localized high-density bodies and result in gravity anomalies.

In this case, the understanding of gravitational potential can help us to remove such noise, which may arise from geological sources, and focus on the more fundamental and necessary details. Further applications of the gravitational field are listed as follows. The differential form of the gravitational field is used to calculate the spatial variations of the gravitational field due to local density contrast. It also provides a framework for understanding gravity anomalies, which are caused by the variation of density in various layers of the Earth.

The differential form is also used to directly link measurable quantities to density distribution. On the other hand, integral forms are used for forward modeling to compute the entire gravitational field from known mass distributions located in the subsurface. The integral form is also used in global studies, such as computing the Earth's geoid and the gravitational potential of the Earth. The integral form is also used in numerical simulation models to estimate the gravitational influence of complex geological structures, which cannot be performed analytically; hence, the integral form, along with numerical simulation, comes to the rescue.

Hence, the integral form used in numerical simulation helps. Thus, we conclude that the equations of the gravitational field are widely used to investigate Earth's internal structure and dynamics. On the surface of the Earth, along with measurements of various components of gravity, the gravitational field and its equations can be used to explore natural resources. These equations are also used to monitor environmental changes, such as ice sheet dynamics and sea level variations, which are very important from a climate dynamics perspective. The following reference can be referred to for more details.

Thank you.